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Research Article

Optimal Harvesting for an Age-Spatial-Structured Population Dynamic Model with External Mortality

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We study an optimal harvesting for a nonlinear age-spatial-structured population dynamic model, where the dynamic system contains an external mortality rate depending on the total population size. The total mortality consists of two types: the natural, and external mortality and the external mortality reflects the effects of external environmental causes. We prove the existence and uniqueness of solutions for the population dynamic model. We also derive a sufficient condition for optimal harvesting and some necessary conditions for optimality in an optimal control problem relating to the population dynamic model. The results may be applied to an optimal harvesting for some realistic biological models.

1. Introduction

Optimal control problems for the age-structured systems are of interest for many areas of application, as harvesting, cost control, birth control, and epidemic disease control [1–5]. Many authors studied some optimal harvesting problems for an age-dependent population dynamic system ([2, 6–8] and references therein). One of the aims of such optimal control problems is to find some conditions of optimality for some objective functionals.

Aniţa [2] considered the optimal harvesting problem for the following nonlinear age-dependent population dynamic model. Let $u(a,t)$ be the population density of age a at time t, and let $\mu(a)$ and $v(t)$ be the natural death rate of individuals of age a and the harvesting rate, respectively. The evolution of an age-structured population subject to harvesting is described as a partial differential equation:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(a,t) + \frac{\partial u}{\partial a}(a,t) + \mu(a)u(a,t) + \Phi(P(t))u(a,t) = -v(t)u(a,t), \\
&\qquad\qquad\qquad\qquad \text{in } (0,A)\times(0,T)\ (T>0) \\
&u(0,t) = \int_0^A \beta(a)u(a,t)da, \quad \text{in } (0,T) \\
&P(t) = \int_0^A u(a,t)da, \quad \text{in } (0,T),
\end{aligned} \tag{1.1}$$

where $P(t)$ and $\Phi(P(t))$ stand for the total population and the external mortality rate, respectively, $\beta(a)$ is the natural fertility-rate, and A is the maximal age of the individual.

Now let us make the previous model a generalized model considering the location and the external environmental cause. Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $A>0$, $T>0$. We denote by $u(x,t,a)$ the distribution of individuals of age $a\geq 0$ at time $t\geq 0$ and location x in $\overline{\Omega}$. Let $\beta(x,t,a)\geq 0$ the natural fertility rate and $\mu(x,t,a,u)\geq 0$ be the natural death rate of individuals of age a at time t and location x and density u, where we note that the death rate μ depends on the density u but most of study for models of the age structured population has been done with the death rate dependnt on the time and age only [2, 4]. For a more realistic situation, it is natural to assume that the death rate depends on the density as well as the time and age. Moreover, we set that the total mortality consists of a natural mortality and an external mortality and that the total population has a special weight $w(x,a)$ at age a and location x:

$$P_w(x,t) = \int_0^A w(x,\alpha)u(x,t,\alpha)d\alpha, \tag{1.2}$$

where A is the maximal age of the individual. In this total population P_w, the weight function w gives the effects differently on each age under the external environmental causes: the virus, the climate change, the earthquake, and storm waves. So the external mortality rate $\Phi(P_w(x,t))$ reflects the long-term or short-term effects of external environments such as the virus, the climate change, and the earthquake. We also assume that the flux of population, as emigration, takes the form $k\nabla_x u(x,t,a)$ with $k>0$, where ∇_x is the gradient vector with respect to the location variable x.

Now we consider the following nonlinear age-spatial-structured population dynamics model with external mortality:

$$\begin{aligned}
&\frac{\partial u}{\partial t}(x,t,a) + \frac{\partial u}{\partial a}(x,t,a) - k\Delta_x u(x,t,a) + \mu(x,t,a,u(x,t,a))u(x,t,a) + \Phi(P_w(x,t))u(x,t,a) \\
&\quad = -v(x,t,a)u(x,t,a) \quad \text{in } Q = \Omega\times(0,T)\times(0,A),
\end{aligned}$$

$$\frac{\partial u}{\partial \eta}(x,t,a) = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,T) \times (0,A),$$
$$u(x,0,a) = u_0(x,a) \quad \text{in } \Omega \times (0,A),$$
$$u(x,t,0) = \int_0^A \beta(x,t,a)u(x,t,a)da \quad \text{in } \Omega \times (0,T),$$
$$P_w(x,t) = \int_0^A w(x,\alpha)u(x,t,\alpha)d\alpha \quad \text{in } \Omega \times (0,T), \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $v(x,t,a)$ is a harvesting rate, and w is a positive function in $L^\infty(\Omega \times (0,A))$.

We study an optimal control problem relating to the dynamic system (1.3) as follows:

$$\text{Find } v^* \in \mathcal{U} \quad \text{such that } J\left(u^{v^*}, v^*\right) = \sup_{v \in \mathcal{U}} J(u^v, v), \tag{P}$$

where $J(u^v,v) \equiv \int_Q v(x,t,a)g(x,t,a)u^v(x,t,a)dx\,dt\,da$, g is a given bounded weight function u^v is the solution of the dynamic control system (1.3), and $\mathcal{U}$ is the set of controllers given by

$$\mathcal{U} = \left\{ v \in L^2(Q) : \nu_1(x,t,a) \le v(x,t,a) \le \nu_2(x,t,a) \text{ a.e., } (x,t,a) \in Q \right\} \tag{1.4}$$

for some $\nu_1, \nu_2 \in L^\infty(Q)$, $0 \le \nu_1(x,t,a) \le \nu_2(x,t,a)$, a.e., in Q. This problem (**P**) is called the primal problem. The objective functional $J(u^v,v)$ in (**P**) represents the profit from harvesting, that is, the profit term is the proportion of the species harvested multiplied by the selling price dependent on age a at time t and location x. In a biological system, we may apply the dynamic system (1.3) to the fish, animal, and plant dynamic models.

The purpose of this paper is to prove the existence and compactness of solutions for the dynamic system (1.3) and to investigate an optimal harvesting problem (**P**) for a nonlinear age-spatial-structured population dynamic model with external mortality. The optimal approach introduced in this work may be applicable in the realistic biological models with field data beyond the theoretical model.

The paper is organized as follows. In Section 2, we obtain the existence, uniqueness, and compactness of solutions for the dynamic system (1.3). In Section 3, we derive a sufficiently condition for the optimal control problem (**P**). Finally, a necessary condition for the optimal control problem (**P**) is given in Section 4.

2. Existence, Uniqueness, and Compactness of Solutions

In this work, we assume the following:

($H1$) The fertility rate $\beta \in L^\infty(Q)$, $\beta(x,t,a) \ge 0$ a.e., $(x,t,a) \in Q$.

($H2$) The mortality rate $\mu \in L^\infty(\overline{\Omega} \times [0,T] \times [0,A) \times L^\infty(Q))$ and μ is increasing and Lipschitz continuous with respect to the variable u.

$(H3)$ $\Phi : [0,\infty) \to [0,\infty)$ is bounded and Lipschitz continuous, that is, there exists a constant $L > 0$ such that

$$\left|\Phi(\psi_1) - \Phi(\psi_2)\right| \leq L\left|\psi_1 - \psi_2\right| \tag{2.1}$$

and $\Phi : [0,\infty) \to [0,\infty)$ is continuously differentiable.

$(H4)$ $u_0 \in L^\infty(\Omega \times (0,A)), u_0(x,a) \geq 0$ a.e., $(x,a) \in \Omega \times (0,A)$.

$(H5)$ $g \in L^\infty(Q), g(x,t,a) \geq 0$ a.e., $(x,t,a) \in Q$.

$(H6)$ w is a nonnegative bounded and measurable function in $L^\infty(\Omega \times (0,A))$ with $0 \leq w(x,a) \leq 1$ for all $(x,a) \in \Omega \times (0,A)$.

The existence of a solution u to the dynamic system (1.3) is given by the following lemma (see also [1]). Here we assume that a function $u \in L^2(Q)$ belongs to $C(\overline{S}; L^2(\Omega)) \cap AC(S; L^2(\Omega)) \cap L^2(S; H^1(\Omega)) \cap L^2_{\text{loc}}(S; L^2(\Omega))$, for almost any characteristic line S; $a - t =$ constant, $(t,a) \in (0,T) \times (0,A)$. In addition, we assume that $\operatorname{esssup}|\partial u/\partial a| < \infty$ or $\operatorname{esssup}|\partial u/\partial t| < \infty$, which may be a natural biological condition for population dynamics.

Lemma 2.1. *Let the assumptions* $(H1)$*–*$(H6)$ *hold. For any* $v \in \mathcal{U}$, *the dynamic system* (1.3) *admits a unique and nonnegative solution* u^v *which belongs to* $L^\infty(Q)$.

Proof. We will use the Banach fixed-point theorem for proof. Let $L^p_+(Q) = \{u \in L^p(Q) : u \geq 0$ a.e., in $Q\}$. Denote by ζ the mapping $\zeta : \widetilde{u} \mapsto u^{\widetilde{u},v}$, where $u^{\widetilde{u},v}$ is the solution of

$$\begin{gathered}
\frac{\partial u}{\partial t}(x,t,a) + \frac{\partial u}{\partial a}(x,t,a) - k\Delta_x u(x,t,a) + \mu(x,t,a,u(x,t,a))u(x,t,a) + \Phi\left(\widetilde{P}(x,t)\right)u(x,t,a) \\
= -v(x,t,a)u(x,t,a) \quad \text{in } Q = \Omega \times (0,T) \times (0,A), \\
\frac{\partial u}{\partial \eta}(x,t,a) = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,T) \times (0,A), \\
u(x,0,a) = u_0(x,a), \quad \text{in } \Omega \times (0,A), \\
u(x,t,0) = \int_0^A \beta(x,t,a)u(x,t,a)da \quad \text{in } \Omega \times (0,T), \\
\widetilde{P}_w(x,t) = \int_0^A w(x,\alpha)\widetilde{u}(x,t,\alpha)d\alpha \quad \Omega \times (0,T).
\end{gathered} \tag{2.2}$$

Then, the mapping ζ is well defined form $L_+^2(Q)$ to $L_+^2(Q)$ (see Lemma 2 of [9]). For any $\tilde{u}_1, \tilde{u}_2 \in L^2(Q)$, we denote $\tilde{P}_w^i(x,t) = \int_0^A w(x,\alpha)\tilde{u}_i(x,t,\alpha)d\alpha$, with $(x,t) \in \Omega \times (0,T)$ and $i \in \{1,2\}$. By definition of ζ, we get the following equation:

$$\begin{aligned}
&\int_{Q_t}\bigg[\frac{\partial}{\partial t}(\zeta\tilde{u}_1 - \zeta\tilde{u}_2) + \frac{\partial}{\partial a}(\zeta\tilde{u}_1 - \zeta\tilde{u}_2) - k\Delta_x(\zeta\tilde{u}_1 - \zeta\tilde{u}_2) + \mu(x,s,a,\zeta\tilde{u}_1)\zeta\tilde{u}_1 - \mu(x,s,a,\zeta\tilde{u}_2)\zeta\tilde{u}_2 \\
&\quad + \Phi\left(\tilde{P}_w^1\right)\zeta\tilde{u}_1 - \Phi\left(\tilde{P}_w^2\right)\zeta\tilde{u}_2 + v(\zeta\tilde{u}_1 - \zeta\tilde{u}_2)\bigg](\zeta\tilde{u}_1 - \zeta\tilde{u}_2)dx\,ds\,da \\
&\quad = \int_{Q_t}\bigg[\frac{\partial}{\partial t}(\zeta\tilde{u}_1 - \zeta\tilde{u}_2) + \frac{\partial}{\partial a}(\zeta\tilde{u}_1 - \zeta\tilde{u}_2) - k\Delta_x(\zeta\tilde{u}_1 - \zeta\tilde{u}_2) \\
&\qquad + (\mu(x,s,a,\zeta\tilde{u}_1) - \mu(x,s,a,\zeta\tilde{u}_2))\zeta\tilde{u}_1 + \mu(x,s,a,\zeta\tilde{u}_2)(\zeta\tilde{u}_1 - \zeta\tilde{u}_2) \\
&\qquad + \left(\Phi\left(\tilde{P}_w^1\right) - \Phi\left(\tilde{P}_w^2\right)\right)\zeta\tilde{u}_1 + \Phi\left(\tilde{P}_w^2\right)(\zeta\tilde{u}_1 - \zeta\tilde{u}_2) + v(\zeta\tilde{u}_1 - \zeta\tilde{u}_2)\bigg](\zeta\tilde{u}_1 - \zeta\tilde{u}_2)dx\,ds\,da \\
&\quad = 0,
\end{aligned} \tag{2.3}$$

where $Q_t = \Omega \times (0,t) \times (0,A), t \in (0,T)$. Using the conditions $(H2)$ and $(H3)$, we get after some calculations that

$$\|(\zeta\tilde{u}_1 - \zeta\tilde{u}_2)(t)\|^2_{L^2(\Omega\times(0,A))} \le C\int_0^t \|(\tilde{u}_1 - \tilde{u}_2)(s)\|^2_{L^2(\Omega\times(0,A))}ds, \tag{2.4}$$

where C is a positive constant. For sufficiently small t, we get the existence of a unique fixed point for ζ. Since the solution u^v satisfies

$$0 \le u^v(x,t,a) \le \overline{u}(x,t,a) \quad \text{a.e., in } Q \tag{2.5}$$

and $\overline{u} \in L_+^\infty(Q)$ is the solution of the dynamic system (1.3) corresponding to $\mu = 0, \Phi = 0$, we complete the proof. □

For $v \in \mathcal{U}$, denote

$$P_w^v(x,t) = \int_0^A w(x,\alpha)u^v(x,t,a)\,d\alpha \quad \text{in } \Omega \times (0,T). \tag{2.6}$$

Lemma 2.2. *The set* $\{P_w^v; v \in \mathcal{U}\}$ *is relatively compact in* $L^2(\Omega \times (0,T))$.

Proof. For any $\varepsilon > 0$ small enough, we get that

$$P_w^{v,\varepsilon}(x,t) = \int_0^{A-\varepsilon} w(x,\alpha)u^v(x,t,\alpha)\,d\alpha \quad \text{in } \Omega \times (0,T) \tag{2.7}$$

is a solution of

$$\begin{gathered}\frac{\partial P_w^{v,\varepsilon}}{\partial t} - k\Delta_x P_w^{v,\varepsilon} \\ = -\int_0^{A-\varepsilon} w(x,\alpha)\partial_\alpha u^v(x,t,\alpha)d\alpha - \int_0^{A-\varepsilon} \mu(x,t,\alpha,u^v(x,t,\alpha))w(x,\alpha)u^v(x,t,\alpha)d\alpha \\ -\int_0^{A-\varepsilon} \Phi(P_w^v(x,t,\alpha))w(x,\alpha)u^v(x,t,\alpha)d\alpha - \int_0^{A-\varepsilon} v(x,t,\alpha)w(x,\alpha)u^v(x,t,\alpha)d\alpha, \\ \frac{\partial P_w^{v,\varepsilon}}{\partial \eta} = 0 \quad \text{a.e., } \partial\Omega \times (0,T), \\ P_w^{v,\varepsilon}(x,0) = \int_0^{A-\varepsilon} w(x,\alpha)u_0(x,\alpha)d\alpha \quad \text{in } \Omega.\end{gathered} \tag{2.8}$$

Using the condition $(H6)$, we obtain

$$\begin{aligned}&\frac{\partial P_w^{v,\varepsilon}}{\partial t} - k\Delta_x P_w^{v,\varepsilon} \\ &\quad = -\int_0^{A-\varepsilon} w(x,\alpha)\frac{\partial u^v}{\partial \alpha}(x,t,\alpha)d\alpha - \int_0^{A-\varepsilon} \mu(x,t,\alpha,u^v(x,t,\alpha))w(x,\alpha)u^v(x,t,\alpha)d\alpha \\ &\qquad -\int_0^{A-\varepsilon} \Phi(P_w^v(x,t,\alpha))w(x,\alpha)u^v(x,t,\alpha)d\alpha - \int_0^{A-\varepsilon} v(x,t,\alpha)w(x,\alpha)u^v(x,t,\alpha)d\alpha \\ &\quad \le \left|\int_0^{A-\varepsilon} w(x,\alpha)\frac{\partial u^v}{\partial \alpha}(x,t,\alpha)d\alpha\right| + \left|\int_0^{A-\varepsilon} \mu(x,t,\alpha,u^v(x,t,\alpha))u^v(x,t,\alpha)d\alpha\right| \\ &\qquad + \left|\int_0^{A-\varepsilon} \Phi(P_w^v(x,t,\alpha))w(x,\alpha)u^v(x,t,\alpha)d\alpha\right| + \left|\int_0^{A-\varepsilon} v(x,t,\alpha)w(x,\alpha)u^v(x,t,\alpha)d\alpha\right| \\ &\quad \le \sup_{\alpha\in[0,A-\varepsilon]}\left|\frac{\partial u^v}{\partial \alpha}\right|(A-\varepsilon) + \left|\int_0^{A-\varepsilon} \mu(x,t,\alpha,u^v(x,t,\alpha))u^v(x,t,\alpha)d\alpha\right| \\ &\qquad + \left|\int_0^{A-\varepsilon} \Phi(P_w^v(x,t,\alpha))u^v(x,t,\alpha)d\alpha\right| + \left|\int_0^{A-\varepsilon} v(x,t,\alpha)u^v(x,t,\alpha)d\alpha\right| \le C,\end{aligned} \tag{2.9}$$

where we have used the fact that $\{vu^v\}$ and $\{\mu(\cdot,\cdot,\cdot,u^v)u^v\}$ are bounded in $L^\infty(\Omega \times (0,T) \times (0,A-\varepsilon))$, $\{\Phi(P_w^v)u^v\}$ is bounded in $L^\infty(\Omega\times(0,T)\times(0,A-\varepsilon))$ and $\{u^v(\cdot,\cdot,A-\varepsilon)\}$ is bounded in $L^\infty(\Omega \times (0,T))$.

Therefore, $\{(\partial P_w^{v,\varepsilon}/\partial t)-k\Delta_x P_w^{v,\varepsilon}\}$ is bounded in $L^\infty(\Omega\times(0,T))$. By Aubin's compactness theorem that for any $\varepsilon > 0$, the set $\{P_w^{v,\varepsilon} : v \in \mathcal{U}\}$ is relatively compact in $L^2(\Omega \times (0,T))$. On the other hand, we get also

$$\left|P_w^{v,\varepsilon}(x,t) - P_w^v(x,t)\right| \le \int_{A-\varepsilon}^{A} w(x,\alpha)u^v(x,t,\alpha)d\alpha \le \varepsilon\|\overline{u}\|_{L^\infty(Q)}. \tag{2.10}$$

Combining these two results, we conclude the relative compactness of $\{P_w^v : v \in \mathcal{U}\}$ in $L^2(Q)$. □

3. Existence of the Optimal Solution

Now, we show the existence of the optimal solution for the primal problem (**P**).

Theorem 3.1. *Let the assumptions* $(H1)$*–*$(H6)$ *hold. Then, the primal problem* (**P**) *has at least one optimal pair.*

Proof. Let $d = \sup_{v\in\mathcal{U}} J(u^v, v)$. Then, we have

$$0 \leq J(u^v, v) \leq \int_Q \nu_2(x,t,a) g(x,t,a) \widehat{u}(x,t,a) dx\, dt\, da, \tag{3.1}$$

where $\widehat{u} \in L^\infty(Q)$ is the solution of the dynamic system (1.3) corresponding to $\mu = 0$ and $\Phi = 0$. Now let $\{v_n\}_{n\in N^*} \subset \mathcal{U}$ be a sequence such that

$$d - \frac{1}{n} < J(u^{v_n}, v_n) \leq d. \tag{3.2}$$

Since $0 \leq u^{v_n}(x,t,a) \leq \widehat{u}(x,t,a)$ a.e., in Q, we conclude that there exists a subsequence such that

$$u^{v_n} \longrightarrow u^* \quad \text{weakly in } L^2(Q). \tag{3.3}$$

For a strong convergence to u^*, we consider the sequence $\{\overline{u}_n\}_{n\in N^*}$ such that

$$\overline{u}_n(x,t,a) = \sum_{i=n+1}^{k_n} \lambda_i^n u^{v_i}(x,t,a), \quad \lambda_i^n \geq 0, \ \sum_{i=n+1}^{k_n} \lambda_i^n = 1, \tag{3.4}$$

where $k_n > 0$ is an increasing sequence of integer numbers.

Let the totality $T_1 = \{u \mid u = \sum_{i=n+1}^{k_n} \lambda_i^n u^{v_i},\ \lambda_i^n \geq 0, \sum_{i=n+1}^{k_n} \lambda_i^n = 1\}$, and we assume that $0 \in T_1$. For any $\varepsilon > 0$, suppose that $\|u^* - \xi\|_{L^2(Q)} > \varepsilon > 0$ for every $\xi \in T_1$. Then, the set $T = \{y \in L^2(Q); \|y - \xi\|_{L^2(Q)} \leq \varepsilon/2 \ \text{ for some } \xi \in T_1\}$ is a convex neighborhood of 0 of $L^2(Q)$ and $\|u^* - y\|_{L^2(Q)} > \varepsilon/2$ for all $y \in T$.

Let $p(y)$ be the Minkowski functional of T. Note here that if we choose $u^* = \delta^{-1}u_0$ with $p(u_0) = 1$ and $0 < \delta < 1$, then we get $p(u^*) = p(\delta^{-1}u_0) = \delta^{-1}p(u_0) = \delta^{-1} > 1$.

Consider a real linear subspace $X_1 = \{\xi \in L^2(Q); \xi = \gamma u_0, -\infty < \gamma < \infty\}$ and put $f_1(\xi) = \gamma$ for $\xi = \gamma u_0 \in X_1$. This real linear functional f_1 on X_1 satisfies $f_1(\xi) \leq p(\xi)$ on X_1. Thus, by the Hahn-Banach extension theorem, there exists a real linear extension f of f_1 defined on the real linear space $L^2(Q)$ such that $f(\xi) \leq p(\xi)$ on $L^2(Q)$. T is a neighborhood of 0, the Minkowski functional $p(\xi)$ is continuous in ξ. Hence, f is a continuous real linear functional defined on the real linear normed space $L^2(Q)$. Moreover, we have

$$\sup_{\xi\in T_1} f(\xi) \le \sup_{\xi\in T} f(\xi) \le \sup_{\xi\in T} p(\xi) = 1 < \delta^{-1} = f\left(\delta^{-1}u_0\right) = f(u^*). \tag{3.5}$$

This is contradiction to $u^* = w - \lim_{n\to\infty} u^{v_n}$. Therefore, $\overline{u}_n$ converges strongly to u^* in $L^2(Q)$. Consider now the sequence of controls:

$$\overline{v}_n(x,t,a) = \begin{cases} \dfrac{\sum_{i=n+1}^{k_n} \lambda_i^n v_i(x,t,a) u^{v_i}(x,t,a)}{\sum_{i=n+1}^{k_n} \lambda_i^n u^{v_i}(x,t,a)} & \text{if } \displaystyle\sum_{i=n+1}^{k_n} \lambda_i^n u^{v_i}(x,t,a) \neq 0, \\ \nu_1(x,t,a) & \text{if } \displaystyle\sum_{i=n+1}^{k_n} \lambda_i^n u^{v_i}(x,t,a) = 0. \end{cases} \tag{3.6}$$

This control $\overline{v}_n$ is an element of the set $\mathcal{U}$. So we can take a subsequence, also denoted by $\{\overline{v}_n\}_{n\in N^*}$ such that

$$\overline{v}_n \longrightarrow v^* \quad \text{weakly in } L^2(Q). \tag{3.7}$$

By Lemma 2.2, we obtain

$$P_w^{v_n} \longrightarrow P_w^* \quad \text{in } L^2(\Omega\times(0,T)) \tag{3.8}$$

and since $u^{v_n} \to u^*$ weakly in $L^2(Q)$, we get

$$P_w^*(x,t) = \int_0^A w(x,\alpha) u^*(x,t,\alpha)\, d\alpha. \tag{3.9}$$

Obviously, $\overline{u}_n$ is a solution of

$$\begin{aligned} &\frac{\partial u}{\partial t}(x,t,a) + \frac{\partial u}{\partial a}(x,t,a) - k\Delta_x u(x,t,a) + \mu(x,t,a,u(x,t,a))u(x,t,a) \\ &\quad + \sum_{i=n+1}^{k_n} \lambda_i^n \Phi\left(P_w^{v_i}(x,t)\right) u^{v_i}(x,t,a) = -\overline{v}_n(x,t,a)u(x,t,a), \quad \text{in } Q, \end{aligned} \tag{3.10}$$

$$\begin{gathered} \frac{\partial u}{\partial \eta}(x,t,a) = 0 \quad \text{on } \Sigma, \\ u(x,t,0) = \int_0^A \beta(x,t,a)u(x,t,a)\,da \quad \text{in } \Omega\times(0,T), \\ u(x,0,a) = u_0(x,a) \quad \text{in } \Omega\times(0,A), \\ P_w^{v_i}(x,t) = \int_0^A w(x,\alpha)u^{v_i}(x,t,\alpha)\,d\alpha \quad \text{in } \Omega\times(0,T). \end{gathered} \tag{3.11}$$

By conditions of Φ and λ, we get

$$\begin{aligned}
&\left\| \sum_{i=n+1}^{k_n} \lambda_i^n \Phi\left(P_w^{v_i}\right) u^{v_i} - \Phi(P_w^*) u^* \right\|_{L^2(Q)} \\
&\quad = \left\| \sum_{i=n+1}^{k_n} \lambda_i^n \Phi(P_w^{v_i}) u^{v_i} - \sum_{n+1}^{k_n} \lambda_i^n \Phi(P_w^*) u^* \right\|_{L^2(Q)} \\
&\quad \leq \left\| \sum_{i=n+1}^{k_n} \lambda_i^n \Phi(P_w^{v_i}) (u^{v_i} - u^*) \right\|_{L^2(Q)} + \left\| \sum_{i=n+1}^{k_n} \lambda_i^n u^* \left(\Phi(P_w^{v_i}) - \Phi(P_w^*)\right) \right\|_{L^2(Q)} \\
&\quad \leq M \left\| \sum_{i=n+1}^{k_n} \lambda_i^n u^{v_i} - u^* \right\|_{L^2(Q)} + \|u^*\|_{L^2(Q)} \sum_{i=n+1}^{k_n} \lambda_i^n \left\| \Phi(P_w^{v_i}) - \Phi(P_w^*) \right\|_{L^2(Q)} \\
&\quad \leq M \left\| \sum_{i=n+1}^{k_n} \lambda_i^n u^{v_i} - u^* \right\|_{L^2(Q)} + \|u^*\|_{L^2(Q)} \sum_{i=n+1}^{k_n} \lambda_i^n L \left\| P_w^{v_i} - P_w^* \right\|_{L^2(Q)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{3.12}$$

where $M = \sup \|\Phi(\cdot)\|_{L^2(Q)}$ and L is the Lipschitz constant. Therefore, we have

$$\sum_{i=n+1}^{k_n} \lambda_i^n \Phi\left(P_w^{v_i}\right) u^{v_i} \longrightarrow \Phi(P_w^*) u^* \quad \text{in } L^2(Q). \tag{3.13}$$

By (H_3) and $\beta \in L^\infty(Q)$, we obtain

$$\int_0^A \beta(x,t,a) \overline{u}(x,t,a) da \longrightarrow \int_0^A \beta(x,t,a) u^*(x,t,a) da. \tag{3.14}$$

Since $\overline{u}_n \rightarrow u^*$ in $L^2(Q)$, we have

$$\sum_{i=n+1}^{k_n} \lambda_i^n \Phi\left(P_w^{v_i}(x,t)\right) u^{v_i}(x,t,a) \longrightarrow \Phi(P_w^*(x,t)) u^*(x,t,a). \tag{3.15}$$

Passing to the limit in (3.10), we obtain that u^* is the solution of the dynamic system (1.3) corresponding to v^*. Therefore, we have

$$\sum_{i=n+1}^{k_n} \lambda_i^n \int_Q v_i(x,t,a) g(x,t,a) u^{v_i}(x,t,a) dx\, dt\, da \longrightarrow J(u^*, v^*) = d \quad \text{as } n \longrightarrow \infty. \tag{3.16}$$

□

4. Necessary Conditions for Optimality

In this section, we study a necessary condition of optimality for the primal problem (**P**).

Theorem 4.1. *Let the assumptions $(H1)$–$(H6)$ hold. Suppose that (u^*, v^*) is an optimal pair for the primal problem* $(\mathbf{P})$. *If p is the solution of*

$$\begin{aligned}
&-\frac{\partial p}{\partial t}(x,t,a) - \frac{\partial p}{\partial a}(x,t,a) - k\Delta_x p(x,t,a) + \mu_u(x,t,a,u^*(x,t,a))u^*(x,t,a)p(x,t,a) \\
&\quad + \mu(x,t,a,u^*(x,t,a))p(x,t,a) + \Phi(P_w^*(x,t))p(x,t,a) \\
&\quad + w(x,a)\int_0^A \Phi_P(P_w^*(x,t))u^*(x,t,\alpha)p(x,t,\alpha)d\alpha - \beta(x,t,a)p(x,t,0) \\
&= -v^*(x,t,a)g(x,t,a) - v^*(x,t,a)p(x,t,a) \quad \text{in } Q,
\end{aligned} \tag{4.1}$$

$$\frac{\partial p}{\partial \eta}(x,t,a) = 0 \quad \text{on } \Sigma, \tag{4.2}$$

$$p(x,T,a) = p(T) = 0 \quad \text{in } \Omega \times (0,A), \tag{4.3}$$

$$p(x,t,A) = p(A) = 0 \quad \text{in } \Omega \times (0,T), \tag{4.4}$$

$$P_w^*(x,t) = \int_0^A w(x,\alpha)u^*(x,t,\alpha)d\alpha \quad \text{in } \Omega \times (0,T), \tag{4.5}$$

then we have

$$v^*(x,t,a) = \begin{cases} \nu_1(x,t,a) & \text{if } (g+p)(x,t,a) < 0 \text{ and } u^* \neq 0, \\ \forall \nu(x,t,a) \in [\nu_1(x,t,a), \nu_2(x,t,a)] & \text{if } (g+p)(x,t,a) = 0 \text{ or } u^* = 0, \\ \nu_2(x,t,a) & \text{if } (g+p)(x,t,a) > 0 \text{ and } u^* \neq 0. \end{cases} \tag{4.6}$$

Here, ν_1, ν_2 are the given functions in the control set $\mathcal{U}$ which is introduced in the introduction, μ_u is the derivative of μ with respect to u, and Φ_P is the derivative of Φ with respect to $P = P_w^*$.

Proof. Since (u^*, v^*) is an optimal pair for the primal problem $(\mathbf{P})$ we get

$$\begin{aligned}
&\int_Q v^*(x,t,a)g(x,t,a)u^*(x,t,a)\,dx\,dt\,da \\
&\quad \geq \int_Q (v^*(x,t,a) + \lambda v(x,t,a))g(x,t,a)u^{v^*+\lambda v}(x,t,a)\,dx\,dt\,da
\end{aligned} \tag{4.7}$$

for all $\lambda > 0$ and for all $v \in L^\infty(Q)$ such that

$$\begin{aligned}
&v(x,t,a) \leq 0 \quad \text{if } v^*(x,t,a) = \nu_2(x,t,a), \\
&v(x,t,a) \geq 0 \quad \text{if } v^*(x,t,a) = \nu_1(x,t,a).
\end{aligned} \tag{4.8}$$

Let $v^\lambda = v^* + \lambda v$ and u^λ be the solution of the dynamic system (1.3) corresponding to v^λ. Then the above equality implies

$$\int_Q v^*(x,t,a)g(x,t,a)\frac{u^\lambda(x,t,a)-u^*(x,t,a)}{\lambda}\,dx\,dt\,da + \int_Q v(x,t,a)g(x,t,a)u^\lambda(x,t,a)\,dx\,dt\,da \le 0. \tag{4.9}$$

Let $z(x,t,a) = \lim_{\lambda\to 0}((u^\lambda(x,t,a)-u^*(x,t,a))/\lambda)$ be the solution to

$$\begin{aligned}
&\frac{\partial z}{\partial t}(x,t,a) + \frac{\partial z}{\partial a}(x,t,a) - k\Delta_x z(x,t,a) + \mu_u(x,t,a,u^*(x,t,a))u^*(x,t,a)z(x,t,a)\\
&\quad + \mu(x,t,a,u^*(x,t,a))z(x,t,a) + \Phi_P(P_w^*(x,t))u^*(x,t,a)\left(\int_0^A w(x,\alpha)z(x,t,\alpha)\,d\alpha\right)\\
&\quad + \Phi(P_w^*(x,t))z(x,t,a)\\
&= -v^*(x,t,a)z(x,t,a) - v(x,t,a)u^*(x,t,a) \quad \text{in } Q,\\
&\frac{\partial z}{\partial \eta} = 0 \quad \text{in } \Sigma,\\
&z(x,t,0) = \int_0^A \beta(x,t,a)z(x,t,a)da, \quad \text{in } \Omega\times(0,T),\\
&z(x,0,a) = 0 \quad \text{in } \Omega\times(0,A),\\
&P_w^*(x,t) = \int_0^A w(x,\alpha)u^*(x,t,\alpha)d\alpha \quad \text{in } \Omega\times(0,T).
\end{aligned} \tag{4.10}$$

Since $u^\lambda(x,t,a) \to u^*(x,t,a)$ in $L^\infty(0,T;L^2(\Omega)\times(0,A))$ as $\lambda \to 0$, after some simple calculations and passing to the limit $\lambda \to 0$ in (4.9) then we obtain

$$\int_Q v^*(x,t,a)g(x,t,a)z(x,t,a)\,dx\,dt\,da + \int_Q v(x,t,a)g(x,t,a)u^*(x,t,a)\,dx\,dt\,da \le 0 \tag{4.11}$$

for all $\lambda > 0$ and for all $v \in L^\infty(Q)$ such that

$$\begin{aligned}
v(x,t,a) &\le 0 \quad \text{if } v^*(x,t,a) = \nu_2(x,t,a)\\
v(x,t,a) &\ge 0 \quad \text{if } v^*(x,t,a) = \nu_1(x,t,a).
\end{aligned} \tag{4.12}$$

Multiplying (4.1) by z and integrating over Q, we get

$$\int_Q v(x,t,a)u^*(x,t,a)(g+p)(x,t,a)\,dx\,dt\,da \le 0 \tag{4.13}$$

for all $\lambda > 0$ and for all $v \in L^\infty(Q)$ such that

$$\begin{aligned} v(x,t,a) \le 0 \quad &\text{if } v^*(x,t,a) = \nu_2(x,t,a) \\ v(x,t,a) \ge 0 \quad &\text{if } v^*(x,t,a) = \nu_1(x,t,a). \end{aligned} \tag{4.14}$$

By inequality (4.13), we get $u^*(g+p) \in N_{\mathcal{U}}(v^*)$, where $N_{\mathcal{U}}(v^*)$ is the normal cone at $\mathcal{U}$ in v^*. Therefore, if $u^*(x,t,a) \neq 0$, then

$$v^*(x,t,a) = \begin{cases} \nu_1(x,t,a) & \text{if } (g+p)(x,t,a) < 0, \\ \forall \nu(x,t,a) \in [\nu_1(x,t,a), \nu_2(x,t,a)] & \text{if } (g+p)(x,t,a) = 0, \\ \nu_2(x,t,a) & \text{if } (g+p)(x,t,a) > 0, \end{cases} \tag{4.15}$$

and if $u^*(x,t,a) = 0$, then v^* is any arbitrary value belonging to the interval $[\nu_1(x,t,a), \nu_2(x,t,a)]$. This completes the proof of Theorem 4.1. □

From now on, let $T > 0$, $A > 0$ and Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. We consider an optimal control problem: find

$$\text{seek } v^* \in \mathcal{U} \quad \text{such that } J(u^*, v^*) = \max_{v \in \mathcal{U}} J(u,v) \tag{P1}$$

subject to

$$\begin{gathered} \frac{\partial u}{\partial t}(x,t,a) + \frac{\partial u}{\partial a}(x,t,a) - k\Delta_x u(x,t,a) + \mu(x,t,a,u(x,t,a))u(x,t,a) \\ + \left(\int_0^A w(x,\alpha)u(x,t,\alpha)d\alpha\right)u(x,t,a) = -v(x,t,a)u(x,t,a) \\ \text{in } Q = \Omega \times (0,T) \times (0,A), \\ \frac{\partial u}{\partial \eta}(x,t,a) = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,T) \times (0,A), \\ u(x,0,a) = u_0(x,a) \quad \text{in } \Omega \times (0,A), \\ u(x,t,0) = \int_0^A \beta(x,t,a)u(x,t,a)da \quad \text{in } \Omega \times (0,T), \end{gathered} \tag{4.16}$$

where $J(u,v) = \int_Q v(x,t,a)g(x,t,a)u(x,t,a)dx\,dt\,da$.

We note that this is a special case of the optimal control problem by the dynamic system (1.3) ($\Phi = I$). Then, we obtain the necessary condition for optimality.

Theorem 4.2. *Let the assumptions* $(H1), (H2), (H4), (H5)$, *and* $(H6)$ *hold. Suppose that* (u^*, v^*) *is an optimal pair for the problem* (**P1**). *If* p *is a solution of the adjoint system* (AE):

$$\begin{gathered}
\begin{aligned}
&-\frac{\partial p}{\partial t}(x,t,a) - \frac{\partial p}{\partial a}(x,t,a) - k\Delta_x p(x,t,a) + \mu_u(x,t,a,u^*(x,t,a))u^*(x,t,a)p(x,t,a) \\
&\quad + \mu(x,t,a,u^*(x,t,a))p(x,t,a) + \left(\int_0^A w(x,\alpha)d\alpha\right)p(x,t,a) \\
&\quad + w(x,a)\int_0^A u^*(x,t,\alpha)p(x,t,\alpha)d\alpha - \beta(x,t,a)p(x,t,0) \\
&= -v^*(x,t,a)g(x,t,a) - v^*(x,t,a)p(x,t,a) \quad \text{in } Q,
\end{aligned} \\
\frac{\partial p}{\partial \eta}(x,t,a) = 0 \quad \text{on } \Sigma, \\
p(x,T,a) = p(T) = 0 \quad \text{in } \Omega \times (0,A), \\
p(x,t,A) = p(A) = 0 \quad \text{in } \Omega \times (0,T),
\end{gathered} \tag{4.17}$$

then we have

$$v^*(x,t,a) = \begin{cases} \nu_1(x,t,a) & \text{if } (\mathrm{g}+\mathrm{p})(\mathrm{x},\mathrm{t},\mathrm{a}) < 0 \text{ and } \mathrm{u}^* \neq 0, \\ \forall \nu(x,t,a) \in [\nu_1(x,t,a), \nu_2(x,t,a)] & \text{if } (\mathrm{g}+\mathrm{p})(\mathrm{x},\mathrm{t},\mathrm{a}) = 0 \text{ or } \mathrm{u}^* = 0, \\ \nu_2(x,t,a) & \text{if } (\mathrm{g}+\mathrm{p})(\mathrm{x},\mathrm{t},\mathrm{a}) > 0 \text{ and } \mathrm{u}^* \neq 0. \end{cases} \tag{4.18}$$

Remark 4.3. We may consider natural death rates

$$\mu(u) = \lambda e^{\lambda u}, \quad u \geq 0,\ \lambda > 0; \qquad \mu(u) = c_0 + c_1\sqrt{u}, \quad u \geq 0,\ c_i \geq 0\ (i = 1,2) \tag{4.19}$$

as examples which satisfy our hypotheses. Here μ is the increasing function of u. It is natural to assume that the mortality rate depends on density of individuals u as well as the total population. Also, we can consider the weight function which gives an effect on age as follows:

$$w(\alpha) = \frac{1}{\sqrt{2\pi}}e^{-\alpha^2/2}, \quad 0 \leq \alpha \leq A. \tag{4.20}$$

Remark 4.4. Let a functional K and the set of controllers $\mathcal{V}$ be defined by

$$\begin{gathered}
K\big(p(x,t,a), u(x,t,a)\big) = \sup_{v \in \mathcal{U}}\big\{v(x,t,a)g(x,t,a)u(x,t,a) + p(x,t,a)v(x,t,a)u(x,t,a)\big\}, \\
\mathcal{V} = \left\{p \in W^{1,\infty}(Q) \mid p \text{ is a positive solution of (AE) in Theorem 4.2}\right\},
\end{gathered} \tag{4.21}$$

respectively.

We introduce another control problem (**D**) corresponding to the primal problem (**P**), which is called the dual problem:

$$\inf_{p\in\mathcal{U}}\int_Q K\big(p(x,t,a),u(x,t,a)\big)+p(x,t,a) \times\left(\frac{\partial u(x,t,a)}{\partial t}+\frac{\partial u(x,t,a)}{\partial a}-k\Delta_x u(x,t,a)+\mu(x,t,a,u(x,t,a)) \times u(x,t,a)+\Phi(P_w(x,t))u(x,t,a)\right)dx\,dt\,da \tag{D}$$

subject to the adjoint system (4.17).

Then, we can establish a duality theorem saying that the primal problem (**P**) is equal to the dual problem (**D**), which is the result in [10].

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