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Research Article

Existence of Positive Periodic Solutions for n-Dimensional Nonautonomous System

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In this paper we consider the existence, multiplicity, and nonexistence of positive periodic solutions for n-dimensional nonautonomous functional differential system $\mathbf{x}'(t) = \mathbf{H}(t, \mathbf{x}(t)) - \lambda \mathbf{B}(t)\mathbf{F}(\mathbf{x}(t - \tau(t)))$, where h_i are ω-periodic in t and there exist ω-periodic functions $\alpha_i, \beta_i \in C(R, R_+)$ such that $\alpha_i(t) \le (h_i(t, \mathbf{x})/x_i) \le \beta_i(t)$, $\int_0^\omega \alpha_i(t)dt > 0$, for $\mathbf{x} \in R_+^n$ all with $x_i > 0$, and $t \in R$, $\lim_{x_i \to 0^+}(h_i(t, \mathbf{x})/x_i)$ exist for $t \in R$; $b_i \in C(R, R_+)$ are ω-periodic functions and $\int_0^\omega b_i(t)dt > 0$; $f_i \in C(R_+^n, R_+)$, $f_i(\mathbf{x}) > 0$ for $\|\mathbf{x}\| > 0$; $\tau \in (R, R)$ is an ω-periodic function. We show that the system has multiple or no positive ω-periodic solutions for sufficiently large or small $\lambda > 0$, respectively.

1. Introduction

In this paper, we consider the first-order n-dimensional nonautonomous functional differential system

$$\mathbf{x}'(t) = \mathbf{H}(t, \mathbf{x}(t)) - \lambda \mathbf{B}(t)\mathbf{F}(\mathbf{x}(t - \tau(t))), \quad (1)$$

where $\lambda > 0$ is a parameter;

$$\begin{gathered} \mathbf{x} = [x_1, x_2 \ldots, x_n]^\top, \\ \mathbf{B}(t) = \operatorname{diag}[b_1(t), b_2(t), \ldots, b_n(t)]; \\ \mathbf{H}(t, \mathbf{x}) = [h_1(t, \mathbf{x}), h_2(t, \mathbf{x}), \ldots, h_n(t, \mathbf{x})]^\top, \\ \mathbf{F}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]^\top. \end{gathered} \quad (2)$$

Let

$$R = (-\infty, +\infty), \qquad R_+ = [0, +\infty), \qquad R_+^n = \prod_{i=1}^n R_+, \quad (3)$$

and for any $\mathbf{x} = [x_1, x_2 \ldots, x_n]^\top \in R_+^n$, the norm of $\mathbf{x}$ is defined as $\|\mathbf{x}\| = \max_{1 \le i \le n}|x_i|$.

Throughout this paper, we use $i = 1, 2, \ldots, n$, unless otherwise stated.

For the system (1), we assume that

(H_1) $\tau \in (R, R)$ is an ω-periodic function, $b_i \in C(R, R_+)$ are ω-periodic functions, and

$$\int_0^\omega b_i(t)\,dt > 0; \quad (4)$$

(H_2) $f_i \in C(R_+^n, R_+)$, $f_i(\mathbf{x}) > 0$ for $\|\mathbf{x}\| > 0$; $h_i \in C(R \times R_+^n, R_+)$, h_i are ω-periodic in t and there exist ω-periodic functions $\alpha_i, \beta_i \in C(R, R_+)$ such that

$$\alpha_i(t) \le \frac{h_i(t, \mathbf{x})}{x_i} \le \beta_i(t), \qquad \int_0^\omega \alpha_i(t)\,dt > 0, \quad \forall \mathbf{x} \in R_+^n \text{ with } x_i > 0, \quad t \in R. \quad (5)$$

In addition, $\lim_{x_i \to 0^+}(h_i(t, \mathbf{x})/x_i)$ exist for $t \in R$.

We note that in (1) $\mathbf{F}(\mathbf{x})$ may have a singularity near $\mathbf{x} = \mathbf{0}$; that is,

$$\lim_{\mathbf{x} \to \mathbf{0}^+} f_i(\mathbf{x}) = \infty. \quad (6)$$

As we well know, the system (1) is sufficiently general to include particular mathematical models which describe multiple population dynamics. Recently, due to the theoretical and practical significance, the existence of positive periodic solution of some particular cases of periodic system (1) has been extensively studied; see, for example, [1–15]. Cheng and Zhang [1], Kang and Cheng [2], Kang et al. [3], Kang and Zhang [4], and Liu et al. [5] studied the existence, multiplicity, and nonexistence of positive periodic solutions. The existence of positive periodic solutions of the scalar functional differential equation

$$x'(t) = a(t) g(x(t)) x(t) - \lambda b(t) f(x(t-\tau(t))) \quad (7)$$

has been studied by Wang [6]. By employing behaviours of the quotient $f(x)/x$ as $x \to 0^+$ and $x \to \infty$, several interesting results on the existence and nonexistence of positive periodic solutions of (7) have been obtained. In [7], Weng and Sun studied more general scalar periodic functional differential equation

$$x'(t) = h(t,x) - \lambda b(t) f(x(t-\tau(t))), \quad (8)$$

where the existence theorems of positive periodic solutions of (8) are obtained by employing the behaviours of $f(x)/x$ at any point $x \in (0,+\infty)$ and $x \to 0^+, x \to \infty$. The result in [7] generalized and improved those in [6]. O'Regan and Wang [8] investigated the n-dimensional periodic system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{g}(\mathbf{x}(t)) - \lambda \mathbf{B}(t)\mathbf{F}(\mathbf{x}(t-\tau(t))). \quad (9)$$

By employing behaviours of $f(\mathbf{x})/\|\mathbf{x}\|$ as $\|\mathbf{x}\| \to 0^+$ and $\|\mathbf{x}\| \to \infty$, under quite general conditions, several existence theorems of positive periodic solutions are proved.

A solution $\mathbf{x} = [x_1, x_2 \dots, x_n]^\top$, $t \in R$ of (1) is said to be positive if its all components $x_i(t)$ are positive; $\mathbf{x}$ is said to be ω-periodic ($\omega > 0$) if $x_i(t) = x_i(t+\omega)$, $t \in R$.

2. Preliminary

Lemma 1 (see [9]). *Let E be a Banach space and K a cone in E. For $r > 0$, define $K_r = \{u \in K : \|x\| < r\}$. Assume that $T : \overline{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K : \|x\| = r\}$.*

(i) *If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 0. \quad (10)$$

(ii) *If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then*

$$i(T, K_r, K) = 1. \quad (11)$$

Lemma 2 (see [9, 10]). *Let X be a Banach space and K a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. Let*

$$T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K \quad (12)$$

be a completely continuous operator such that one of the following conditions is satisfied:

(a) *$\|Ty\| \leq \|y\|$ for $y \in K \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|$ for $y \in K \cap \partial\Omega_2$;*

(b) *$\|Ty\| \geq \|y\|$ for $y \in K \cap \partial\Omega_1$ and $\|Ty\| \leq \|y\|$ for $y \in K \cap \partial\Omega_2$.*

Then T has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In order to apply Lemmas 1 and 2 to system (1), we take

$$X = \{\mathbf{x}(t) : \mathbf{x}(t) \in C(R, R^n), \mathbf{x}(t+\omega) = \mathbf{x}(t), t \in R\}, \quad (13)$$

endowed with the norm $\|\mathbf{x}\| = \max_{1\leq i\leq n}|x_i|_0$, where $|x_i|_0 = \sup_{t\in[0,\omega]}|x_i(t)|$; then X is a Banach space.

Define the operator

$$T_\lambda : X \longrightarrow X \quad (14)$$

by

$$(T_\lambda \mathbf{x})(t) = ((T_\lambda \mathbf{x})_1(t), (T_\lambda \mathbf{x})_2(t), \dots, (T_\lambda \mathbf{x})_n(t))^\top, \quad (15)$$

where

$$(T_\lambda \mathbf{x})_i(t) = \lambda \int_t^{t+\omega} G_i(t,s) b_i(s) f_i(\mathbf{x}(s-\tau(s))) ds, \quad (16)$$

$$G_i(t,s) = \frac{\exp\left(-\int_t^s (h_i(\theta,\mathbf{x}(\theta))/x_i(\theta)) d\theta\right)}{1-\exp\left(-\int_0^\omega (h_i(\theta,\mathbf{x}(\theta))/x_i(\theta)) d\theta\right)}, \quad t \leq s \leq t+\omega. \quad (17)$$

Let $m = \min_{1\leq i\leq n}\min_{t,s\in[0,\omega]}G_i(t,s)$ and $M = \max_{1\leq i\leq n}\max_{t,s\in[0,\omega]}G_i(t,s)$, clearly;

$$0 < m \leq G_i(t,s) \leq M, \quad t \leq s \leq t+\omega. \quad (18)$$

Define a set by

$$K = \left\{\mathbf{x}(t) = (x_1, x_2 \dots, x_n)^\top \in X : x_i(t) \geq \sigma\|\mathbf{x}\|, t \in [0,\omega]\right\}, \quad \text{where } \sigma = \frac{m}{M}. \quad (19)$$

We use the following notations.

Let $r > 0$ be a constant, and $\mathbf{x} \in K$, defining

$$\Omega_r = \{\mathbf{x} \in X : \|\mathbf{x}\| < r\}, \qquad \partial\Omega_r = \{\mathbf{x} \in X : \|\mathbf{x}\| = r\},$$

$$\rho^u(r) := \max_{1\leq i\leq n}\sup_{\|\mathbf{x}\|=r}\frac{f_i(\mathbf{x})}{\|\mathbf{x}\|}, \qquad \rho^l(r) := \min_{1\leq i\leq n}\inf_{\|\mathbf{x}\|=r}\frac{f_i(\mathbf{x})}{\|\mathbf{x}\|},$$

$$f_i^0 = \lim_{\|\mathbf{x}\|\to 0^+}\frac{f_i(\mathbf{x})}{\|\mathbf{x}\|}, \qquad f_i^\infty = \lim_{\|\mathbf{x}\|\to\infty}\frac{f_i(\mathbf{x})}{\|\mathbf{x}\|},$$

$$\mathbf{F}_0 = \max_{1\leq i\leq n}\{f_i^0\}, \qquad \mathbf{F}_\infty = \max_{1\leq i\leq n}\{f_i^\infty\},$$

$$I_0 = \text{number of zeros in the set } \{\mathbf{F}_0, \mathbf{F}_\infty\},$$

$$I_\infty = \text{number of infinities in the set } \{\mathbf{F}_0, \mathbf{F}_\infty\},$$

$$b^u := \max_{1\leq i\leq n}\int_0^\omega b_i(t)\,dt, \qquad b^l := \min_{1\leq i\leq n}\int_0^\omega b_i(t)\,dt. \quad (20)$$

Lemma 3. *Assume that (H_1)-(H_2) hold; then $T_\lambda(K) \subset K$ and $T_\lambda : K \to K$ is continuous and completely continuous.*

Proof. In view of the definition of K, for $\mathbf{x} \in K$, we have

$$\begin{aligned}(T_\lambda \mathbf{x})_i(t+\omega) &= \lambda \int_{t+\omega}^{t+2\omega} G_i(t,s)\, b_i(s)\, f_i(\mathbf{x}(s-\tau(s)))\, ds \\ &= \lambda \int_t^{t+\omega} G_i(t+\omega, s+\omega)\, b_i(s+\omega) \\ &\quad \times f_i(\mathbf{x}(s+\omega-\tau(s+\omega)))\, ds \\ &= \lambda \int_t^{t+\omega} G_i(t,s)\, b_i(s)\, f_i(\mathbf{x}(s-\tau(s)))\, ds \\ &= (T_\lambda \mathbf{x})_i(t).\end{aligned} \tag{21}$$

It is easy to see that $\int_t^{t+\omega} b_i(s) f_i(\mathbf{x}(s-\tau(s)))ds$ is a constant because of the periodicity of $b_i(t) f_i(\mathbf{x}(t-\tau(t)))$.

Notice that, for $\mathbf{x} \in K$ and $t \in [0,\omega]$,

$$\begin{aligned}(T_\lambda \mathbf{x})_i(t) &= \lambda \int_t^{t+\omega} G_i(t,s)\, b_i(s)\, f_i(\mathbf{x}(s-\tau(s)))\, ds \\ &\ge \lambda m \int_t^{t+\omega} b_i(s)\, f_i(\mathbf{x}(s-\tau(s)))\, ds \\ &= \lambda m \int_0^{\omega} b_i(s)\, f_i(\mathbf{x}(s-\tau(s)))\, ds \\ &= \frac{m}{M} \lambda M \int_0^{\omega} b_i(s)\, f_i(\mathbf{x}(s-\tau(s)))\, ds \\ &\ge \frac{m}{M} \max_{t\in[0,\omega]} \left|(T_\lambda \mathbf{x})_i(t)\right|.\end{aligned} \tag{22}$$

Thus $T_\lambda(K) \subset K$ and it is easy to show that $T_\lambda : K \to K$ is continuous and completely continuous. □

Lemma 4. *Assume that (H_1)-(H_2) hold; then a function $\mathbf{x}(t) \in K$ is a positive ω-periodic solution of* (1) *if and only if $T_\lambda \mathbf{x} = \mathbf{x}$, $\mathbf{x} \in K$.*

Proof. If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in K$ and $T_\lambda \mathbf{x} = \mathbf{x}$, then

$$\begin{aligned}x_i'(t) &= \frac{d}{dt}\left(\lambda \int_t^{t+\omega} G_i(t,s)\, b_i(s)\, f_i(\mathbf{x}(s-\tau(s)))\, ds\right) \\ &\ge \lambda G_i(t,t+\omega)\, b_i(t+\omega)\, f_i(\mathbf{x}(t+\omega-\tau(t+\omega))) \\ &\quad - \lambda G_i(t,t)\, b_i(t)\, f_i(\mathbf{x}(t-\tau(t))) + \frac{h_i(t,\mathbf{x})}{x_i(t)} (T_\lambda \mathbf{x})_i(t) \\ &= \lambda \left[G_i(t,t+\omega) - G_i(t,t)\right] b_i(t)\, f_i(\mathbf{x}(t-\tau(t))) \\ &\quad + h_i(t,\mathbf{x}) \\ &= h_i(t,\mathbf{x}) - \lambda b_i(t)\, f_i(\mathbf{x}(t-\tau(t))).\end{aligned} \tag{23}$$

Thus $\mathbf{x}$ is a positive ω-periodic solution of (1). On the other hand, if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a positive ω-periodic solution of (1), then $x_i'(t) = h_i(t,\mathbf{x}) - \lambda b_i(t) f_i(\mathbf{x}(t-\tau(t)))$ and

$$\begin{aligned}(T_\lambda \mathbf{x})_i(t) &= \lambda \int_t^{t+\omega} G_i(t,s)\, b_i(s)\, f_i(\mathbf{x}(s-\tau(s)))\, ds \\ &= \int_t^{t+\omega} G_i(t,s) \left(h_i(s,\mathbf{x}) - x_i'(s)\right) ds \\ &= \int_t^{t+\omega} G_i(t,s)\, h_i(s,\mathbf{x})\, ds - \int_t^{t+\omega} G_i(t,s)\, x_i'(s)\, ds \\ &= \int_t^{t+\omega} G_i(t,s) h_i(s,\mathbf{x}) ds - G_i(t,s) x_i(s)\Big|_t^{t+\omega} \\ &\quad - \int_t^{t+\omega} G_i(t,s)\, h_i(s,\mathbf{x})\, ds = x_i(t).\end{aligned} \tag{24}$$

Thus, $T_\lambda \mathbf{x} = \mathbf{x}$; furthermore, in view of the proof of Lemma 3, we also have $x_i(t) \ge (m/M) \max_{t\in[0,\omega]} |(T_\lambda \mathbf{x})_i(t)|$ for $t \in [0,\omega]$. That is, $\mathbf{x}$ is a fixed point of T_λ in K. □

Lemma 5. *Assume that (H_1)-(H_2) hold; for any $\eta > 0$ and $\mathbf{x} \in K$, if there exists a component f_i of $\mathbf{F}$ such that $f_i(\mathbf{x}(t)) \ge x_i(t)\eta$, then*

$$\left\|T_\lambda \mathbf{x}\right\| \ge \frac{\lambda \eta m^2 b^l}{M} \|\mathbf{x}\|. \tag{25}$$

Proof. Since $\mathbf{x} \in K$ and $f_i(\mathbf{x}(t)) \ge x_i(t)\eta$, we have

$$\begin{aligned}(T_\lambda \mathbf{x})_i(t) &= \lambda \int_0^{\omega} G_i(t,s)\, b_i(s)\, f_i(\mathbf{x}(s-\tau(s)))\, ds \\ &\ge \lambda m \int_0^{\omega} b_i(s)\, f_i(\mathbf{x}(s-\tau(s)))\, ds \\ &\ge \lambda \eta m \int_0^{\omega} b_i(s)\, x_i(s-\tau(s))\, ds \\ &\ge \frac{\lambda \eta m^2 b^l}{M} \|\mathbf{x}\|.\end{aligned} \tag{26}$$

Thus $\|T_\lambda \mathbf{x}\| \ge (\lambda \eta m^2 b^l / M)\|\mathbf{x}\|$. The proof is complete. □

For each $i = 1, 2, \ldots, n$, let $\widehat{f}(r) : R_+ \to R_+$ be the function given by

$$\widehat{f}_i(r) = \max\left\{f_i(\mathbf{x}) : \mathbf{x} \in R_+^n, \|\mathbf{x}\| \le r, t \in [0,\omega]\right\}. \tag{27}$$

Let $\widehat{f}_i^0 = \lim_{r\to 0}(\widehat{f}_i(r)/r)$ and $\widehat{f}_i^\infty = \lim_{r\to\infty}(\widehat{f}_i(r)/r)$.

Lemma 6 (see [11]). *Assume (H_2) holds. Then $\widehat{f}_i^0 = f_i^0$ and $\widehat{f}_i^\infty = f_i^\infty$.*

Lemma 7. *Assume that (H_1)-(H_2) hold and let $r > 0$. If $\mathbf{x} \in \partial\Omega_r$ and there exists an $\epsilon > 0$, such that $\widehat{f}_i(r) \le \epsilon r$ for $t \in [0,\omega]$, then*

$$\left\|T_\lambda \mathbf{x}\right\| \le \lambda \epsilon M b^\mu \|\mathbf{x}\|. \tag{28}$$

Proof. From the definition of T_λ, for $\mathbf{x} \in \partial\Omega_r$, we have

$$\begin{aligned}(T_\lambda \mathbf{x})_i(t) &= \int_0^\omega G_i(t,s) b_i(s) f_i(\mathbf{x}(s-\tau(s)))\,ds \\ &\le \lambda M \int_0^\omega b_i(s) f_i(\mathbf{x}(s-\tau(s)))\,ds \\ &\le \lambda M \int_0^\omega b_i(s) \hat{f}_i(r)\,ds \\ &\le \lambda \epsilon M b^\mu \|\mathbf{x}\|.\end{aligned} \quad (29)$$

That is, $\|T_\lambda \mathbf{x}\| \le \lambda\epsilon M b^\mu \|\mathbf{x}\|$. The proof is complete. □

The following two lemmas are weak forms of Lemmas 5 and 7.

Lemma 8. *Assume that* (H_1)-(H_2) *hold. If* $\mathbf{x} \in \partial\Omega_r, r > 0$, *then*

$$\|T_\lambda \mathbf{x}\| \ge \lambda m b^l \widehat{m}_r, \quad (30)$$

where $\widehat{m}_r = \min\{f_i(\mathbf{x}) : \mathbf{x} \in R^n_+, mr/M \le \|\mathbf{x}\| \le r\}$.

Lemma 9. *Assume that* (H_1)-(H_2) *hold. If* $\mathbf{x} \in \partial\Omega_r, r > 0$, *then*

$$\|T_\lambda \mathbf{x}\| \le \lambda M b^\mu \widehat{M}(r), \quad (31)$$

where $\widehat{M}_r = \max\{f_i(\mathbf{x}) : \mathbf{x} \in R^n_+, \|\mathbf{x}\| \le r\}$.

3. The Main Results

Theorem 10. *Assume that* (H_1)-(H_2) *hold and there exist positive constants* r_1 *and* r_2 *with* $r_1 < r_2$ *such that*

$$m b^l \rho^l(r_2) > M b^\mu \rho^\mu(r_1); \quad (32)$$

then for

$$\frac{1}{m b^l \rho^l(r_2)} \le \lambda \le \frac{1}{M b^\mu \rho^\mu(r_1)}, \quad (33)$$

the system (1) *has at least a positive* ω*-periodic solution* $\mathbf{x}(t)$ *satisfying* $r_1 \le \|\mathbf{x}\| \le r_2$.

Proof. From (32) for λ satisfying (33) we have that

$$\lambda m b^l \rho^l(r_2) \ge 1, \qquad \lambda M b^\mu \rho^\mu(r_1) < 1, \quad (34)$$

or

$$\lambda m b^l \rho^l(r_2) > 1, \qquad \lambda M b^\mu \rho^\mu(r_1) \le 1. \quad (35)$$

Let $\mathbf{x} \in K$ and $\|\mathbf{x}\| = r_1$; by (16) and (34), we have

$$\begin{aligned}(T_\lambda \mathbf{x})_i(t) &= \lambda \int_t^{t+\omega} G_i(t,s) b_i(s) f_i(\mathbf{x}(s-\tau(s)))\,ds \\ &\le \lambda M \int_t^{t+\omega} b_i(s) f_i(\mathbf{x}(s-\tau(s)))\,ds \\ &\le \lambda M \int_t^{t+\omega} b_i(s) \frac{f_i(\mathbf{x}(s-\tau(s)))}{\|\mathbf{x}\|} \|\mathbf{x}\|\,ds \\ &\le \lambda M b^\mu \rho^\mu(r_1) \|\mathbf{x}\| \le \|\mathbf{x}\|.\end{aligned} \quad (36)$$

That is, $\|T_\lambda \mathbf{x}_i\| \le \|\mathbf{x}\|$. This implies that $\|T_\lambda \mathbf{x}\| \le \|\mathbf{x}\|$ for $\mathbf{x} \in K \cap \partial\Omega_{r_1}$.

If $\mathbf{x} \in K$ and $\|\mathbf{x}\| = r_2$, by (16) and (35), we have

$$\begin{aligned}(T_\lambda \mathbf{x})_i(t) &= \lambda \int_t^{t+\omega} G_i(t,s) b_i(s) f_i(\mathbf{x}(s-\tau(s)))\,ds \\ &\ge \lambda m \int_t^{t+\omega} b_i(s) f_i(\mathbf{x}(s-\tau(s)))\,ds \\ &\ge \lambda m \int_t^{t+\omega} b_i(s) \frac{f_i(\mathbf{x}(s-\tau(s)))}{\|\mathbf{x}\|} \|\mathbf{x}\|\,ds \\ &\ge \lambda m b^l \rho^l(r_2) \|\mathbf{x}\| \ge \|\mathbf{x}\|.\end{aligned} \quad (37)$$

This implies that $\|T_\lambda \mathbf{x}\| \ge \|\mathbf{x}\|$ for $\mathbf{x} \in K \cap \partial\Omega_{r_2}$. By Lemma 2(a), T_λ has a fixed point in $K \cap (\overline{\Omega}_{r_2} \setminus \Omega_{r_1})$. It follows from Lemma 2 that (1) has an ω-periodic solution $\mathbf{x}$ with $r_1 \le \|\mathbf{x}\| \le r_2$. The proof is complete. □

Theorem 11. *Assume that* (H_1)-(H_2) *hold and there exist positive constants* r_1 *and* r_2 *with* $r_1 < r_2$ *such that*

$$m b^l \rho^l(r_1) > M b^\mu \rho^\mu(r_2); \quad (38)$$

then for

$$\frac{1}{m b^l \rho^l(r_1)} \le \lambda \le \frac{1}{M b^\mu \rho^\mu(r_2)}, \quad (39)$$

the system (1) *has at least a positive* ω*-periodic solution* $\mathbf{x}(t)$ *satisfying* $r_1 \le \|\mathbf{x}\| \le r_2$.

Proof. The proof of Theorem 11 is similar to that of Theorem 10, so we omit it. The proof is complete. □

Theorem 12. *Assume that* (H_1)-(H_2) *hold.*

(a) *If* $I_0 = 1$ *or* $I_0 = 2$, *then* (1) *has* I_0 *positive* ω*-periodic solution(s) for* $\lambda > 1/mb^l\widehat{m}_1 > 0$.

(b) *If* $I_\infty = 1$ *or* $I_\infty = 2$, *then* (1) *has* I_∞ *positive* ω*-periodic solution(s) for* $0 < \lambda < 1/Mb^\mu\widehat{M}_1$.

(c) *If* $I_0 = 0$ *or* $I_\infty = 0$, *then* (1) *has no positive* ω*-periodic solution for sufficiently large or small* $\lambda > 0$, *respectively.*

Proof. (a) Choose a number $r_1 = 1$. By Lemma 8 we infer that there exists $\lambda_0 = 1/mb^l\widehat{m}_1 > 0$ such that

$$\|T_\lambda(\mathbf{x})\| \ge \|\mathbf{x}\| \quad \text{for } \mathbf{x} \in \partial\Omega_{r_1},\ \lambda > \lambda_0. \quad (40)$$

If $\mathbf{F}_0 = 0$, this implies that $f_i^0 = 0$. It follows from Lemma 6 that $\hat{f}_i^0 = 0$; therefore, we choose $0 < r_2 < r_1$ so that $\hat{f}_i(r_2) \le \epsilon r_2$, where the constant $\epsilon > 0$ satisfies $\lambda\epsilon M b^\mu < 1$. By Lemma 7, it follows that

$$\|T_\lambda \mathbf{x}\| \le \lambda\epsilon M b^\mu \|\mathbf{x}\| < \|\mathbf{x}\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_2},\ t \in [0,\omega]; \quad (41)$$

it follows from Lemma 1 that

$$i\left(T_\lambda, \Omega_{r_1}, K\right) = 0, \qquad i\left(T_\lambda, \Omega_{r_2}, K\right) = 1. \tag{42}$$

Thus $i(T_\lambda, \Omega_{r_1} \setminus \overline{\Omega}_{r_2}, K) = -1$, which implies T_λ has a fixed point in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$, which is positive ω-periodic solution of (1) for $\lambda > \lambda_0$.

If $\mathbf{F}_\infty = 0$, then $f_i^\infty = 0$. It follows from Lemma 6 that $\widehat{f}_i^\infty = 0$. Therefore, we choose $r_3 > 2r_1$ such that $\widehat{f}_i(t, r_3) \le \epsilon r_3$, where the constant $\epsilon > 0$ satisfies $\lambda \epsilon M b^\mu < 1$. By Lemma 7, it follows that

$$\left\|T_\lambda \mathbf{x}\right\| \le \lambda \epsilon M b^\mu \left\|\mathbf{x}\right\| < \left\|\mathbf{x}\right\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_3},\ t \in [0, \omega]; \tag{43}$$

it follows from Lemma 1 that

$$i\left(T_\lambda, \Omega_{r_1}, K\right) = 0, \qquad i\left(T_\lambda, \Omega_{r_3}, K\right) = 1; \tag{44}$$

thus $i(T_\lambda, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K) = 1$, which implies T_λ has a fixed point in $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$, which is positive ω-periodic solution of (1) for $\lambda > \lambda_0$.

If $\mathbf{F}_0 = \mathbf{F}_\infty = 0$, it is easy to see from the above proof that T_λ has fixed points $\mathbf{x}_1$ in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ and $\mathbf{x}_2$ in $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ such that

$$r_2 < \left\|\mathbf{x}_1\right\| < r_1 < \left\|\mathbf{x}_2\right\| < r_3. \tag{45}$$

Consequently, (1) has two positive ω-periodic solutions for $\lambda > \lambda_0$.

(b) Choose a number $r_1 = 1$; by Lemma 9 we infer that there exists a $\lambda_0 = 1/M b^\mu \widehat{M}_1 > 0$ such that

$$\left\|T_\lambda(\mathbf{x})\right\| < \left\|\mathbf{x}\right\| \quad \text{for } \mathbf{x} \in \partial\Omega_{r_1},\ 0 < \lambda < \lambda_0. \tag{46}$$

If $\mathbf{F}_0 = \infty$, there exists a component f_i such that $f_i^0 = \infty$. Therefore there is a positive number $r_2 < r_1$ such that

$$f_i(\mathbf{x}) \ge \eta \left\|\mathbf{x}\right\|, \quad \text{for } \left\|\mathbf{x}\right\| \le r_2,\ t \in [0, \omega], \tag{47}$$

where the constant $\eta > 0$ satisfies $\lambda \eta m^2 b^l M > 1$. Lemma 5 implies that

$$\left\|T_\lambda \mathbf{x}\right\| \ge \lambda \eta M b^\mu \left\|\mathbf{x}\right\| > \left\|\mathbf{x}\right\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_2},\ t \in [0, \omega]. \tag{48}$$

It follows from Lemma 1 that

$$i\left(T_\lambda, \Omega_{r_1}, K\right) = 1, \qquad i\left(T_\lambda, \Omega_{r_2}, K\right) = 0. \tag{49}$$

Thus $i(T_\lambda, \Omega_{r_1} \setminus \overline{\Omega}_{r_2}, K) = 1$ which implies T_λ has a fixed point in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$, which is positive ω-periodic solution of (1) for $0 < \lambda < \lambda_0$.

If $\mathbf{F}_\infty = \infty$, there exists a component f_i such that $f_i^\infty = \infty$. Therefore there is a positive number $\widehat{H}$ such that

$$f_i(\mathbf{x}) \ge \eta \left\|\mathbf{x}\right\|, \quad \text{for } \left\|\mathbf{x}\right\| \ge \widehat{H}, \tag{50}$$

where the constant $\eta > 0$ satisfies $\lambda \eta m^2 b^l M > 1$. Let $r_3 = \max\{2r_1, M\widehat{H}/m\}$; if $\mathbf{x} \in \partial\Omega_{r_3}$, then

$$\min_{t \in [0,\omega]} x_i(t) \ge \frac{m}{M} \left\|\mathbf{x}\right\| = \frac{m}{M} r_3 \ge \widehat{H}. \tag{51}$$

Hence,

$$f_i(\mathbf{x}) \ge \eta x_i(t). \tag{52}$$

Again, it follows from Lemma 5 that

$$\left\|T_\lambda \mathbf{x}\right\| \ge \lambda \eta M b^\mu \left\|\mathbf{x}\right\| > \left\|\mathbf{x}\right\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_3},\ t \in [0, \omega]. \tag{53}$$

It follows from Lemma 1 that

$$i\left(T_\lambda, \Omega_{r_1}, K\right) = 1, \qquad i\left(T_\lambda, \Omega_{r_3}, K\right) = 0. \tag{54}$$

Thus $i(T_\lambda, \Omega_{r_3} \setminus \overline{\Omega}_{r_1}, K) = -1$, which implies T_λ has a fixed point in $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$, which is positive ω-periodic solution of (1) for $0 < \lambda < \lambda_0$.

If $\mathbf{F}_0 = \mathbf{F}_\infty = \infty$, it is easy to see from the above proof that T_λ has fixed points $\mathbf{x}_1$ in $\Omega_{r_1} \setminus \overline{\Omega}_{r_2}$ and $\mathbf{x}_2$ in $\Omega_{r_3} \setminus \overline{\Omega}_{r_1}$ such that

$$r_2 < \left\|\mathbf{x}_1\right\| < r_1 < \left\|\mathbf{x}_2\right\| < r_3. \tag{55}$$

Consequently, (1) has two positive ω-periodic solutions for $0 < \lambda < \lambda_0$.

(c) If $I_0 = 0$, then $\mathbf{F}_0 > 0$ and $\mathbf{F}_\infty > 0$; there exist two components f_i and f_j such that

$$f_i^0 > 0, \qquad f_i^\infty > 0. \tag{56}$$

It is easy to show (see [11]) that positive numbers η, r_1 exist such that

$$f_i(\mathbf{x}) \ge \eta \left\|\mathbf{x}\right\|, \quad \left\|\mathbf{x}\right\| \le r_1, \tag{57}$$

$$f_j(\mathbf{x}) \ge \eta \left\|\mathbf{x}\right\|, \quad \left\|\mathbf{x}\right\| \ge \frac{m}{M} r_1. \tag{58}$$

Assume $\mathbf{y}(t) = [y_1, y_2, \ldots, y_n]$ is a positive ω-periodic solution of (1), we will show that this leads to a contradiction for $\lambda > \lambda_0 = M/\eta m^2 b^l$. In fact, if $\|\mathbf{y}\| \le r_1$, (57) implies that

$$f_i(\mathbf{y}) \ge \eta y_i(t), \quad \text{for } t \in [0, \omega]. \tag{59}$$

On the other hand, if $\|\mathbf{y}\| > r_1$, then

$$\min_{t \in [0,\omega]} y_i(t) \ge \sigma \left\|\mathbf{y}\right\| \ge \sigma r_1, \tag{60}$$

which together with (58), implies that

$$f_j(\mathbf{y}) \ge \eta y_j(t), \quad \text{for } t \in [0, \omega]. \tag{61}$$

Since $T_\lambda(\mathbf{y}) = \mathbf{y}(t)$, for $t \in [0, \omega]$, it follows from Lemma 5 that, for $\lambda > \lambda_0$,

$$\left\|\mathbf{y}\right\| = \left\|T_\lambda(\mathbf{y})\right\| \ge \frac{\lambda \eta m^2 b^l}{M} \left\|\mathbf{y}\right\| > \left\|\mathbf{y}\right\|, \tag{62}$$

which is a contradiction.

If $I_\infty = 0$, then $\mathbf{F}_0 < \infty$ and $\mathbf{F}_\infty < \infty$; there exist two components f_i and f_j such that $f_i^0 < \infty$ and $f_j^\infty < \infty$. It is easy to show (see [11]) that positive numbers ϵ exist such that

$$f_i(\mathbf{x}) \le \epsilon \left\|\mathbf{x}\right\|. \tag{63}$$

Assume that $\mathbf{y}(t) = [y_1, y_2, \ldots, y_n]$ is a positive ω-periodic solution of (1); we will show that this leads to a contradiction for $0 < \lambda < \lambda_0 = 1/\epsilon M b^{\mu}$. In fact, for $0 < \lambda < \lambda_0$, since $T_\lambda(\mathbf{y}) = \mathbf{y}(t)$, we find

$$\|\mathbf{y}\| = \|T_\lambda(\mathbf{y})\| \leq \lambda \epsilon M b^{\mu} \|\mathbf{y}\| < \|\mathbf{y}\|, \tag{64}$$

which is a contradiction. The proof is complete. □

Theorem 13 is a direct consequence of the proof of Theorem 12(c). Under the conditions of Theorem 13, we are able to give explicit intervals of λ such that (1) has no positive ω-periodic solution.

Theorem 13. *Assume that* (H_1)-(H_2) *hold.*

(a) *If there is a* $c_1 > 0$ *such that* $f_i(\mathbf{x}) \geq c_1\|\mathbf{x}\|$, $\mathbf{x} \in R_+^n$, *then there exists a* $\lambda_0 = M/m^2 c_1 b^l$ *such that, for all* $\lambda > \lambda_0$, (1) *has no positive* ω*-periodic solution.*

(b) *If there is a* $c_2 > 0$ *such that* $f_i(\mathbf{x}) \leq c_2\|\mathbf{x}\|$, $\mathbf{x} \in R_+^n$, *then there exists a* $\lambda_0 = 1/M c_2 b^{\mu}$ *such that, for all* $0 < \lambda < \lambda_0$, (1) *has no positive* ω*-periodic solution.*

Theorem 14. *Assume that* (H_1)-(H_2) *hold and* $I_0 = I_\infty = 0$. *If*

$$\frac{M}{\max\{\mathbf{F}_0, \mathbf{F}_\infty\} m^2 b^l} < \lambda < \frac{1}{\min\{\mathbf{F}_0, \mathbf{F}_\infty\} M b^{\mu}}, \tag{65}$$

then (1) *has a positive* ω*-periodic solution.*

Proof. (a) If $\mathbf{F}_\infty > \mathbf{F}_0$, then there exist two components f_i and f_j such that $f_i^0 < f_j^\infty$. It is easy to see that there exists an $0 < \epsilon < f_j^\infty$ such that

$$\frac{M}{m^2 b^l \left(f_j^\infty - \epsilon\right)} < \lambda < \frac{1}{M b^{\mu} \left(f_i^0 + \epsilon\right)}. \tag{66}$$

Now, turning to f_i^0 and f_j^∞, there is an $r_1 > 0$ such that

$$f_i(\mathbf{x}) \leq \left(f_i^0 + \epsilon\right)\|\mathbf{x}\|, \quad \text{for } 0 < \|\mathbf{x}\| < r_1. \tag{67}$$

Thus

$$f_i(\mathbf{x}) \leq \left(f_i^0 + \epsilon\right)\|\mathbf{x}\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_1},\ t \in [0, \omega]. \tag{68}$$

We have by Lemma 7 that

$$\|T_\lambda \mathbf{x}\| \leq \lambda \left(f_i^0 + \epsilon\right) M b^{\mu} \|\mathbf{x}\| < \|\mathbf{x}\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_1}, \quad t \in [0, \omega]. \tag{69}$$

On the other hand, there is an $\widehat{H} > r_1$ such that

$$f_j(\mathbf{x}) \geq \left(f_j^\infty - \epsilon\right)\|\mathbf{x}\|, \quad \text{for } \|\mathbf{x}\| \geq \widehat{H}. \tag{70}$$

Let $r_2 = \max\{2r_1, m\widehat{H}/M\}$. It follows that

$$x_i(t) \geq \frac{m}{M}\|\mathbf{x}\| \geq \widehat{H}, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_2},\ t \in [0, \omega]. \tag{71}$$

Thus $f_i(\mathbf{x}) \leq (f_j^\infty - \epsilon)\|\mathbf{x}\|$, for $\mathbf{x} \in \partial\Omega_{r_2}$ and $t \in [0, \omega]$. In view of Lemma 7, we have

$$\|T_\lambda \mathbf{x}\| \geq \lambda \left(f_j^\infty - \epsilon\right) \frac{m^2 b^l}{M} \|\mathbf{x}\| > \|\mathbf{x}\|, \quad \text{for } \mathbf{x} \in \partial\Omega_{r_2}, \quad t \in [0, \omega]. \tag{72}$$

It follows from Lemma 1 that

$$i\left(T_\lambda, \Omega_{r_1}, K\right) = 1, \qquad i\left(T_\lambda, \Omega_{r_2}, K\right) = 0. \tag{73}$$

Thus $i(T_\lambda, \Omega_{r_2} \setminus \overline{\Omega}_{r_1}, K) = -1$. Hence, T_λ has a fixed point in $\Omega_{r_2} \setminus \overline{\Omega}_{r_1}$. Consequently, (1) has a positive ω-periodic solution.

(b) If $\mathbf{F}_\infty > \mathbf{F}_0$. The remaining part of the proof is similar to that of Theorem 14(a); therefore it is omitted. The proof is complete. □

4. Remarks

Remark 15. Based on the condition $H2$ of [8], we may obtain the inequality sequence of

$$l a_i(t) \leq A(t) G(\mathbf{x}) \leq L a_i(t), \quad i = 1, 2, \ldots, n. \tag{74}$$

Comparatively, in terms of condition of H_1 presented in this paper, we have the inequality sequence of

$$\alpha_i(t) x_i \leq h_i(t, \mathbf{x}) \leq \beta_i(t) x_i, \quad i = 1, 2, \ldots, n; \tag{75}$$

it is clear that result of this paper can be applied to even more wider domain. Additionally, an extra requirement of

$$0 < l < G(\mathbf{x}) < L, \tag{76}$$

included in $H2$ of [8], is not demanded here.

For example, when letting

$$H(t, \mathbf{x}) = A(t) G(\mathbf{x}) = \begin{bmatrix} a_1(t) & 0 \\ 0 & a_2(t) \end{bmatrix} \begin{bmatrix} (2 + \sin x_1) x_1 \\ (2 + \cos x_2) x_2 \end{bmatrix} \tag{77}$$

and then

$$G(\mathbf{x}) = \begin{bmatrix} (2 + \sin x_1) x_1 \\ (2 + \cos x_2) x_2 \end{bmatrix}, \tag{78}$$

we can obtain the formula of

$$h_1(t, \mathbf{x}) = a_1(t)(2 + \sin x_1) x_1, \quad h_2(t, \mathbf{x}) = a_2(t)(2 + \cos x_2) x_2. \tag{79}$$

According to condition H_1, the formulas would be further achieved as

$$a_1(t) \leq \frac{h_1(t, \mathbf{x})}{x_1} = \frac{a_1(t)(2 + \sin x_1) x_1}{x_1} \leq 3a_1(t), \quad a_2(t) \leq \frac{h_2(t, \mathbf{x})}{x_2} = \frac{a_2(t)(2 + \cos x_2) x_2}{x_2} \leq 3a_2(t). \tag{80}$$

However, there do not exist $l > 0$ such that

$$0 < l < g_1(x_1) = (2 + \sin x_1) x_1. \tag{81}$$

Therefore, function of $H(t, \mathbf{x}) = A(t)G(\mathbf{x})$ cannot satisfy the conditions proposed in [8].

Remark 16. The technique of proof of theorems in this paper differs from that of theorems in [8]. What is more, the condition of Theorems 10 and 11 is obviously weaker here than proposed in [8].

Remark 17. In this paper the application scope of Theorems 12–14 is relatively wider than that of Theorems 1.1–1.3 in [6], respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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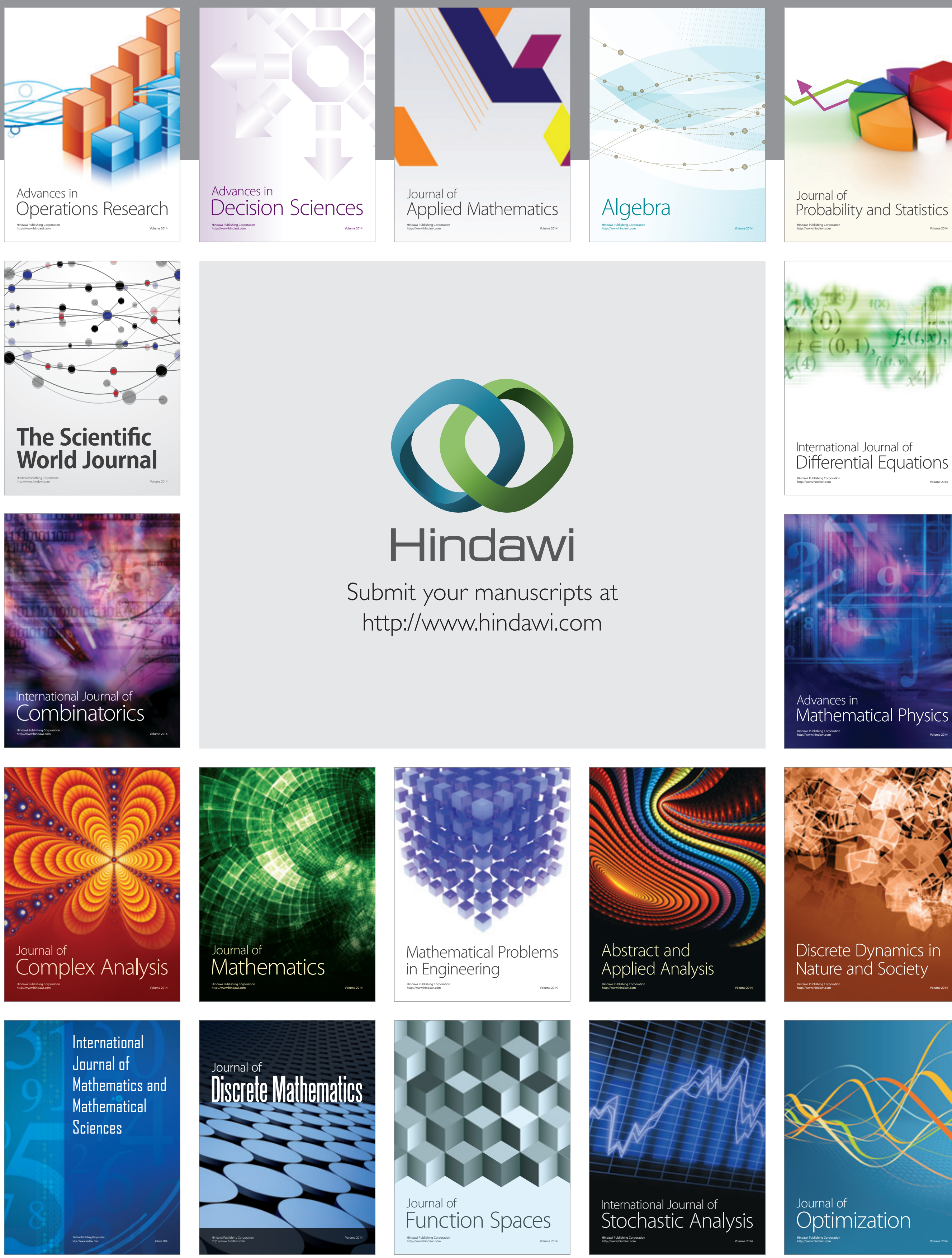

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