Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 296796, 7 pages
doi:10.1155/2011/296796

Letter to the Editor

Analyzing "Homotopy Perturbation Method for Solving Fourth-Order Boundary Value Problem"

Afgan Aslanov

Department of Mathematics and Computing, Beykent University, Istanbul, Turkey

Correspondence should be addressed to Afgan Aslanov, afganaslanov@yahoo.com

Received 29 March 2011; Accepted 8 April 2011

We analyze a previous paper by S. T. Mohyud-Din and M. A. Noor (2007) and show the mistakes in it. Then, we demonstrate a more efficient method for solving fourth-order boundary value problems.

1. Problem

Let us consider the fifth-order boundary problem of the type

$$u^{(4)}(x) = f\left(u, u', u'', u'''\right) + g(x) \tag{1.1}$$

with the boundary conditions

$$u(0) = \alpha_1, \qquad u'(0) = \alpha_2, \qquad u(1) = \beta_1, \qquad u'(1) = \beta_2, \tag{1.2}$$

where f and g are continuous functions and α_1, α_2, β_1, and β_2 are real constants.

The homotopy perturbation method (HPM) is employed in [1] for solving such problems. The purpose of this paper is to point out the mistakes in paper [1] and demonstrate more efficient method for solving the problems of type (1.1)-(1.2).

First of all, we show the mistakes.

(1) In Example 3.2 the approximate solution

$$u_{\text{approx}}(x) = 512 + 480x + 224.00000000226055x^2 + \ldots \tag{1.3}$$

of the problem

$$u^{(4)}(x) = u(x) + u''(x) + e^{x}(x-3),$$
$$u(0) = 1, \qquad u'(0) = 0, \qquad u(1) = 0, \qquad u'(1) = -e \tag{1.4}$$

(see formula (3.19) in [1]) has no relationship with the exact solution $u_{\text{exact}}(x) = (1-x)e^{x}$. And "error" is at least $u_{\text{approx}}(0) - u_{\text{exact}}(0) = 512 - e = 509.28$.

It is strange that the authors demonstrated the reliability of the errors in the table (see Table 3.2 in [1]).

(2) In Example 3.3 the approximate solution

$$u_{\text{approx}}(x) = 11.3472 - 262.827x - 3.40768x^{3} + \dots \tag{1.5}$$

of the problem

$$u^{(4)} = \sin x + \sin^{2}x - \left(u''(x)\right)^{2},$$
$$u(0) = 0, \qquad u'(0) = 1, \qquad u(1) = \sin 1, \qquad u'(1) = \cos 1 \tag{1.6}$$

(see formula (3.28) in [1]) has no relationship with the exact solution $u = \sin x$, since $u_{\text{approx}}(0) = 11.3472$, $u'_{\text{approx}}(0) = -262.827$.

2. Homotopy Perturbation Method

The basic ideas of the standard HPM were given by He [2, 3], and a new interpretation of HPM was given by He [4]. We introduce a new reliable procedure for choosing the initial approximation in HPM. To do so, we consider the following general nonlinear differential equation

$$Lu + Nu = f(u,x) \tag{2.1}$$

with some initial boundary conditions, where L and N are, respectively, the linear and nonlinear operators.

According to HPM, we construct a homotopy which satisfies the following relations:

$$H(u,p) = Lu - Lv_{0} + pLv_{0} + p\left[Nu - f(u,x)\right] = 0, \tag{2.2}$$

where $p \in [0,1]$ is an embedding parameter and v_{0} is an initial approximation. When we put $p = 0$ and $p = 1$ in (2.2), we obtain

$$H(u,0) = Lu - Lv_{0}, \qquad H(u,1) = Lu + Nu - f(u,x), \tag{2.3}$$

respectively. In topology, this is called deformation and $Lu - Lv_{0}$ and $Lu + Nu - f(u,x)$ are called homotopics.

The solution of (2.2) is expressed as

$$u(x) = u_0(x) + pu_1(x) + p^2u_2(x) + \dots. \tag{2.4}$$

Hence, the approximate solution of (1.5) can be expressed as

$$u(x,t) = u_0(x) + u_1(x) + u_2(x) + \dots. \tag{2.5}$$

Mohyud-Din and Noor [1] tried to rewrite the problem as a system of integral equations and then HPM applied for each equation. In the use of HPM, what we are mainly concerned about are the auxiliary operator L and the initial guess v_0. We take $L = (d/dx^4)(\cdot)$, and

$$v_0 = \alpha_1 + \alpha_2 x + Ax^3 + Bx^4 + L^{-1}\left(g(x)\right), \tag{2.6}$$

where A and B are yet to be determined. Then using the boundary conditions $u(1) = \beta_1$, $u'(1) = \beta_2$ we determine A and B.

3. Applications

Here we apply the HPM to solve correctly the problems in [1].

Example 3.1 (see [1, Example 3.2]). We have

$$u^{(4)}(x) = u(x) + u''(x) + e^x(x-3) \tag{3.1}$$

with boundary conditions

$$u(0) = 1, \qquad u'(0) = 0, \qquad u(1) = 0, \qquad u'(1) = -e. \tag{3.2}$$

We construct a homotopy which satisfies the relation

$$u^{(4)}(x) - v_0^{(4)}(x) + p\left[v_0^{(4)}(x) - u(x) - u''(x) - e^x(x-3)\right] = 0, \tag{3.3}$$

where

$$v_0 = 1 + Ax^2 + Bx^3 + L^{-1}(e^x(x-3)). \tag{3.4}$$

Now substituting (3.4) into (3.3), we obtain

$$\begin{aligned} &u_0^{(4)} + pu_1^{(4)} + p^2u_2^{(4)} + \cdots - v_0^{(4)}(x) \\ &\quad + p\left[v_0^{(4)}(x) - u_0 - pu_1 - p^2u_2 - \cdots - u_0'' - pu_1'' - p^2u_2'' - \cdots - e^x(x-3)\right] = 0, \end{aligned} \tag{3.5}$$

and, equating the coefficients of a like powers of p, we get a system of equations:

$$\begin{gathered}
u_0^{(4)}(x) - v_0^{(4)}(x) = 0, \qquad u_0(0) = 1, \qquad u_0'(0) = 0, \qquad u_0''(0) = A, \qquad u_0'''(0) = B, \\
u_1^{(4)} + v_0^{(4)}(x) - u_0 - u_0'' - e^x(x-3) = 0, \qquad u_1(0) = 0, \\
u_1'(0) = 0, \qquad u_1''(0) = 0, \qquad u_1'''(0) = 0, \\
u_2^{(4)} - u_1 - u_1'' = 0, \qquad u_2(0) = 0, \qquad u_2'(0) = 0, \qquad u_2''(0) = 0, \qquad u_2'''(0) = 0, \\
u_3^{(4)} - u_2 - u_2'' = 0, \qquad u_3(0) = 0, \qquad u_3'(0) = 0, \qquad u_3''(0) = 0, \qquad u_3'''(0) = 0, \ldots.
\end{gathered} \tag{3.6}$$

Solving (3.6), we get

$$\begin{aligned}
u_0 &= 1 + Ax^2 + Bx^3 + L^{-1}(e^x(x-3)) \\
&= 1 + Ax^2 + Bx^3 + 6x - 7e^x + xe^x + \frac{5}{2}x^2 + \frac{2}{3}x^3 + 7 \\
&= 6x - 7e^x + Ax^2 + Bx^3 + xe^x + \frac{5}{2}x^2 + \frac{2}{3}x^3 + 8, \\
u_1 &= 20 + x^5\left(\frac{1}{20}B + \frac{1}{10}\right) + x^4\left(\frac{1}{12}A + \frac{13}{24}\right) + x^6\left(\frac{1}{360}A + \frac{1}{144}\right) \\
&\quad + x^7\left(\frac{1}{840}B + \frac{1}{1260}\right) + \frac{1}{6}14x^3 + \frac{1}{2}16x^2 + e^x(2x-20) + 18x, \\
u_2 &= L^{-1}(u_1 + u_1'') \\
&= L^{-1}\left(32x - 36e^x + Ax^2 + \frac{A}{6}x^4 + Bx^3 + \frac{A}{360}x^6 + \frac{B}{10}x^5 + \frac{B}{840}x^7\right. \\
&\qquad \left. + 4xe^x + \frac{29}{2}x^2 + \frac{13}{3}x^3 + \frac{3}{4}x^4 + \frac{2}{15}x^5 + \frac{1}{144}x^6 + \frac{1}{1260}x^7 + 36\right) \\
&= 52 + x^6\left(\frac{1}{360}A + \frac{29}{720}\right) + x^7\left(\frac{1}{840}B + \frac{13}{2520}\right) + x^8\left(\frac{1}{10\,080}A + \frac{1}{2240}\right) \\
&\quad + x^9\left(\frac{B}{30\,240} + \frac{1}{22\,680}\right) + x^{10}\left(\frac{A}{1814\,400} + \frac{1}{725\,760}\right) + x^{11}\left(\frac{B}{6652\,800} + \frac{1}{9979\,200}\right) \\
&\quad + \frac{1}{6}40x^3 + \frac{1}{2}44x^2 + e^x(4x-52) + 48x + \frac{3}{2}x^4 + \frac{4}{15}x^5, \ldots.
\end{aligned} \tag{3.7}$$

Using only three-term approximation, we have

$$\begin{aligned}
u = u_0 + u_1 + u_2 &= 6x - 7e^x + Ax^2 + Bx^3 + xe^x + \frac{5}{2}x^2 + \frac{2}{3}x^3 + 8 \\
&\quad + 20 + x^5\left(\frac{1}{20}B + \frac{1}{10}\right) + x^4\left(\frac{1}{12}A + \frac{13}{24}\right) + x^6\left(\frac{1}{360}A + \frac{1}{144}\right)
\end{aligned}$$

$$+x^7\left(\frac{1}{840}B+\frac{1}{1260}\right)+\frac{1}{6}14x^3+\frac{1}{2}16x^2+e^x(2x-20)+18x$$
$$+52+x^6\left(\frac{A}{360}+\frac{29}{720}\right)+x^7\left(\frac{B}{840}+\frac{13}{2520}\right)+x^8\left(\frac{A}{10\,080}+\frac{1}{2240}\right)$$
$$+x^9\left(\frac{B}{30\,240}+\frac{1}{22\,680}\right)+x^{10}\left(\frac{A}{1814\,400}+\frac{1}{725\,760}\right)+x^{11}\left(\frac{B}{6652\,800}+\frac{1}{9979\,200}\right)$$
$$+\frac{1}{6}40x^3+\frac{1}{2}44x^2+e^x(4x-52)+48x+\frac{3}{2}x^4+\frac{4}{15}x^5. \tag{3.8}$$

Now it follows from conditions $u(1)=0$, $u'(1)=-e$ that $A=-0.467\,17$ and $B=-0.383\,54$ and, therefore,

$$u=u_0+u_1+u_2=80+72x-79e^x+7xe^x+32.033x^2+9.283\,2x^3+2.002\,7x^4$$
$$+0.347\,49x^5+4.462\,7\times10^{-2}x^6+5.039\,2\times10^{-3}x^7+4.000\,8\times10^{-4}x^8 \tag{3.9}$$
$$+3.140\,9\times10^{-5}x^9+1.120\,4\times10^{-6}x^{10}+4.255\,8\times10^{-8}x^{11},$$

or, in power series form,

$$u=1-0.467\,x^2-0.383\,47x^3-0.122\,3x^4-0.019\,18x^5-6.762\times10^{-3}x^6$$
$$-9.132\times10^{-4}x^7+4.000\,8\times10^{-4}x^8+3.140\,9\times10^{-5}x^9. \tag{3.10}$$

Higher accuracy level can be attained by evaluating some more terms of $u(x)$.

Example 3.2 (see [1, Example 3.3]). We have

$$u^{(4)}(x)=\sin x+\sin^2 x-\left(u''(x)\right)^2 \tag{3.11}$$

with boundary conditions

$$u(0)=0,\qquad u'(0)=1,\qquad u(1)=\sin 1,\qquad u'(1)=\cos 1 \tag{3.12}$$

(the exact solution of the problem is $u=\sin x$).

We construct a homotopy which satisfies the relation

$$u^{(4)}(x)-v_0^{(4)}(x)+p\left[v_0^{(4)}(x)+\left(u''(x)\right)^2-\sin x-\sin^2 x\right]=0, \tag{3.13}$$

where

$$v_0=x+Ax^2+Bx^3+L^{-1}\left(\sin x+\sin^2 x\right). \tag{3.14}$$

Substituting (3.14) into (3.13), we obtain

$$u_0^{(4)} + pu_1^{(4)} + p^2u_2^{(4)} + \cdots - v_0^{(4)}(x) + p\left[v_0^{(4)}(x) + \left(u_0'' + pu_1'' + p^2u_2'' + \cdots\right)^2 - \sin x - \sin^2 x\right] = 0, \tag{3.15}$$

and, equating the coefficients of a like powers of p, we get a system of equations:

$$\begin{gathered}
u_0^{(4)}(x) - v_0^{(4)}(x) = 0, \qquad u_0(0) = 1, \qquad u_0(0) = 0, \\
u_0'(0) = 1, \qquad u_0''(0) = A, \qquad u_0'''(0) = B, \\
u_1^{(4)} + v_0^{(4)}(x) + \left(u_0''\right)^2 - \sin x - \sin^2 x = 0, \qquad u_1(0) = 0, \\
u_1'(0) = 0, \qquad u_1''(0) = 0, \qquad u_1'''(0) = 0, \\
u_2^{(4)} + 2u_0''u_1'' = 0, \qquad u_2(0) = 0, \qquad u_2'(0) = 0, \qquad u_2''(0) = 0, \qquad u_2'''(0) = 0, \\
u_3^{(4)} + \left(u_1''\right)^2 + 2u_0''u_2'' = 0, \qquad u_3(0) = 0, \qquad u_3'(0) = 0, \qquad u_3''(0) = 0, \qquad u_3'''(0) = 0, \ldots.
\end{gathered} \tag{3.16}$$

Solving (3.16) we get

$$\begin{aligned}
u_0 &= x + Ax^2 + Bx^3 + L^{-1}\left(\sin x + \sin^2 x\right) \\
&= x + Ax^2 + Bx^3 + \frac{1}{32} + \sin x - \frac{1}{32}\cos 2x + \frac{1}{6}x^3 - \frac{1}{16}x^2 - x + \frac{1}{48}x^4, \\
u_1 &= -\frac{1}{1260}Ax^7 - x^8\left(\frac{1}{5040}A + \frac{1}{1680}B\right) - x^9\left(\frac{1}{6048}B - \frac{1}{90\,720}A\right) \\
&\quad + x^{10}\left(\frac{A}{113\,400} + \frac{B}{100\,800} - \frac{1}{181\,440}\right) - \frac{1}{6}A^2x^4 - \frac{1}{40}B^2x^6 - \frac{1}{10}ABx^5 + O\left(x^{11}\right), \\
u_2 &= x^{14}\left(\frac{1}{7567\,560}A^2 + \frac{1}{30\,270\,240}AB + \frac{1}{23\,284\,800}A - \frac{1}{1441\,440}B^2 + \frac{1}{2522\,520}B\right) \\
&\quad + x^{13}\left(\frac{1}{21\,621\,600}A^2 - \frac{17}{5405\,400}AB + \frac{1}{1853\,280}A - \frac{1}{739\,200}B^2 + \frac{43}{43\,243\,200}B\right) \\
&\quad + x^{11}\left(-\frac{23}{4989\,600}A^2 + \frac{1}{15\,400}AB + \frac{13}{158\,400}B^2\right) + x^{10}\left(\frac{17}{226\,800}A^2 + \frac{1}{4200}BA\right) \\
&\quad + x^{12}\left(-\frac{1}{249\,480}A^2 - \frac{13}{3326\,400}AB + \frac{1}{907\,200}A + \frac{1}{44\,352}B^2\right) \\
&\quad + x^9\left(\frac{1}{3780}A^2 + \frac{1}{336}B^3\right) + \frac{1}{45}A^3x^6 + \frac{4}{105}A^2Bx^7 + \frac{9}{560}AB^2x^8.
\end{aligned} \tag{3.17}$$

Using only three-term approximation we have

$$\begin{aligned} u = {} & x + Ax^2 + Bx^3 + \frac{1}{32} + \sin x - \frac{1}{32}\cos 2x + \frac{1}{6}x^3 - \frac{1}{16}x^2 - x + \frac{1}{48}x^4 \\ & - \frac{1}{1260}Ax^7 - x^8\left(\frac{1}{5040}A + \frac{1}{1680}B\right) - x^9\left(\frac{1}{6048}B - \frac{1}{90\,720}A\right) \\ & + x^{10}\left(\frac{1}{113\,400}A + \frac{1}{100\,800}B - \frac{1}{181\,440}\right) - \frac{1}{6}A^2x^4 - \frac{1}{40}B^2x^6 - \frac{1}{10}ABx^5 \\ & + x^9\left(\frac{1}{3780}A^2 + \frac{1}{336}B^3\right) + \frac{1}{45}A^3x^6 + \frac{4}{105}A^2Bx^7 + \frac{9}{560}AB^2x^8. \end{aligned} \tag{3.18}$$

Now by using the conditions $u(1) = \sin 1$, $u'(1) = \cos 1$, we have a system of equations of degree three. Solving this system numerically (applying some standard computer programs) we have that $A = 5.861\,1 \times 10^{-3}$, $B = -0.174\,55$ and the series solution

$$\begin{aligned} u = {} & x + 5.861 \times 10^{-3}x^2 - 0.174\,55x^3 - 6.333\,3 \times 10^{-6}x^4 + 8.435\,6 \times 10^{-3}x^5 \\ & + 2.016\,1 \times 10^{-3}x^6 - 2.032\,9 \times 10^{-4}x^7 - 9.280\,3 \times 10^{-5}x^8 + O\left(x^9\right). \end{aligned} \tag{3.19}$$

4. Conclusion

In this paper we have used the homotopy perturbation method for finding the solution of fourth-order linear and nonlinear boundary value problems. We presented a simple way to choose L and v_0 when we use the homotopy perturbation method. In most cases, our simple choice yields very good approximation of exact solution.

References

[1] S. T. Mohyud-Din and M. A. Noor, “Homotopy perturbation method for solving fourth-order boundary value problems,” *Mathematical Problems in Engineering*, vol. 2007, Article ID 98602, 15 pages, 2007.

[2] J. H. He, “A coupling method of a homotopy technique and a perturbation technique for non-linear problems,” *International Journal of Non-Linear Mechanics*, vol. 35, no. 1, pp. 37–43, 2000.

[3] J. H. He, “Some asymptotic methods for strongly nonlinear equations,” *International Journal of Modern Physics B*, vol. 20, no. 10, pp. 1141–1199, 2006.

[4] J. H. He, “New interpretation of homotopy perturbation method,” *International Journal of Modern Physics B*, vol. 20, no. 18, pp. 2561–2568, 2006.