Research Article

A Class of Spectral Element Methods and Its A Priori/A Posteriori Error Estimates for 2nd-Order Elliptic Eigenvalue Problems

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This paper discusses spectral and spectral element methods with Legendre-Gauss-Lobatto nodal basis for general 2nd-order elliptic eigenvalue problems. The special work of this paper is as follows. (1) We prove a priori and a posteriori error estimates for spectral and spectral element methods. (2) We compare between spectral methods, spectral element methods, finite element methods and their derived *p*-version, *h*-version, and *hp*-version methods from accuracy, degree of freedom, and stability and verify that spectral methods and spectral element methods are highly efficient computational methods.

1. Introduction

As we know, finite element methods are local numerical methods for partial differential equations and particularly well suitable for problems in complex geometries, whereas spectral methods can provide a superior accuracy, at the expense of domain flexibility. Spectral element methods combine the advantages of the above methods (see [1]). So far, spectral and spectral element methods are widely applied to boundary value problems (see [1, 2]), as well as applied to symmetric eigenvalue problems (see [3]). However, it is still a new subject to apply them to nonsymmetric elliptic eigenvalue problems.

A posteriorii error estimates and highly efficient computational methods for finite elements of eigenvalue problems are the subjects focused on by the academia these years; see [3–16], and among them, for nonsymmetric 2nd-order elliptic eigenvalue problems, [5, 15] provide a posteriori error estimates and adaptive algorithms, [9] the function value recovery techniques and [8, 10] two-level discretization schemes.

Based on the work mentioned above, this paper shall further apply spectral and spectral element methods to nonsymmetric elliptic eigenvalue problems. This paper will mainly perform the following work.

(1) We prove a priori and a posteriori error estimates of spectral and spectral element methods with Legendre-Gauss-Lobatto nodal basis, respectively, for the general 2nd-order elliptic eigenvalue problems.

(2) We compare between spectral methods, spectral element methods with Legendre-Gauss-Lobatto nodal basis, finite element methods, and their derived *p*-version, *h*-version, and *hp*-version methods from accuracy, degree of freedom, and stability and verify that spectral methods and spectral element methods are highly efficient computational methods for nonsymmetric 2nd-order elliptic eigenvalue problems.

This paper is organized as follows. Section 2 introduces basic knowledge of second elliptic eigenvalue problems. Sections 3 and 4 are devoted to a priori and a posteriori error estimates of spectral and spectral element methods, respectively. In Section 5, some numerical experiments are performed by the methods mentioned above.

2. Preliminaries

Consider the 2nd-order elliptic boundary value problem

$$Lw = -\nabla \cdot (D\nabla w) + \mathbf{b} \cdot \nabla w + cw = f, \quad \text{in } \Omega,$$
$$w = 0, \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset R^d$ $(d = 2, 3)$ is a bounded domain, $\mathbf{b}$ and c are a real-valued vector function and a real-valued function, respectively, and D is a positive scalar function with $D(x) \geq D_0 > 0$ $(\forall x \in \Omega)$.

We denote the complex Sobolev spaces with norm $\|\cdot\|_m$ by $H^m(\Omega)$, $H_0^1(\Omega) = \{v \in H^1(\Omega), v|_{\partial\Omega} = 0\}$. Let $(\cdot, \cdot)$ and $\|\cdot\|_{0,\Omega}$ be a inner product and a norm in the complex space $L^2(\Omega)$, respectively.

In this paper, C denotes a generic positive constant independent of the polynomial degrees and mesh scales, which may not be the same at different occurrences.

Define the bilinear form $a(\cdot, \cdot)$ as follows:

$$a(w, v) = \int_\Omega D\nabla w \nabla \bar{v} + (\mathbf{b} \cdot \nabla w)\bar{v}$$
$$+ cw\bar{v}, \quad \forall w, v \in H_0^1(\Omega). \tag{2}$$

We assume that $f \in L^2(\Omega)$, D, $\mathbf{b}$, and c are bounded functions on Ω, namely $D, c \in L^\infty(\Omega)$, $\mathbf{b} \in (L^\infty(\Omega))^d$. Furthermore, we assume that $\nabla \cdot \mathbf{b}$ exists and satisfies

$$-\frac{1}{2}\nabla \cdot \mathbf{b} + c \geq 0, \quad \text{in } \Omega. \tag{3}$$

Under these assumptions, the bilinear form $a(\cdot, \cdot)$ is continuous in $H_0^1(\Omega)$ and $H_0^1(\Omega)$-elliptic; that is, there exist two constants $A, B > 0$ independent of w, v such that

$$|a(w, v)| \leq A\|w\|_{1,\Omega}\|v\|_{1,\Omega}, \quad \forall w, v \in H_0^1(\Omega),$$
$$\text{Re } a(v, v) \geq B\|v\|_{1,\Omega}^2, \quad \forall v \in H_0^1(\Omega). \tag{4}$$

The corresponding variational formulation of (1) is given as follows: find $w \in H_0^1(\Omega)$, such that

$$a(w, v) = (f, v), \quad \forall v \in H_0^1(\Omega). \tag{5}$$

The adjoint problem of (5) is as follows: find $w^* \in H_0^1(\Omega)$, such that

$$a(v, w^*) = (v, f), \quad \forall v \in H_0^1(\Omega). \tag{6}$$

As the general 2nd-order elliptic boundary value problems, we assume that the regularity estimates for problem (5) and its adjoint problem (6) hold, respectively. Namely

$$\|w\|_{r_1+1,\Omega} \leq C\|f\|_{0,\Omega}, \quad 0 < r_1 \leq 1, \tag{7}$$

$$\|w^*\|_{r_2+1,\Omega} \leq C\|f\|_{0,\Omega}, \quad 0 < r_2 \leq 1. \tag{8}$$

We assume that $K_h = \{\kappa\}$ is a regular rectangle (resp. cuboid) or simplex partition of the domain Ω and satisfies $\overline{\Omega} = \bigcup \overline{\kappa}$. We associate with the partition a polynomial degree vector $\mathbf{N} = \{N_\kappa\}$, where N_κ is the polynomial degree in κ. Let h_κ be the diameter of the element κ, and let $h = \max_{\kappa \in K_h} h_\kappa$.

We define spectral and spectral element spaces as follows:

$$S_N(\Omega) = \{v \in P_N(\Omega), v|_{\partial\Omega} = 0\},$$
$$S_{N,h}(\Omega) = \{v \in C(\overline{\Omega}) : v|_\kappa \in P_{N_\kappa}(\kappa), \forall \kappa \in K_h, v|_{\partial\Omega} = 0\}, \tag{9}$$

where $P_N(\Omega)$ and $P_{N_\kappa}(\kappa)$ are polynomial spaces of degree N (resp. degree N in every direction) in Ω and degree N_κ (resp. degree N_κ in every direction) in the element κ, respectively.

The spectral approximation of (5) is as follows: find $w_N \in S_N(\Omega)$, such that

$$a(w_N, v) = (f, v), \quad \forall v \in S_N(\Omega). \tag{10}$$

The spectral element approximation of (5) is as follows: find $w_{N,h} \in S_{N,h}(\Omega)$, such that

$$a(w_{N,h}, v) = (f, v), \quad \forall v \in S_{N,h}(\Omega). \tag{11}$$

We assume that $f \in L^2(\Omega)$ and derive from Lax-Milgram theorem that the variational formations (5), (6), (10), and (11) have a unique solution, respectively.

Define the interpolation operators

$$I_{N_\kappa, h_\kappa} : C(\kappa) \longrightarrow P_{N_\kappa}(\kappa),$$
$$I_N : C(\Omega) \longrightarrow S_N(\Omega), \tag{12}$$

as the interpolations in the element κ and the domain Ω, respectively, with the tensorial Legendre-Gauss-Lobatto (LGL) interpolation nodes.

Define the interpolation operator

$$I_{N,h} : C(\Omega) \longrightarrow S_{N,h}(\Omega), \quad \text{satisfying } I_{N,h}|_\kappa = I_{N_\kappa, h_\kappa}. \tag{13}$$

We quote from [2] (see (5.8.27) therein) the interpolation estimates for spectral and spectral element methods with LGL Nodal-basis as follows.

For all $v \in H^{s_\kappa}(\kappa)$, $s_\kappa \geq (d+1)/2$,

$$\left\|v - I_{N_\kappa, h_\kappa} v\right\|_{1,\kappa} \leq C(s_\kappa) h_\kappa^{\min(N_\kappa+1, s_\kappa)-1} N_\kappa^{-s_\kappa+1} \|v\|_{s_\kappa, \kappa}, \tag{14}$$

$$\left\|v - I_{N_\kappa, h_\kappa} v\right\|_{0,\kappa} \leq C(s_\kappa) h_\kappa^{\min(N_\kappa+1, s_\kappa)} N_\kappa^{-s_\kappa} \|v\|_{s_\kappa, \kappa}. \tag{15}$$

For all $v \in H^s(\Omega)$, $s \geq (d+1)/2$,

$$\|v - I_N v\|_{1,\Omega} \leq C(s) N^{-s+1} \|v\|_{s,\Omega}, \tag{16}$$

$$\|v - I_N v\|_{0,\Omega} \leq C(s) N^{-s} \|v\|_{s,\Omega}. \tag{17}$$

We assume that the solution of boundary value problem (5) $w \in H_0^1(\Omega) \cap H^m(\Omega)$ $(m > 1)$, that w_N and $w_{N,h}$ are the

solutions of (10) and (11), respectively; then we derive from Céa lemma and the interpolation estimates that

$$\|w_N - w\|_{1,\Omega} \le C(m) N^{-m+1} \|w\|_{m,\Omega}, \tag{18}$$

$$\|w_{N,h} - w\|_{1,\Omega} \le \left\{ \sum_\kappa C(s_\kappa) h_\kappa^{2\{\min(N_\kappa+1,s_\kappa)-1\}} \right. \tag{19}$$
$$\left. \times N_\kappa^{2(-s_\kappa+1)} \|w\|_{s_\kappa,\kappa}^2 \right\}^{1/2},$$

where $s_\kappa \ge (d+1)/2, \forall \kappa \in K_h$.

Particularly, if $N_\kappa = N, \forall \kappa \in K_h$, then we have

$$\|w_{N,h} - w\|_{1,\Omega} \le C(m) h^{\min(N+1,m)-1} \tag{20}$$
$$\times N^{1-m} \|w\|_{m,\Omega}.$$

Note that (18) is also suited to spectral methods with modal basis (see [1, 2]).

Using Aubin-Nitsche technique, we deduce from the regularity estimate (8) and the estimates (18)–(20) the priori estimates of boundary value problem (5) for spectral and spectral element methods; that is,

$$\|w_N - w\|_{0,\Omega} \le C N^{-m-r_2+1} \|w\|_{m,\Omega}, \tag{21}$$

$$\|w_{N,h} - w\|_{0,\Omega} \le C N^{-m-r_2+1} h^{r_2+\min(N+1,m)-1} \|w\|_{m,\Omega}. \tag{22}$$

3. Spectral and Spectral-Element Approximations and Error Estimates for Eigenvalue Problems

3.1. Spectral and Spectral-Element Approximations for Eigenvalue Problems. Consider the following eigenvalue problem corresponding to the boundary value problem (1):

$$Lu = \lambda u, \quad \text{in } \Omega,$$
$$u = 0, \quad \text{on } \partial\Omega. \tag{23}$$

The variational formation of (23) is given by the following: find $\lambda \in \mathbb{C}, 0 \ne u \in H_0^1(\Omega)$, such that

$$a(u,v) = \lambda(u,v), \quad \forall v \in H_0^1(\Omega). \tag{24}$$

The spectral approximation scheme of (24) is given by the following: find $\lambda_N \in \mathbb{C}, 0 \ne u_N \in S_N(\Omega)$, such that

$$a(u_N, v_N) = \lambda_N(u_N, v_N), \quad \forall v_N \in S_N(\Omega). \tag{25}$$

The spectral element approximation scheme of (24) is given by the following: find $\lambda_{N,h} \in \mathbb{C}, 0 \ne u_{N,h} \in S_{N,h}(\Omega)$, such that

$$a(u_{N,h}, v_{N,h}) = \lambda_{N,h}(u_{N,h}, v_{N,h}), \quad \forall v_{N,h} \in S_{N,h}(\Omega). \tag{26}$$

Define the solution operators $T : L^2(\Omega) \to H_0^1(\Omega), T_N : L^2(\Omega) \to S_N(\Omega)$, and $T_{N,h} : L^2(\Omega) \to S_{N,h}(\Omega)$ by

$$a(Tf,v) = (f,v), \quad \forall f \in L^2(\Omega), \ \forall v \in H_0^1(\Omega),$$
$$a(T_N f, v_N) = (f, v_N), \quad \forall f \in L^2(\Omega),$$
$$\forall v_N \in S_N(\Omega), \tag{27}$$
$$a(T_{N,h} f, v_{N,h}) = (f, v_{N,h}), \quad \forall f \in L^2(\Omega),$$
$$\forall v_{N,h} \in S_{N,h}(\Omega).$$

Obviously (see [17]), the equivalent operator forms for (24) and (26) are the following.

Find $\lambda \in \mathbb{C}, 0 \ne u \in H_0^1(\Omega)$, such that

$$Tu = \lambda^{-1}u. \tag{28}$$

Find $\lambda_{N,h} \in \mathbb{C}, 0 \ne u_{N,h} \in S_{N,h}(\Omega)$, such that

$$T_{N,h} u_{N,h} = \lambda_{N,h}^{-1} u_{N,h}. \tag{29}$$

The adjoint problem of the eigenvalue problem (23) is

$$L^* u^* = \lambda^* u^*, \quad \text{in } \Omega,$$
$$u^* = 0, \quad \text{on } \partial\Omega, \tag{30}$$

where $L^* u^* = -\nabla \cdot (D\nabla u^*) - \mathbf{b} \cdot \nabla u^* + (c - \nabla \cdot \mathbf{b})u^*$.

The variational formation of (30) is given by the following: find $\lambda^* \in \mathbb{C}, 0 \ne u^* \in H_0^1(\Omega)$, such that

$$a^*(u^*, v) := \overline{a(v, u^*)} = \lambda^*(u^*, v), \quad \forall v \in H_0^1(\Omega). \tag{31}$$

The spectral element approximation scheme of (31) is given by the following: find $\lambda_{N,h}^* \in \mathbb{C}, 0 \ne u_{N,h}^* \in S_{N,h}(\Omega)$, such that

$$a^*(u_{N,h}^*, v_{N,h}) = \lambda_{N,h}^*(u_{N,h}^*, v_{N,h}), \quad \forall v_{N,h} \in S_{N,h}(\Omega). \tag{32}$$

We can likewise define the equivalent operator forms for the eigenvalue problems (31) and (32) as

$$T^* u^* = \lambda^{*-1} u^*,$$
$$T_{N,h}^* u_{N,h}^* = \lambda_{N,h}^{*-1} u_{N,h}^*. \tag{33}$$

Let λ be an eigenvalue of (23). There exists a smallest integer μ, called the ascent of λ, such that $\ker((\lambda^{-1} - T)^\mu) = \ker((\lambda^{-1} - T)^{\mu+1})$. $q = \dim \ker((\lambda^{-1} - T)^\mu)$ is called the algebraic multiplicity of λ. The functions in $\ker((\lambda^{-1} - T)^\mu)$ are called generalized eigenfunctions of T corresponding to λ. Likewise the ascent, algebraic multiplicity and generalized eigenfunctions of $\lambda_{N,h}$, λ^* and $\lambda_{N,h}^*$ can be defined.

Let λ be an eigenvalue of (23) with the algebraic multiplicity q and the ascent μ. Assume $\|T_{N,h} - T\|_{1,\Omega} \to 0 \, (N \to \infty, h \to 0)$; then there are eigenvalues $\lambda_{j,N,h} \, (j = 1, 2, \ldots, q)$ of (26) which converge to λ. Let $M(\lambda)$ be the space spanned by all generalized eigenfunctions corresponding to λ of T,

and let $M_{N,h}(\lambda)$ be the space spanned by all generalized eigenfunctions corresponding to all eigenvalues of $T_{N,h}$ that converge to λ.

In view of adjoint problems (31) and (32), the definitions of $M^*(\lambda^*)$ and $M_{N,h}^*(\lambda^*)$ are analogous to $M(\lambda)$ and $M_{N,h}(\lambda)$. Let $\widehat{M}(\lambda) = \{v \in M(\lambda) : \|v\|_{1,\Omega} = 1\}$, and let $\widehat{M^*}(\lambda^*) = \{v \in M^*(\lambda^*) : \|v\|_{1,\Omega} = 1\}$.

Note that when $\mathbf{b} = 0$, both (24) and (26) are symmetric. Thus, the ascent $\mu = 1$ of λ, and the ascent $l = 1$ of $\lambda_{N,h}$.

3.2. A Priori Error Estimates.
We will analyze a prior error estimates for spectral element methods which are suitable for spectral methods with mesh fineness h not considered.

Assume that R and U are two closed subspace in $H_0^1(\Omega)$. Denote

$$\delta(R,U) = \sup_{\substack{v \in R \\ \|v\|_{1,\Omega}=1}} \mathrm{dist}(v,U),$$

$$\theta(R,U) = \max(\delta(R,U), \delta(U,R)). \tag{34}$$

We say that $\theta(R,U)$ is the gap between R and U. Denote

$$\varepsilon_{N,h} = \varepsilon_{N,h}(\lambda) = \sup_{u \in \widehat{M}(\lambda)} \inf_{v \in S_{N,h}(\Omega)} \|u - v\|_{1,\Omega},$$

$$\varepsilon_{N,h}^* = \varepsilon_{N,h}^*(\lambda^*) = \sup_{u \in \widehat{M^*}(\lambda^*)} \inf_{v \in S_{N,h}(\Omega)} \|u - v\|_{1,\Omega}. \tag{35}$$

We give the following four lemmas from Theorem 8.1–8.4 in [17], which are applications to spectral element methods.

Lemma 1. *Assume* $\|T_{N,h} - T\|_{1,\Omega} \to 0 \ (N \to \infty, h \to 0)$. *For small enough h and big enough N, there holds*

$$\theta(M(\lambda), M_{N,h}(\lambda)) \leq C\varepsilon_{N,h}. \tag{36}$$

Lemma 2. *Assume* $\|T_{N,h} - T\|_{1,\Omega} \to 0 \ (N \to \infty, h \to 0)$; *then*

$$\left| \lambda^{-1} - \frac{1}{q} \sum_{j=1}^{q} \lambda_{j,N,h}^{-1} \right| \leq C\varepsilon_{N,h}\varepsilon_{N,h}^*. \tag{37}$$

Lemma 3. *Assume that* $\|T_{N,h} - T\|_{1,\Omega} \to 0 \ (N \to \infty, h \to 0)$; *then there holds*

$$\left| \lambda - \lambda_{j,N,h} \right| \leq C\left(\varepsilon_{N,h}\varepsilon_{N,h}^*\right)^{1/\mu} \quad (j = 1, 2, \dots, q). \tag{38}$$

Since $\ker((\lambda^{-1} - T)^l) \ (l \geq 1)$ *is a finite dimensional space, there exists a direct-sum decomposition* $H_0^1(\Omega) = \ker((\lambda^{-1} - T)^l) \oplus M_l$. *We define the operator E_l as a projection along M_l from $H_0^1(\Omega)$ to $\ker((\lambda^{-1} - T)^l)$.*

Lemma 4. *Assume* $\|T_{N,h} - T\|_{1,\Omega} \to 0 \ (N \to \infty, h \to 0)$. *Let $\lambda_{N,h}$ be an eigenvalue of $T_{N,h}$ and $\lim_{N\to\infty, h\to 0} \lambda_{N,h} = \lambda$. $u_{N,h}$ satisfies $(\lambda_{N,h}^{-1} - T_{N,h})^k u_{N,h} = 0$ and $\|u_{N,h}\|_{1,\Omega} = 1$, where $k \leq \mu$ is a positive integer. Then, for every integer $l \in [k, \mu]$, there holds*

$$\left\| u_{N,h} - E_l u_{N,h} \right\|_{1,\Omega} \leq C\varepsilon_{N,h}^{(l-k+1)/\mu}. \tag{39}$$

We assume that in this section, for the sake of simplicity, $N_\kappa = N, \forall \kappa \in K_h$.

Theorem 5. *If $M(\lambda) \subset H^{t_1}(\Omega)$ and $M^*(\lambda^*) \subset H^{t_2}(\Omega)$, then there holds the following error estimates:*

$$\left| \frac{1}{q} \sum_{j=1}^{q} \lambda_{j,N,h} - \lambda \right|$$
$$\leq C \left(\frac{h^{\tau_1+\tau_2-2}}{N^{t_1+t_2-2}} \right) \sup_{u \in \widehat{M}(\lambda)} \|u\|_{t_1,\Omega} \sup_{v \in \widehat{M^*}(\lambda^*)} \|v\|_{t_2,\Omega}, \tag{40}$$

$$\left| \lambda_{j,N,h} - \lambda \right|$$
$$\leq C \left(\left(\frac{h^{\tau_1+\tau_2-2}}{N^{t_1+t_2-2}} \right) \sup_{u \in \widehat{M}(\lambda)} \|u\|_{t_1,\Omega} \sup_{v \in \widehat{M^*}(\lambda^*)} \|v\|_{t_2,\Omega} \right)^{1/\mu} \tag{41}$$
$$(j = 1, 2, \dots, q),$$

$$\theta(M(\lambda), M_{N,h}(\lambda)) \leq C \frac{h^{\tau_1-1}}{N^{t_1-1}} \sup_{u \in \widehat{M}(\lambda)} \|u\|_{t_1,\Omega}. \tag{42}$$

Let $\|u_{N,h}\|_{1,\Omega} = 1$, and let $(\lambda_{N,h}^{-1} - T_{N,h})^{l_1} u_{N,h} = 0$, for some $l_1 \leq \mu$. Then, for every integer $l_2 \ (l_1 \leq l_2 \leq \mu)$, there exists a function u', such that $(\lambda^{-1} - T)^{l_2} u' = 0$ and

$$\left\| u_{N,h} - u' \right\|_{1,\Omega} \leq C \left(\left(\frac{h^{\tau_1-1}}{N^{t_1-1}} \right) \sup_{u \in \widehat{M}(\lambda)} \|u\|_{t_1,\Omega} \right)^{(l_2-l_1+1)/\mu}, \tag{43}$$

where $\tau_1 = \min(N+1, t_1)$, $\tau_2 = \min(N+1, t_2)$.

Proof. We derive from the error estimate (20) that

$$\|T_{N,h} - T\|_{1,\Omega}$$
$$= \sup_{f \in H_0^1(\Omega)} \frac{\|(T - T_{N,h})f\|_{1,\Omega}}{\|f\|_{1,\Omega}} \tag{44}$$
$$\leq C(1 + r_1) h^{r_1} N^{-r_1} \longrightarrow 0 \quad (N \longrightarrow \infty, h \longrightarrow 0).$$

By (14),

$$\varepsilon_{N,h} = \varepsilon_{N,h}(\lambda) = \sup_{u \in \widehat{M}(\lambda)} \inf_{v \in S_{N,h}(\Omega)} \|u - v\|_{1,\Omega}$$
$$\leq C \left(\frac{h^{\tau_1-1}}{N^{t_1-1}} \right) \sup_{u \in \widehat{M}(\lambda)} \|u\|_{t_1,\Omega}. \tag{45}$$

Analogically,

$$\varepsilon_{N,h}^* \leq C \left(\frac{h^{\tau_2-1}}{N^{t_2-1}} \right) \sup_{u \in \widehat{M^*}(\lambda^*)} \|u\|_{t_2,\Omega}. \tag{46}$$

Plugging the two inequalities above into (36), (38), and (39) yields (42), (41), and (43), respectively. We find from (37) that

$$
\begin{aligned}
\left| \frac{1}{q} \sum_{j=1}^{q} \lambda_{j,N,h} - \lambda \right| &= \left| \frac{1}{q} \sum_{j=1}^{q} \frac{\lambda_{j,N,h}^{-1} - \lambda^{-1}}{\lambda^{-1} \lambda_{j,N,h}^{-1}} \right| \\
&\leq C \left| \frac{1}{q} \sum_{j=1}^{q} \lambda_{j,N,h}^{-1} - \lambda^{-1} \right| \leq C \varepsilon_{N,h} \varepsilon_{N,h}^{*},
\end{aligned}
\tag{47}
$$

combining with (45) and (46) yields (40). □

Supposing that $\|T_{N,h} - T\|_{0,\Omega} \rightarrow 0 \, (N \rightarrow \infty, h \rightarrow 0)$, $\rho(T)$ is a regular set of T, and $\Gamma \subset \rho(T)$ is a closed Jordan curve enclosing λ^{-1}.

Denote

$$
\begin{aligned}
R(z) &= (T - z)^{-1}, \\
R(T_{N,h}, z) &= (T_{N,h} - z)^{-1}.
\end{aligned}
\tag{48}
$$

Define the spectral projection operators

$$
\begin{aligned}
E &= \frac{-1}{2i\pi} \int_{\Gamma} R(T, z) \, dz : H_0^1(\Omega) \longrightarrow M(\lambda), \\
E_{N,h} &= \frac{-1}{2i\pi} \int_{\Gamma} R(T_{N,h}, z) \, dz : H_0^1(\Omega) \longrightarrow M_{N,h}(\lambda).
\end{aligned}
\tag{49}
$$

We give the following lemma by referring to [18, 19] (see proposition 5.3 in [18] and theorem 1.3.2 in [19]).

Lemma 6. *If $\|T_{N,h} - T\|_{0,\Omega} \rightarrow 0 \, (N \rightarrow \infty, h \rightarrow 0)$, then there holds that $E_{N,h} \rightarrow E(p)$, $R(T_{N,h}, z)$ is uniformly bounded with N and h, and*

$$
\left\| (E_{N,h} - E) v \right\|_{0,\Omega} \leq C \max_{z \in \Gamma} \left\| (T - T_{N,h}) R(z) v \right\|_{0,\Omega},
$$
$$
\forall v \in H_0^1(\Omega),
$$
$$
\left\| (E_{N,h} - E) v \right\|_{0,\Omega} \leq C \max_{z \in \Gamma} \left\| (T - T_{N,h}) R(T_{N,h}, z) v \right\|_{0,\Omega},
$$
$$
\forall v \in H_0^1(\Omega).
\tag{50}
$$

Theorem 7. *Under the assumptions of Theorem 5, further assume that the ascent of λ is $\mu = 1$. Let $(\lambda_{N,h}, u_{N,h})$ be an eigenpair of (26) with $\|u_{N,h}\|_{0,\Omega} = 1$; then there exists an eigenpair (λ, u) of (24), such that*

$$
\left\| u_{N,h} - u \right\|_{1,\Omega} \leq \frac{C h^{\tau_1 - 1} \|u\|_{t_1,\Omega}}{N^{t_1 - 1}},
\tag{51}
$$

$$
\left\| u_{N,h} - u \right\|_{0,\Omega} \leq \frac{C h^{r_2 + \tau_1 - 1} \|u\|_{t_1,\Omega}}{N^{r_2 + t_1 - 1}},
\tag{52}
$$

$$
\left| \lambda_{N,h} - \lambda \right|
$$
$$
\leq C \left(\left(\frac{h^{\tau_1 + \tau_2 - 2}}{N^{t_1 + t_2 - 2}} \right) \sup_{u \in \widetilde{M}(\lambda)} \|u\|_{t_1,\Omega} \sup_{v \in \widetilde{M}^*(\lambda^*)} \|v\|_{t_2,\Omega} \right),
\tag{53}
$$

where $\tau_1 = \min(N+1, t_1)$ and $\tau_2 = \min(N+1, t_2)$.

Let (λ, u) be an eigenpair of (24). If $\lambda_{N,h}$ is an eigenvalue of (26) convergence to λ, then there exists $u_{N,h} \in \ker(\lambda_{N,h}^{-1} - T_{N,h})$, such that (51)–(53) hold.

Proof. We deduce (53) immediately from (41). We derive from (22) and (7) that

$$
\left\| Tf - T_{N,h} f \right\|_{0,\Omega} \leq C N^{-r_1 - r_2} h^{r_1 + r_2} \|f\|_{0,\Omega};
\tag{54}
$$

thus, $\|T - T_{N,h}\|_{0,\Omega} \rightarrow 0 \, (N \rightarrow \infty, h \rightarrow 0)$. Taking $u = E u_{N,h}$ and by virtue of $R(T_{N,h}, z) u_{N,h} = (\lambda_{N,h}^{-1} - z)^{-1} u_{N,h}$, Lemma 6 and (22), we have

$$
\begin{aligned}
\left\| u - u_{N,h} \right\|_{0,\Omega} &= \left\| E u_{N,h} - E_{N,h} u_{N,h} \right\|_{0,\Omega} \\
&\leq C \left\| (T - T_{N,h}) u_{N,h} \right\|_{0,\Omega} \\
&\leq C \left(\left\| (T - T_{N,h}) u \right\|_{0,\Omega} \right. \\
&\quad \left. + \left\| (T - T_{N,h}) (u_{N,h} - u) \right\|_{0,\Omega} \right),
\end{aligned}
\tag{55}
$$

from which follows

$$
\begin{aligned}
\left\| u - u_{N,h} \right\|_{0,\Omega} &\leq C \left\| (T - T_{N,h}) u \right\|_{0,\Omega} \\
&\leq \frac{C h^{r_2 + \tau_1 - 1} \|u\|_{t_1,\Omega}}{N^{r_2 + t_1 - 1}},
\end{aligned}
\tag{56}
$$

which is (52). By direct calculation, we have

$$
\begin{aligned}
&\left\| u - u_{N,h} \right\|_{1,\Omega} \\
&= \left\| \lambda T u - \lambda_h T_{N,h} u_{N,h} \right\|_{1,\Omega} \\
&\leq \left\| \lambda T u - \lambda T_{N,h} u \right\|_{1,\Omega} + \left\| \lambda T_{N,h} u - \lambda_h T_{N,h} u_{N,h} \right\|_{1,\Omega} \\
&\leq \left\| (T - T_{N,h}) (\lambda u) \right\|_{1,\Omega} + C \left\| \lambda u - \lambda_{N,h} u_{N,h} \right\|_{0,\Omega}.
\end{aligned}
\tag{57}
$$

Plugging (20), (52), and (53) into (57) yields (51).

If (λ, u) is an eigenpair of (24), let $u_{N,h} = E_{N,h} u$; by the same argument we can prove (51) and (52). □

4. A Posteriori Error Estimates

Based on [20], we will discuss a posteriori error estimates. We further assume that $\Omega \subset R^2$, the partition K_h is γ-shape regular, and the polynomial degree of neighboring elements are comparable; that is, there exists $\gamma > 0$, such that for all $\kappa, \kappa' \in K_h, \overline{\kappa} \cap \overline{\kappa'} \neq \emptyset$,

$$
\gamma^{-1} h_\kappa \leq h_{\kappa'} \leq \gamma h_\kappa,
$$
$$
\gamma^{-1} (N_\kappa + 1) \leq N_{\kappa'} + 1 \leq \gamma (N_\kappa + 1).
\tag{58}
$$

We refer to the *hp*-clément interpolation estimates given by [20, 21] (see theorems 2.2 and 2.3, respectively), which generalize the well-known clément type interpolation operators studied in [22] and [23] to the hp context.

Lemma 8. *Assume that the partition K_h is γ-shape regular and the polynomial distribution $\mathbf{N}$ is comparable. Then there*

exists a positive constant $C = C(\gamma)$ and the clément operator $I : H_0^1(\Omega) \to S_{N,h}(\Omega)$, such that

$$\|v - Iv\|_{0,\kappa} \le C \frac{h_\kappa}{N_\kappa} \|\nabla v\|_{0,\omega_\kappa}, \tag{59}$$

$$\|v - Iv\|_{0,e} \le C \sqrt{\frac{h_e}{N_e}} \|\nabla v\|_{0,\omega_e}, \tag{60}$$

where h_e is the length of the edge e and $N_e = \max(N_{\kappa_1}, N_{\kappa_2})$, where κ_1, κ_2 are elements sharing the edge e and ω_κ, ω_e are patches covering κ and e with a few layers, respectively.

Define interval $\widehat{I} = (0, 1)$ and weight function $\Phi_{\widehat{I}}(x) := x(1 - x)$. Denote the reference square and triangle element by $\widehat{\kappa} = (0, 1)^2$ and $\widehat{\kappa} = \{(x, y) | 0 < x < 1, 0 < y < \sqrt{3}(1/2 - |1/2 - x|)\}$, respectively. Define weight function $\Phi_{\widehat{\kappa}}(x) := \text{dist}(x, \partial\widehat{\kappa})$.

The following three lemmas are given by [20]. Lemmas 9–10 provide the polynomial inverse estimates in standard interval and element, while Lemma 11 provides a result for the extension from an edge to the element.

Lemma 9. *Let* $-1 < \alpha < \beta, \sigma \in [0, 1]$. *Then there exists* $C = C(\alpha, \beta)$, *such that for all* $N \in \mathbb{N}$ *and all univariate polynomials* π_N *of degree* N,

$$\int_{\widehat{I}} \Phi_{\widehat{I}}^\alpha(x) |\pi_N(x)|^2 dx$$
$$\le C N^{2(\beta-\alpha)} \int_{\widehat{I}} \Phi_{\widehat{I}}^\beta |\pi_N(x)|^2 dx. \tag{61}$$

Lemma 10. *Let* $-1 < \alpha < \beta, \sigma \in [0, 1]$. *Then there exist* $C_1 = C(\alpha, \beta), C_2 = C_\sigma > 0$, *such that for all* $N \in \mathbb{N}$ *and all polynomials* π_N *of degree bi-N,*

$$\int_{\widehat{\kappa}} \Phi_{\widehat{\kappa}}^\alpha |\pi_N|^2 dxdy$$
$$\le C_1 N^{2(\beta-\alpha)} \int_{\widehat{\kappa}} \Phi_{\widehat{\kappa}}^\beta |\pi_N|^2 dxdy, \tag{62}$$

$$\int_{\widehat{\kappa}} \Phi_{\widehat{\kappa}}^{2\sigma} |\nabla \pi_N|^2 dxdy$$
$$\le C_2 N^{2(2-\sigma)} \int_{\widehat{\kappa}} \Phi_{\widehat{\kappa}}^\sigma |\pi_N|^2 dxdy. \tag{63}$$

Lemma 11. *Let* $\alpha \in (1/2, 1]$. $\widehat{e} := (0, 1) \times \{0\}$, $\Phi_{\widehat{e}} := x(1 - x)$; *then there exists* $C_\alpha > 0$ *such that for all* $N \in \mathbb{N}, \varepsilon \in (0, 1]$, *and all univariate polynomials* π *of degree* N, *there exists an extension* $v_{\widehat{e}} \in H^1(\widehat{\kappa})$ *and holds*

$$v_{\widehat{e}}|_{\widehat{e}} = \pi \cdot \Phi_{\widehat{e}}^\alpha, \qquad v_{\widehat{e}}|_{\partial\widehat{\kappa}\backslash\widehat{e}} = 0, \tag{64}$$

$$\|v_{\widehat{e}}\|_{0,\widehat{\kappa}}^2 \le C_\alpha \varepsilon \|\pi \Phi_{\widehat{e}}^{\alpha/2}\|_{0,\widehat{e}}^2, \tag{65}$$

$$\|\nabla v_{\widehat{e}}\|_{0,\widehat{\kappa}}^2 \le C_\alpha \left(\varepsilon N^{2(2-\alpha)} + \varepsilon^{-1} \right) \|\pi \Phi_{\widehat{e}}^{\alpha/2}\|_{0,\widehat{e}}^2. \tag{66}$$

It is easy to know that the three lemmas above hold for complex-valued polynomials.

Let $D_\kappa, \mathbf{b}_\kappa$, and c_κ be the interpolations of $D, \mathbf{b}$, and c in κ with the polynomial degree N_κ (resp. degree N_κ in every direction), respectively, or the $L^2(\kappa)$-projection on the space of polynomials with degree N_κ. For convenient argument, here and hereafter we assume that (λ, u) and $(\lambda^* = \overline{\lambda}, u^*)$ are the eigenpairs of the eigenvalue problem (24) and its adjoint problem (31), respectively. $(\lambda_{N,h}, u_{N,h})$ and $(\lambda_{N,h}^* = \overline{\lambda_{N,h}}, u_{N,h}^*)$ are the solutions of the corresponding spectral element approximations (26) and (32), respectively.

Denote

$$L_\kappa u_{N,h} : = -\nabla \cdot (D_\kappa \nabla u_{N,h})$$
$$+ \mathbf{b}_\kappa \cdot \nabla u_{N,h} + c_\kappa u_{N,h},$$
$$L_\kappa^* u_{N,h}^* : = -\nabla \cdot \left(D_\kappa \nabla u_{N,h}^* \right)$$
$$- \mathbf{b}_\kappa \cdot \nabla u_{N,h}^* + (c_\kappa - \nabla \cdot \mathbf{b}_\kappa) u_{N,h}^*. \tag{67}$$

Define the local error indicators

$$\eta_{\alpha;\kappa}^2 := \eta_{\alpha;B_\kappa}^2 + \eta_{\alpha;E_\kappa}^2,$$
$$\eta_{\alpha;\kappa}^{*2} := \eta_{\alpha;B_\kappa}^{*2} + \eta_{\alpha;E_\kappa}^{*2}. \tag{68}$$

Their first terms $\eta_{\alpha;B_\kappa}^2, \eta_{\alpha;B_\kappa}^{*2}$ are the weighted element internal residuals given by

$$\eta_{\alpha;B_\kappa}^2 := \frac{h_\kappa^2}{N_\kappa^2} \left\| \left(-L_\kappa u_{N,h} + \lambda_{N,h} u_{N,h} \right) \Phi_\kappa^{\alpha/2} \right\|_{0,\kappa}^2,$$
$$\eta_{\alpha;B_\kappa}^{*2} := \frac{h_\kappa^2}{N_\kappa^2} \left\| \left(-L_\kappa^* u_{N,h}^* + \lambda_{N,h}^* u_{N,h}^* \right) \Phi_\kappa^{\alpha/2} \right\|_{0,\kappa}^2. \tag{69}$$

Their second terms $\eta_{\alpha;E_\kappa}^2, \eta_{\alpha;E_\kappa}^{*2}$ are the weighted element boundary residuals given by

$$\eta_{\alpha;E_\kappa}^2 := \sum_{e \subset \partial\kappa \cap \Omega} \frac{h_e}{2N_e} \left\| D_\kappa \left[\frac{\partial u_{N,h}}{\partial n} \right] \Phi_e^{\alpha/2} \right\|_{0,e}^2,$$
$$\eta_{\alpha;E_\kappa}^{*2} := \sum_{e \subset \partial\kappa \cap \Omega} \frac{h_e}{2N_e} \left\| D_\kappa \left[\frac{\partial u_{N,h}^*}{\partial n} \right] \Phi_e^{\alpha/2} \right\|_{0,e}^2, \tag{70}$$

where we denote the jump of the normal derivatives of $u_{N,h}$ and $u_{N,h}^*$ across the edges by $[\partial u_{N,h}/\partial n]$ and $[\partial u_{N,h}^*/\partial n]$, respectively. h_e is the length of edge e. The weight functions Φ_κ and Φ_e are scaled transformations of the weight functions $\Phi_{\widehat{\kappa}}$ and $\Phi_{\widehat{e}}$; that is, if F_κ is the element map for element κ and e is the image of the edge $\widehat{e}$ under F_κ, then

$$\Phi_\kappa = C_\kappa \Phi_{\widehat{\kappa}} \circ F_\kappa^{-1}, \qquad \Phi_e = C_e \Phi_{\widehat{e}} \circ F_\kappa^{-1}, \tag{71}$$

where we choose $C_\kappa, C_e > 0$, such that

$$\int_\kappa \Phi_\kappa dxdy = \int_\kappa dxdy, \qquad \int_e \Phi_e ds = \int_e ds. \tag{72}$$

We define the global error indicators as follows:

$$\eta_\alpha^2 := \sum_{\kappa \in K_h} \eta_{\alpha;\kappa}^2,$$
$$\eta_\alpha^{*2} := \sum_{\kappa \in K_h} \eta_{\alpha;\kappa}^{*2}. \tag{73}$$

Theorem 12. *Let $\alpha \in [0,1]$. Then there exists a constant $C > 0$ independent of h, $\mathbf{N}$, and κ, such that*

$$\|u - u_{N,h}\|_{1,\Omega}^2 \leq C \sum_{\kappa \in K_h} N_\kappa^{2\alpha} \eta_{\alpha;\kappa}^2$$

$$+ C \sum_{\kappa \in K_h} \left\{ \frac{h_\kappa^2}{N_\kappa^2} \|L_\kappa u_{N,h} - L u_{N,h}\|_{0,\kappa}^2 \right.$$

$$+ \sum_{e \subset \partial\kappa \cap \Omega} \frac{h_e}{N_e} \|D - D_\kappa\|_{0,e}^2 \qquad (74)$$

$$\left. \times \left\| \frac{\partial u_{N,h}}{\partial n} \right\|_{0,\infty,e}^2 \right\}$$

$$+ C\|\lambda u - \lambda_{N,h} u_{N,h}\|_{0,\Omega}^2.$$

Proof. We denote $w := u - u_{N,h} - I(u - u_{N,h})$, where I is hp-clément operator given by Lemma 8. We derive from $H_0^1(\Omega)$-elliptic of $a(\cdot,\cdot)$ that

$$C\|u - u_{N,h}\|_{1,\Omega}^2 \leq a(u - u_{N,h}, w)$$

$$+ a(u - u_{N,h}, I(u - u_{N,h}))$$

$$= \lambda \int_\Omega u\overline{w} - a(u_{N,h}, w)$$

$$+ \int_\Omega (\lambda u - \lambda_{N,h} u_{N,h})$$

$$\times \overline{I(u - u_{N,h})}$$

$$= \int_\Omega (\lambda_{N,h} u_{N,h}) \overline{w}$$

$$+ \int_\Omega (\lambda u - \lambda_{N,h} u_{N,h}) \overline{u - u_{N,h}} \qquad (75)$$

$$- a(u_{N,h}, w),$$

$$a(u_{N,h}, w) = \sum_{\kappa \in \kappa_h} \int_\kappa L u_{N,h} \overline{w}$$

$$- \sum_{\kappa \in \kappa_h} \int_{\partial\kappa} D \frac{\partial u_{N,h}}{\partial n} \overline{w}$$

$$= \sum_{\kappa \in \kappa_h} \int_\kappa L u_{N,h} \overline{w}$$

$$- \frac{1}{2} \sum_{\kappa \in \kappa_h} \sum_{e \subset \partial\kappa \cap \Omega} \int_e D \left[\frac{\partial u_{N,h}}{\partial n} \right] \overline{w}.$$

Therefore,

$$C\|u - u_{N,h}\|_{1,\Omega}^2$$

$$\leq \sum_{\kappa \in K_h} \int_\kappa (-L u_{N,h} + \lambda_{N,h} u_{N,h}) \overline{w}$$

$$+ \frac{1}{2} \sum_{\kappa \in K_h} \sum_{e \subset \partial\kappa \cap \Omega} \int_e D \left[\frac{\partial u_{N,h}}{\partial n} \right] \overline{w}$$

$$+ \int_\Omega (\lambda u - \lambda_{N,h} u_{N,h}) \overline{(u - u_{N,h})}, \qquad (76)$$

which together with

$$\int_e D \left[\frac{\partial u_{N,h}}{\partial n} \right] \overline{w} = \int_e (D - D_\kappa) \left[\frac{\partial u_{N,h}}{\partial n} \right] \overline{w}$$

$$+ \int_e D_\kappa \left[\frac{\partial u_{N,h}}{\partial n} \right] \overline{w},$$

$$\int_\kappa (-L u_{N,h} + \lambda_{N,h} u_{N,h}) \overline{w} = \int_\kappa (-L_\kappa u_{N,h} + \lambda_{N,h} u_{N,h}) \overline{w}$$

$$+ \int_\kappa (L_\kappa u_{N,h} - L u_{N,h}) \overline{w} \qquad (77)$$

and using Cauchy-Schwartz inequality, the hp-clément interpolation estimates in Lemma 8 then yield

$$\|u - u_{N,h}\|_{1,\Omega}^2$$

$$\leq C \left\{ \sum_{\kappa \in K_h} \left[\eta_{0;B_\kappa}^2 + \eta_{0;E_\kappa}^2 + \frac{h_\kappa^2}{N_\kappa^2} \|L_\kappa u_{N,h} - L u_{N,h}\|_{0,\kappa}^2 \right. \right.$$

$$\left. \left. + \sum_{e \subset \partial\kappa \cap \Omega} \frac{h_e}{N_e} \left\| (D - D_\kappa) \left[\frac{\partial u_{N,h}}{\partial n} \right] \right\|_{0,e}^2 \right] \right\}^{1/2}$$

$$\times \|u - u_{N,h}\|_{1,\Omega} + C\|\lambda u - \lambda_{N,h} u_{N,h}\|_{0,\Omega} \|u - u_{N,h}\|_{0,\Omega}. \qquad (78)$$

Using scaled transformation and setting $\alpha = 0$, $\beta = \alpha$ in (61) and (62), we get $\eta_{0;E_\kappa} \leq CN_\kappa^\alpha \eta_{\alpha;E_\kappa}$ and $\eta_{0;B_\kappa} \leq CN_\kappa^\alpha \eta_{\alpha;B_\kappa}$; then this proof concludes. □

For the adjoint eigenvalue problem, we still have the following.

Theorem 13. *Let $\alpha \in [0,1]$. Then there exists a constant $C > 0$ independent of h, $\mathbf{N}$, and κ, such that*

$$\|u^* - u_{N,h}^*\|_{1,\Omega}^2$$

$$\leq C \sum_{\kappa \in K_h} N_\kappa^{2\alpha} \eta_{\alpha;\kappa}^{*2}$$

$$+ C \sum_{\kappa \in K_h} \left\{ \frac{h_\kappa^2}{N_\kappa^2} \|L_\kappa^* u_{N,h}^* - L^* u_{N,h}^*\|^2 \right.$$

$$\left. + \sum_{e \subset \partial\kappa \cap \Omega} \frac{h_e}{N_e} \|D - D_\kappa\|_{0,e}^2 \left\| \frac{\partial u_{N,h}^*}{\partial n} \right\|_{0,\infty,e}^2 \right\}$$

$$+ C\|\lambda^* u^* - \lambda_{N,h}^* u_{N,h}^*\|_{0,\Omega}^2. \qquad (79)$$

Lemma 14. *Let $\alpha \in [0,1], \varepsilon > 0$. Then there exists a constant $C(\varepsilon) > 0$ independent of $h, \mathbf{N}$, and κ, such that*

$$\eta_{\alpha;B_\kappa}^2 \leq C(\varepsilon) \left\{ N_\kappa^{2(1-\alpha)} \|u - u_{N,h}\|_{1,\kappa}^2 \right.$$
$$+ N_\kappa^{\max\{1+2\varepsilon-2\alpha,0\}} \frac{h_\kappa^2}{N_\kappa^2}$$
$$\left. \times \left(\|\lambda_{N,h} u_{N,h} - \lambda u + Lu_{N,h} - L_\kappa u_{N,h}\|_{0,k}^2 \right) \right\}.$$
(80)

Proof. We denote $v_\kappa := (-L_\kappa u_{N,h} + \lambda_{N,h} u_{N,h})\Phi_\kappa^\alpha \in H_0^1(\kappa)$ with $\alpha \in (0,1]$ and extend v_κ to Ω by $v_\kappa = 0$ on $\Omega \setminus \kappa$; then

$$\left\| v_\kappa \Phi_\kappa^{-\alpha/2} \right\|_{0,\kappa}^2 = \int_\kappa (-L_\kappa u_{N,h} + \lambda_{N,h} u_{N,h}) \overline{v_\kappa}$$
$$= -\int_\kappa (L_\kappa u_{N,h}) \overline{v_\kappa} + a(u, v_\kappa)$$
$$+ \int_\kappa (\lambda_{N,h} u_{N,h} - \lambda u) \overline{v_\kappa}$$
$$= a(u - u_{N,h}, v_\kappa)$$
$$+ \int_\kappa (\lambda_{N,h} u_{N,h} - \lambda u + Lu_{N,h} - L_\kappa u_{N,h}) \overline{v_\kappa}$$
$$\leq C \|u - u_{N,h}\|_{1,\kappa} |v_\kappa|_{1,\kappa}$$
$$+ \left\| (\lambda_{N,h} u_{N,h} - \lambda u \right.$$
$$\left. + Lu_{N,h} - L_\kappa u_{N,h}) \Phi_\kappa^{\alpha/2} \right\|_{0,\kappa}$$
$$\times \left\| v_\kappa \Phi_\kappa^{-\alpha/2} \right\|_{0,\kappa}.$$
(81)

We consider the H^1 semi norm for v_κ. Using the polynomial inverse estimates (62)-(63) in Lemma 10, by transformation between the reference element $\widehat{\kappa}$ and κ, we find for $\alpha > 1/2$ that

$$|v_\kappa|_{1,\kappa}^2 \leq 2 \int_\kappa \Phi_\kappa^{2\alpha} |\nabla (\lambda_{N,h} u_{N,h} - L_\kappa u_{N,h})|^2$$
$$+ 2 \int_\kappa |\nabla \Phi_\kappa^\alpha|^2 |\lambda_{N,h} u_{N,h} - L_\kappa u_{N,h}|^2$$
$$\leq C \frac{N_\kappa^{2(2-\alpha)}}{h_\kappa^2} \int_\kappa \Phi_\kappa^\alpha |\lambda_{N,h} u_{N,h} - L_\kappa u_{N,h}|^2$$
$$+ C \frac{1}{h_\kappa^2} \int_\kappa \Phi_\kappa^{2(\alpha-1)} |\lambda_{N,h} u_{N,h} - L_\kappa u_{N,h}|^2$$
$$\leq C \frac{N_\kappa^{2(2-\alpha)}}{h_\kappa^2} \int_\kappa \Phi_\kappa^\alpha |\lambda_{N,h} u_{N,h} - L_\kappa u_{N,h}|^2$$
$$= C N_\kappa^{2(1-\alpha)} \frac{N_\kappa^2}{h_\kappa^2} \| v_\kappa \Phi_\kappa^{-\alpha/2} \|_{0,\kappa}^2.$$
(82)

Note that (62) is applicable since $\alpha > 1/2$ implies $2(\alpha-1) > -1$; thus, we set $\beta = \alpha, \alpha = 2(\alpha-1)$ in (62); then the third inequality above holds.

Since $\eta_{\alpha;B_\kappa} = h_\kappa / N_\kappa \|v_\kappa \Phi_\kappa^{-\alpha/2}\|_{0,\kappa}$, we obtain

$$\eta_{\alpha;B_\kappa} \leq C \left(N_\kappa^{1-\alpha} \|u - u_{N,h}\|_{1,\kappa} \right.$$
$$+ \frac{h_\kappa}{N_\kappa} \|Lu_{N,h} - L_\kappa u_{N,h}$$
$$\left. + \lambda_{N,h} u_{N,h} - \lambda u\|_{0,\kappa} \right).$$
(83)

To obtain an upper bound in the case of $0 \leq \alpha \leq 1/2$, we use the polynomial inverse estimate (62) in Lemma 10; for $\beta > 1/2$, we derive from (62) that

$$\frac{N_\kappa}{h_\kappa} \eta_{\alpha;B_\kappa} = \Phi_\kappa^{\alpha/2} \|(\lambda_{N,h} u_{N,h} - L_\kappa u_{N,h})\|_{0,\kappa}$$
$$\leq C N_\kappa^{\beta-\alpha} \|(\lambda_{N,h} u_{N,h} - L_\kappa u_{N,h}) \Phi_\kappa^{\beta/2}\|_{0,\kappa}$$
$$= C N_\kappa^{\beta-\alpha} \frac{N_\kappa}{h_\kappa} \eta_{\beta;B_\kappa}$$
$$\leq C N_\kappa^{\beta-\alpha} \left(N_\kappa^{1-\beta} \frac{N_\kappa}{h_\kappa} \|u - u_{N,h}\|_{1,\kappa} \right.$$
$$\left. + \|\lambda_{N,h} u_{N,h} - \lambda u + Lu_{N,h} - L_\kappa u_{N,h}\|_{0,\kappa} \right).$$
(84)

Setting $\beta = 1/2 + \varepsilon, \varepsilon > 0$,

$$\eta_{\alpha;B_\kappa} \leq C(\varepsilon) \left\{ N_\kappa^{1-\alpha} \|u - u_{N,h}\|_{1,\kappa} \right.$$
$$+ N_\kappa^{1/2+\varepsilon-\alpha} \frac{h_\kappa}{N_\kappa}$$
$$\left. \times \|\lambda_{N,h} u_{N,h} - \lambda u + Lu_{N,h} - L_\kappa u_{N,h}\|_{0,\kappa} \right\}.$$
(85)

We obtain the desired result immediately from (83) and (85). $\square$

In order to obtain a local upper bound for the error indicator $\eta_{\alpha;\kappa}$, we consider the edge residual term $\eta_{\alpha;E_\kappa}$. we introduce the set

$$\omega_\kappa := \cup \left\{ \kappa' \mid \kappa' \text{ and } \kappa \text{ share at least one edge} \right\}. \tag{86}$$

Lemma 15. *Let $\alpha \in [0,1], \varepsilon > 0$. Then there exists a constant $C(\varepsilon) > 0$ independent of $h, \mathbf{N}$, and κ, such that*

$$\eta_{\alpha;E_\kappa}^2 \leq C(\varepsilon) N_\kappa^{\max(1-2\alpha+2\varepsilon,0)}$$
$$\times \left\{ N_\kappa \|u - u_{N,h}\|_{1,\omega_\kappa}^2 + N_\kappa^{2\varepsilon} \frac{h_\kappa^2}{N_\kappa^2} \right.$$
$$\left. \times \|\lambda_{N,h} u_{N,h} - \lambda u + Lu_{N,h} - L_\kappa u_{N,h}\|_{0,\omega_\kappa}^2 \right\}.$$
(87)

Proof. We will use weight functions on edge and a suitable extension operator. For a given element κ with edge e, we choose the element κ_1 so that $\partial \kappa_1 \cap \partial \kappa = e$. Denote $\overline{\kappa_e} := \overline{\kappa_1} \cup \overline{\kappa}$; we construct a function $w_e \in H_0^1(\kappa_e)$ with $w_e|_e = D_\kappa[\partial u_{N,h}/\partial n]\Phi_e^\alpha$ as follows.

Let $v_{\widehat{e}} = C_e D_\kappa[\partial u_{N,h}/\partial n]\Phi_{\widehat{e}}^\alpha$ (C_e is defined by (71)). Using Lemma 11, we extend $v_{\widehat{e}}$ to $\widehat{\kappa}$, where the polynomial π corresponds to $C_e D_\kappa[\partial u_{N,h}/\partial n]$. Define $w_e|_\kappa$ and $w_e|_{\kappa_1}$ as the affine transformation of $v_{\widehat{e}}$ in $\widehat{\kappa}$; Thus, w_e is a piecewise H^1-function. From (64), we know w_e vanishes on $\partial \kappa_e$; Therefore, $w_e \in H_0^1(\kappa_e)$. It is trivial to extend w_e to Ω, such that $w_e = 0$ in $\Omega \setminus \kappa_e$. We find

$$
\left\| D_\kappa \left[\frac{\partial u_{N,h}}{\partial n} \right] \Phi_e^{\alpha/2} \right\|_{0,e}^2
$$

$$
= \int_e D \left[\frac{\partial u_{N,h}}{\partial n} \right] \overline{w_e} + \int_e (D_\kappa - D) \left[\frac{\partial u_{N,h}}{\partial n} \right] \overline{w_e}
$$

$$
= \int_{\kappa_e} L u_{N,h} \overline{w_e} - a(u_{N,h}, w_e)
$$

$$
+ \int_e (D_\kappa - D) \left[\frac{\partial u_{N,h}}{\partial n} \right] \overline{w_e}
$$

$$
= \int_{\kappa_e} (L u_{N,h} - \lambda u) \overline{w_e} + a(u - u_{N,h}, w_e) \tag{88}
$$

$$
+ \int_e (D_\kappa - D) \left[\frac{\partial u_{N,h}}{\partial n} \right] \overline{w_e}
$$

$$
\leq \| L u_{N,h} - \lambda u \|_{0,\kappa_e} \| w_e \|_{0,\kappa_e}
$$

$$
+ C \| u - u_{N,h} \|_{1,\kappa_e} |w_e|_{1,\kappa_e}
$$

$$
+ \left\| \frac{(D_\kappa - D)}{D_\kappa} \right\|_{0,\infty,e} \left\| D_\kappa \left[\frac{\partial u_{N,h}}{\partial n} \right] \Phi_e^{\alpha/2} \right\|_{0,e}^2.
$$

Therefore,

$$
\left\| D_\kappa \left[\frac{\partial u_{N,h}}{\partial n} \right] \Phi_e^{\alpha/2} \right\|_{0,e}^2
$$

$$
\leq C \| L u_{N,h} - \lambda u \|_{0,\kappa_e} \| w_e \|_{0,\kappa_e} \tag{89}
$$

$$
+ C \| u - u_{N,h} \|_{1,\kappa_e} |w_e|_{1,\kappa_e}.
$$

We consider the case of $\alpha \in (1/2, 1]$ first. Using the affine equivalence and (65)-(66) in Lemma 11, we obtain the upper bounds for $\| w_e \|_{0,\kappa_e}$ and $|w_e|_{1,\kappa_e}$ as follows:

$$
|w_e|_{1,\kappa_e}^2 \leq C \frac{1}{h_\kappa} \left(\varepsilon N_\kappa^{2(2-\alpha)} + \varepsilon^{-1} \right)
$$

$$
\times \left\| D_\kappa \left[\frac{\partial u_{N,h}}{\partial n} \right] \Phi_e^{\alpha/2} \right\|_{0,e}^2, \tag{90}
$$

$$
\| w_e \|_{0,\kappa_e} \leq C h_\kappa \varepsilon \left\| D_\kappa \left[\frac{\partial u_{N,h}}{\partial n} \right] \Phi_e^{\alpha/2} \right\|_{0,e}^2.
$$

It follows from (89)-(90) that

$$
\left\| D_\kappa \left[\frac{\partial u_{N,h}}{\partial n} \right] \Phi_e^{\alpha/2} \right\|_{0,e}
$$

$$
\leq C \left\{ \left(\frac{1}{h_\kappa} \left(\varepsilon N_\kappa^{2(2-\alpha)} + \varepsilon^{-1} \right) \right)^{1/2} \right.
$$

$$
\left. \times \| u - u_{N,h} \|_{1,\kappa_e} + (h_\kappa \varepsilon)^{1/2} \| \| L u_{N,h} - \lambda u \|_{0,\kappa_e} \right\}. \tag{91}
$$

By the definition of $\eta_{\alpha;E_\kappa}^2$ and setting $\alpha = 0$ in Lemma 14, we get

$$
\| L_\kappa u_{N,h} - \lambda_{N,h} u_{N,h} \|_{0,k}^2
$$

$$
\leq C(\varepsilon) \left\{ \frac{N_\kappa^4}{h_\kappa^2} \| u - u_{N,h} \|_{1,k}^2 + N_\kappa^{1+2\varepsilon} \right. \tag{92}
$$

$$
\left. \times \| \lambda_{N,h} u_{N,h} - \lambda u + L u_{N,h} - L_\kappa u_{N,h} \|_{0,k}^2 \right\},
$$

by the triangle inequality

$$
\| L u_{N,h} - \lambda u \|_{0,\kappa_e}
$$

$$
\leq \| L_\kappa u_{N,h} - \lambda_{N,h} u_{N,h} \|_{0,\kappa_e} \tag{93}
$$

$$
+ \| \lambda_{N,h} u_{N,h} - \lambda u + L u_{N,h} - L_\kappa u_{N,h} \|_{0,\kappa_e}.
$$

Combining the three inequalities above and summing, we have

$$
\eta_{\alpha;E_\kappa}^2 \leq C \left\{ \frac{1}{N_\kappa} \left(\varepsilon N_\kappa^{2(2-\alpha)} + \varepsilon^{-1} \right) + N_\kappa^3 \varepsilon \right\}
$$

$$
\times \| u - u_{N,h} \|_{1,\omega_\kappa}^2 + C \varepsilon N_\kappa^{2(1+\varepsilon)} \frac{h_\kappa^2}{N_\kappa^2} \tag{94}
$$

$$
\times \| \lambda_{N,h} u_{N,h} - \lambda u + L u_{N,h} - L_\kappa u_{N,h} \|_{0,\omega_\kappa}^2.
$$

Setting $\varepsilon = 1/N_\kappa^2$ in the above inequality yields the assertion for $\alpha > 1/2$. For the case of $\alpha \in [0, 1/2]$, we set $\beta = 1/2 + \varepsilon$, use (62) in Lemma 10 to get $\eta_{\alpha;E_\kappa} \leq N_\kappa^{\beta-\alpha} \eta_{\beta;E_\kappa}$, and find the desired result. □

Combining Lemmas 14 and 15, we obtain the following theorem.

Theorem 16. *Let $\alpha \in [0,1], \varepsilon > 0$. Then there exists a constant $C > 0$ independent of $h, \mathbf{N}$, and κ, such that*

$$
\eta_{\alpha;\kappa}^2 \leq C(\varepsilon) N_\kappa^{\max(1-2\alpha+2\varepsilon,0)}
$$

$$
\times \left\{ N_\kappa \| u - u_{N,h} \|_{1,\omega_\kappa}^2 + N_\kappa^{2\varepsilon} \frac{h_\kappa^2}{N_\kappa^2} \right. \tag{95}
$$

$$
\left. \times \| \lambda_{N,h} u_{N,h} - \lambda u + L u_{N,h} - L_\kappa u_{N,h} \|_{0,\omega_\kappa}^2 \right\}.
$$

Similarly, we have Theorem 17.

Theorem 17. *Let $\alpha \in [0,1], \varepsilon > 0$. Then there exists a constant $C > 0$ independent of $h, \mathbf{N}$, and κ, such that*

$$
\eta_{\alpha;\kappa}^{*2} \le C(\varepsilon) N_\kappa^{\max(1-2\alpha+2\varepsilon,0)}
$$
$$
\times \left\{ N_\kappa \left\| u^* - u_{N,h}^* \right\|_{1,\omega_\kappa}^2 + N_\kappa^{2\varepsilon} \frac{h_\kappa^2}{N_\kappa^2} \right.
$$
$$
\left. \times \left\| \lambda_{N,h}^* u_{N,h}^* - \lambda^* u^* + L^* u_{N,h}^* - L_\kappa^* u_{N,h}^* \right\|_{0,\omega_\kappa}^2 \right\}. \tag{96}
$$

In order to estimate bounds of $|\lambda - \lambda_{N,h}|$, we also need Lemma 18 (see [8, 10]).

Lemma 18. *Let (λ, u) be an eigenpair of (24), and let $(\lambda^* = \overline{\lambda}, u^*)$ be the associated eigenpair of the adjoint problem (31). Then for all $w, w^* \in H_0^1(\Omega), (w, w^*) \ne 0$,*

$$
\frac{a(w,w^*)}{(w,w^*)} - \lambda
$$
$$
= \frac{a(w-u,w^*-u^*)}{(w,w^*)} - \lambda \frac{(w-u,w^*-u^*)}{(w,w^*)}. \tag{97}
$$

Theorem 19. *Under the assumptions of Theorem 7, we assume that $D, \mathbf{b}$, and c are smooth enough, and let $\alpha \in [0,1]$. Then there exists an eigenpair (λ, u) of (24), such that*

$$
\left\| u - u_{N,h} \right\|_{1,\Omega} \le C \left(\sum_{\kappa \in K_h} N_\kappa^{2\alpha} \eta_{\alpha;\kappa}^2 \right)^{1/2}, \tag{98}
$$

$$
\eta_{\alpha;\kappa}^2 \le C(\varepsilon) N_\kappa^{\max(2-2\alpha+2\varepsilon,1)} \left\| u - u_{N,h} \right\|_{1,\omega_\kappa}^2. \tag{99}
$$

Further let the ascent of $\lambda_{N,h}$ be $l = 1$, and let $(\lambda_{N,h}^*, u_{N,h}^*)$ be the corresponding adjoint eigenpair of (32), then there exists an adjoint eigenpair (λ^*, u^*) of (31), such that

$$
\left| \lambda_{N,h} - \lambda \right| \le C \left(\sum_{\kappa \in K_h} N_\kappa^{2\alpha} \eta_{\alpha;\kappa}^2 \right)^{1/2} \left(\sum_{\kappa \in K_h} N_\kappa^{2\alpha} \eta_{\alpha;\kappa}^{*2} \right)^{1/2}. \tag{100}
$$

Particularly, if the eigenvalue problem (23) is symmetric (i.e., $\mathbf{b} = 0$), then

$$
CC(\varepsilon)^{-1} \sum_{\kappa \in K_h} N_\kappa^{\min(2\alpha-2-2\varepsilon,-1)} \eta_{\alpha;\kappa}^2 \le \left| \lambda_{N,h} - \lambda \right|. \tag{101}
$$

Proof. We know from the assumption $D, c \in H^{t_1}(\kappa), \mathbf{b} \in (H^{t_1}(\kappa))^2$. By the interpolation error estimates (14) and (15), we have

$$
\left\| L_\kappa u_{N,h} - L u_{N,h} \right\|_{0,\kappa} \le C h_\kappa^{\min(N_\kappa+1,t_1)-1} N_\kappa^{-t_1+1}. \tag{102}
$$

From $D \in H^{t_1}(\kappa)$, we know that $D \in H^{t_1-1/2}(e)$. By the interpolation error estimate on edge of element (see formula (5.4.42) in [2]), we get

$$
\left\| D - D_\kappa \right\|_{0,e} \le C h_e^{\min(N_\kappa+1,t_1-1/2)} N_\kappa^{-t_1+1/2}. \tag{103}
$$

Note that the formula (51) gives the optimal orders of convergence; thus, we deduce that the second and third terms on the right side of (74) are higher order infinitesimals. We derive from (52) and (53), and $N = N_\kappa$, that

$$
\left\| \lambda u - \lambda_{N,h} u_{N,h} \right\|_{0,\Omega}
$$
$$
\le \left| \lambda - \lambda_{N,h} \right| \left\| u \right\|_{0,\Omega}
$$
$$
+ \left| \lambda_{N,h} \right| \left\| u - u_{N,h} \right\|_{0,\Omega} \tag{104}
$$
$$
\le \frac{C h^{\tau_1+\tau_2-2}}{N^{t_1+t_2-2}} + \frac{C h^{r_2+\tau_1-1}}{N^{r_2+t_1-1}} \le \frac{C h^{r_2+\tau_1-1}}{N^{r_2+t_1-1}}.
$$

Therefore, the fourth term on the right side of (74) is also a higher order infinitesimal. Up to higher order terms, we get (98). We ignore higher order infinitesimals in (95) and get (99). From Lemma 4 in [10], we know that $(u_{N,h}, u_{N,h}^*) = 1$ and $u_{N,h}^*$ is uniformly bounded with h and N. By the same argument of (98), we can deduce that

$$
\left\| u^* - u_{N,h}^* \right\|_{1,\Omega}^2 \le C \sum_{\kappa \in K_h} N_\kappa^{2\alpha} \eta_{\alpha;\kappa}^{*2}. \tag{105}
$$

From (97), we have

$$
\frac{a\left(u_{N,h}, u_{N,h}^*\right)}{\left(u_{N,h}, u_{N,h}^*\right)} - \lambda
$$
$$
= \frac{a\left(u_{N,h} - u, u_{N,h}^* - u^*\right)}{\left(u_{N,h}, u_{N,h}^*\right)} \tag{106}
$$
$$
- \lambda \frac{\left(u_{N,h} - u, u_{N,h}^* - u^*\right)}{\left(u_{N,h}, u_{N,h}^*\right)};
$$

that is,

$$
\lambda_{N,h} - \lambda = a\left(u_{N,h} - u, u_{N,h}^* - u^*\right)
$$
$$
- \lambda \left(u_{N,h} - u, u_{N,h}^* - u^*\right). \tag{107}
$$

Substituting (98) and (105) into the above equality, we obtain (100).

If the eigenvalue problem (23) is symmetric (i.e., $\mathbf{b} = 0$), then

$$
\lambda_{N,h} - \lambda = a\left(u_{N,h} - u, u_{N,h} - u\right)
$$
$$
- \lambda \left(u_{N,h} - u, u_{N,h} - u\right). \tag{108}
$$

Up to higher order term $\lambda(u_{N,h} - u, u_{N,h} - u)$, by (99) we get (101). $\qquad \square$

Remark 20. Babuška and Osborn [17] have discussed hp finite element approximation with simplex partition for eigenvalue problems. Obviously, the Interpolation estimates (14) and (15) hold for hp finite element with simplex partition (see [24]). Therefore, our theoretical results of spectral methods and spectral methods for eigenvalue problems, which have been discussed in Sections 3 and 4, hold for hp finite element with simplex partition.

TABLE 1: Errors of LGL-SM, modal, and Eq-SM for 1st eigenvalue.

N	DOF	LGL-SM λ_1	Modal-SM λ_1	Eq-SM λ_1
4	9	$5.19E + 00$	$5.19E + 00$	$5.19E + 00$
5	16	$4.51E - 01$	$4.51E - 01$	$4.51E - 01$
6	25	$7.68E - 03$	$7.68E - 03$	$7.68E - 03$
7	36	$1.07E - 05$	$1.07E - 05$	$1.07E - 05$
8	49	$1.21E - 05$	$1.21E - 05$	$1.21E - 05$
9	64	$9.16E - 07$	$9.16E - 07$	$9.16E - 07$
10	81	$2.46E - 08$	$2.46E - 08$	$2.48E - 08$
11	100	$2.91E - 10$	$2.91E - 10$	$4.35E - 09$
12	121	$9.31E - 13$	$1.06E - 12$	$2.79E - 08$
13	144	$5.68E - 14$	$1.28E - 13$	$1.41E - 07$
14	169	$2.84E - 14$	$1.28E - 13$	$2.28E - 06$
15	196	$7.82E - 14$	$2.13E - 14$	$3.60E - 05$

5. Numerical Experiments

In this section, we simply denote spectral methods, spectral element methods, and finite element methods with SM, SEM, and FEM, respectively. And spectral methods with equidistant nodal basis, modal basis, and LGL nodal basis are replaced by Eq-SM, Modal-SM, and LGL-SM, respectively. Note that all these methods employ the tensorial basis.

In our experiment, we compute $1/|(u_{N,h}, u_{N,h}^*)|$ as condition number for simple eigenvalue (see Remark 2.1 in [25]), where $u_{N,h}$ and $u_{N,h}^*$ are eigenfunctions of eigenvalue problem (25) and its adjoint problem (32) normalized with $\|\cdot\|_{0,\Omega}$, respectively.

5.1. Example 1. Consider the nonsymmetric eigenvalue problem

$$-\Delta u + 10u_x + u_y = \lambda u, \quad \text{in } \Omega = (0, 1)^2,$$
$$u = 0, \quad \text{on } \partial\Omega. \tag{109}$$

The first eigenvalue of (109) $\lambda_1 = 101/4 + 2\pi^2$ is a simple eigenvalue. And the corresponding eigenfunctions are sufficiently smooth.

5.1.1. Comparisons between LGL-SM, Modal, and Eq-SM. Figure 1 shows that the condition numbers of the first eigenvalue for LGL-SM, Modal-SM, and Eq-SM coincide with each other at the beginning but perform abnormally with $N > 19$ for Eq-SM. Table 1 tells us that when $N > 11$, the accuracy of first eigenvalue obtained by Eq-SM is not as good as obtained that by LGL-SM and Modal-SM. When $N = 15$, the error of the first eigenvalues obtained by Eq-SM is greater than 1E-5; however, the order of the magnitude of errors for LGL-SM and Modal-SM still keeps below 1E-13. The best result of first eigenvalue error for Eq-SM is merely 1E-9 or so.

5.1.2. LGL-SM and Modal-SM versus hp-SEM. Tables 1 and 2 indicate that increasing the polynomial degree N or
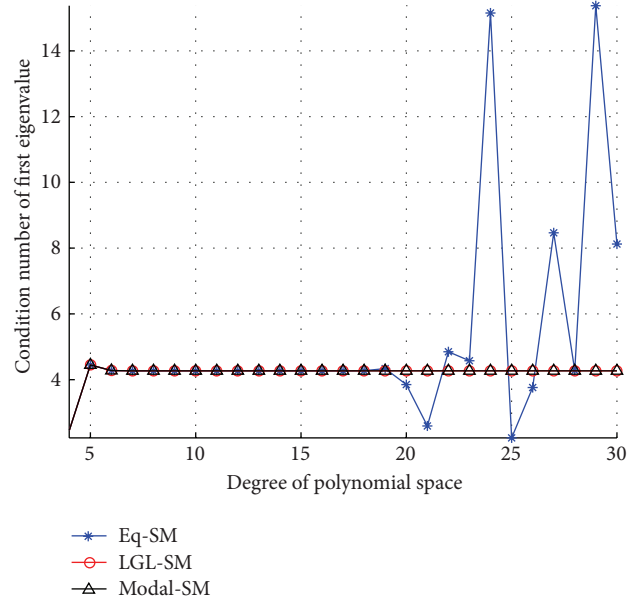

FIGURE 1: Condition number of first eigenvalue for SM.

decreasing the mesh fineness h can decrease the errors of the first eigenvalue. But it is expensive to increase polynomial degree and decrease mesh fineness h at the same time. For $h = 1/4$ and $h = 1/16$, we obtain from Table 2 the first eigenvalue errors $2.8E - 14$ and $1.3E - 13$ and the corresponding degree of freedom 1225 and 6241 for hp-SEM, respectively, Whereas from Table 1, to reach this accuracy, LGL-SM and Modal-SM should merely perform the interpolation approximations with polynomial degree bi-14 and bi-13 or so, and the corresponding degrees of freedom are merely 169 and 144, respectively. Therefore, we conclude that LGL-SM and Modal-SM are highly accurate and efficient for this kind of nonsymmetric eigenvalue problems.

In Figure 2 from [9], when the degree of freedom is up to 1000, the error of linear FEM is about 1E-2; the function value recovery techniques in [9] obviously improves the accuracy up to 1E-5. Comparing Tables 1 and 2 in this paper with Figure 2 in [9], we can also find the advantages of LGL-SM, Modal-SM, and hp-SEM over the function value recovery techniques for FEM given by [9] from accuracy and degree of freedom.

5.1.3. hp-SEM versus hp-FEM. From Table 4, we find that the condition number of the first eigenvalue for hp-version methods (hp-SEM and hp-FEM) stays at 4.27. It is indicated from Tables 2 and 3 that, when N is greater than 7, compared with hp-SEM, the errors of hp-FEM tend to become large, whereas the errors of hp-SEM still keep stable or even stay a decreasing tendency; however, this phenomenon is not apparent for $h = 1/2$.

Remark 21. Condition numbers of 1st eigenvalue for hp-FEM (not listed in Table 4) are almost the same to those for hp-SEM.

TABLE 2: Errors and DOF of hp-SEM for the first eigenvalue.

N	$h = 1/2$		$h = 1/4$		$h = 1/8$		$h = 1/16$	
	Error	DOF	Error	DOF	Error	DOF	Error	DOF
2	5.18E + 00	9	2.54E − 01	49	1.50E − 02	225	9.00E − 04	961
3	7.00E − 03	25	6.10E − 04	121	1.20E − 05	529	1.90E − 07	2209
4	8.40E − 03	49	2.60E − 05	225	9.70E − 08	961	3.70E − 10	3969
5	1.64E − 04	81	1.60E − 07	361	1.50E − 10	1521	1.30E − 13	6241
6	4.10E − 07	121	2.30E − 11	529	9.90E − 13	2209	3.60E − 12	9025
7	3.10E − 08	169	1.70E − 12	729	3.10E − 13	3025	1.60E − 12	12321
8	1.90E − 10	225	1.90E − 13	961	2.10E − 12	3969	4.80E − 12	16129
9	5.50E − 13	289	2.80E − 14	1225	6.00E − 13	5041	1.10E − 12	20449
10	3.80E − 13	361	1.10E − 12	1521	4.40E − 12	6241	1.50E − 11	25281

TABLE 3: Errors of hp-FEM for the first eigenvalue.

N	$h = 1/2$	$h = 1/4$	$h = 1/8$	$h = 1/16$
3	7.00E − 03	6.10E − 04	1.20E − 05	1.90E − 07
4	8.40E − 03	2.60E − 05	9.70E − 08	3.70E − 10
5	1.60E − 04	1.60E − 07	1.50E − 10	1.30E − 12
6	4.10E − 07	2.40E − 11	3.60E − 13	8.60E − 12
7	3.10E − 08	6.10E − 12	1.30E − 11	3.00E − 11
8	1.80E − 10	3.10E − 11	2.30E − 10	2.10E − 10
9	7.50E − 11	3.40E − 11	6.80E − 10	7.40E − 10
10	2.50E − 11	9.90E − 10	8.70E − 09	6.60E − 09
11	2.00E − 09	9.60E − 09	8.90E − 09	5.40E − 07

TABLE 4: Condition number of first eigenvalue for hp-SEM.

N	$h = 1/2$	$h = 1/4$	$h = 1/8$	$h = 1/16$
3	4.284381324	4.270132842	4.269625046	4.269615821
4	4.267343095	4.269607452	4.269615638	4.26961567
5	4.269636446	4.269615725	4.26961567	4.26961567
6	4.269619135	4.26961567	4.26961567	4.26961567
7	4.269615617	4.26961567	4.26961567	4.26961567
8	4.26961567	4.26961567	4.26961567	4.26961567
9	4.26961567	4.26961567	4.26961567	4.26961567

TABLE 5: The Approximate eigenvalues and indicator ψ_0 of P-SEM.

N	$\lambda_{N,h}$	ψ_0
3	28.56900	2.72E + 01
4	31.99175	3.49E + 00
5	34.82082	2.25E − 01
6	34.65087	1.31E − 02
7	34.65057	3.32E − 03
8	34.64765	1.92E − 03
9	34.64567	1.22E − 03
10	34.64432	8.11E − 04
11	34.64335	5.62E − 04
12	34.64265	4.02E − 04
13	34.64212	2.95E − 04
14	34.64171	2.22E − 04
15	34.64139	1.71E − 04
16	34.64114	1.33E − 04
17	34.64094	1.06E − 04
18	34.64078	8.49E − 05

5.1.4. Validity of the Error Indicator. Denote

$$\psi_\alpha = \left(\sum_{\kappa \in K_h} N_\kappa^{2\alpha} \eta_{\alpha;\kappa}^2 \right)^{1/2} \left(\sum_{\kappa \in K_h} N_\kappa^{2\alpha} \eta_{\alpha;\kappa}^{*2} \right)^{1/2}. \quad (110)$$

From Theorem 19, we know that ψ_α is a reliable error indicator for $\lambda_{N,h}$. We choose ψ_0 (setting $\alpha = 0$ in (110)) as a posteriorii error indicator.

In Figures 2 and 3, we denote the true error and est. error with $|\lambda_{N,h} - \lambda|$ and ψ_0, respectively.

As is depicted in Figure 2, when the polynomial degree $N \leq 12$, the error indicator ψ_0 can properly estimate the true errors of LGL-SM for the first eigenvalue, however, also slightly underestimate the true errors. It is easy to see that ψ_0 shows almost the same algebraic decay as the true error with the polynomial degree N (≤ 12) increasing. Nevertheless, the error indicator ψ_0 cannot approximate the true errors if N is large enough, which is caused by round-off errors derived from the bad condition number of eigenvalue. In Figure 3, we give the comparison between the error indicator ψ_0 and the true errors for hp-SEM.

5.2. Example 2. Consider the nonsymmetric eigenvalue problem

$$-\Delta u + 10u_x = \lambda u, \quad \text{in } \Omega = \frac{(-1, 1)^2}{(0, 1)^2}, \quad (111)$$

$$u = 0, \quad \text{on } \partial\Omega.$$

A reference value for the first eigenvalue (simple eigenvalue) of (111) is 34.6397 given by [5]. And the corresponding eigenfunctions have the singularity at the origin. Next, we shall compare the relevant numerical results between P-SEM and the other methods adopted in this paper. Note that here and hereafter P-version methods are for the fixed mesh fineness $h = 1$. Table 5 lists part data of the approximate eigenvalues computed by P-SEM and the corresponding error indicator ψ_0 for reference.
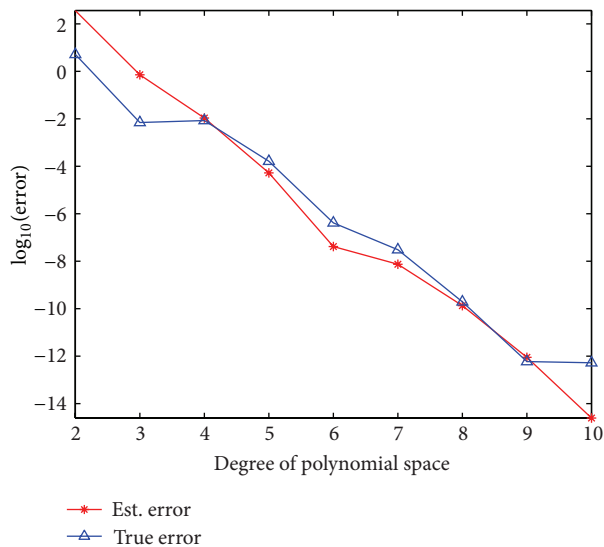
FIGURE 2: The Error indicator ψ_0 of LGL-SM.

FIGURE 4: The Approximate 1st eigenvalue of P-SEM and P-FEM.

FIGURE 3: The Error indicator ψ_0 of hp-SEM ($h = 1/2$).
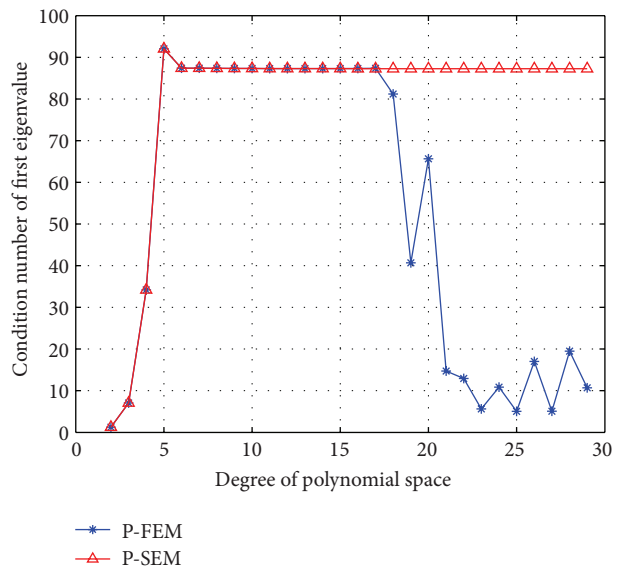
FIGURE 5: Condition number of first eigenvalue for P-SEM and P-FEM.

5.2.1. Stability of P-Version Methods. Figure 4 indicates that the eigenvalues computed by P-FEM will not seriously deviate from the results computed by P-SEM until the interpolation polynomial degree N is up to 19. This phenomenon coincides with the abnormity of condition number of first eigenvalue for P-FEM (see Figure 5). The reason is that the singularities of the eigenfunctions limit the accuracy of both kinds of methods; this is slightly different from the case of the eigenvalue problem with the sufficiently smooth eigenfunctions.

5.2.2. P-SEM versus Other Methods. By calculations, we find that, in the case of the linear FEM, for fixed mesh fineness $h = 1/256$, the approximate eigenvalue is 34.6403 with degree of freedom up to 195585. But P-SEM with the polynomial degree bi-22 can reach this accuracy, and the corresponding
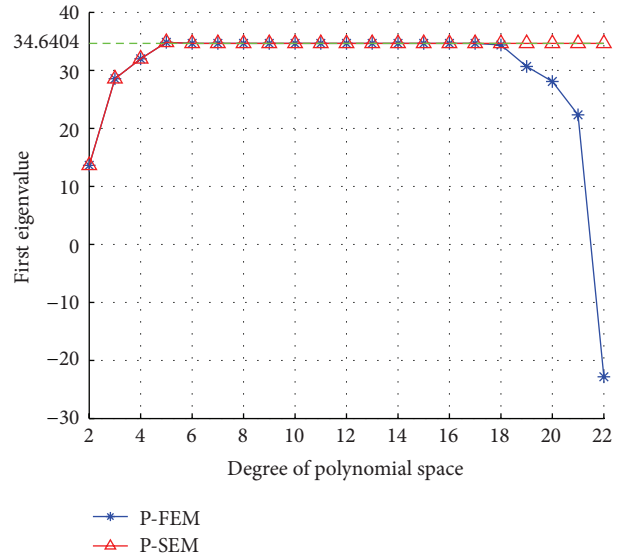
degree of freedom is merely 1365. Compared with the linear FEM, hp-SEM can obtain a higher accuracy with less degrees of freedom as follows: for fixed $h = 1/16$ and $N = 10$, the approximate eigenvalue is 34.63984 with degree of freedom 76161 but P-SEM with polynomial degree bi-44 can reach this accuracy. Therefore, P-SEM is more efficient for the eigenvalue problems with the singular solutions than the other methods.

Acknowledgments

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