Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2010, Article ID 369078, 15 pages
doi:10.1155/2010/369078

Research Article

Differential Subordination Defined by New Generalised Derivative Operator for Analytic Functions

Ma'moun Harayzeh Al-Abbadi and Maslina Darus

School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, 43600 Bangi Selangor (Darul Ehsan), Malaysia

Correspondence should be addressed to Maslina Darus, maslina@ukm.my

Received 9 June 2009; Revised 20 December 2009; Accepted 9 March 2010

Academic Editor: Dorothy Wallace

A new generalised derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m}$ is introduced. This operator generalised many well-known operators studied earlier by many authors. Using the technique of differential subordination, we will study some of the properties of differential subordination. In addition we investigate several interesting properties of the new generalised derivative operator.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad \text{where } a_k \text{ is complex number}, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $S, S^*(\alpha), C(\alpha)$ $(0 \leq \alpha < 1)$ denote the subclasses of $\mathcal{A}$ consisting of functions that are univalent, starlike of order α, and convex of order α in U, respectively. In particular, the classes $S^*(0) = S^*$ and $C(0) = C$ are the familiar classes of starlike and convex functions in U, respectively. A function $f \in C(\alpha)$ if $\mathrm{Re}(1 + zf''/f') > \alpha$. Furthermore a function f analytic in U is said to be convex if it is univalent and $f(U)$ is convex.

Let $\mathcal{H}(U)$ be the class of holomorphic function in unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let

$$\mathcal{A}_n = \left\{ f \in \mathcal{H}(U) : \ f(z) = z + a_{n+1} z^{n+1} + \cdots, \ (z \in U) \right\}, \tag{1.2}$$

with $\mathcal{A}_1 = \mathcal{A}$.

For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ we let

$$\mathcal{H}[a,n] = \left\{ f \in \mathcal{H}(U) : \ f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots, \ (z \in U) \right\}. \tag{1.3}$$

Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ be analytic in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Then the Hadamard product (or convolution) $f * g$ of the two functions f, g is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.4}$$

Next, we state basic ideas on subordination. If f and g are analytic in U, then the function f is said to be subordinate to g, and can be written as

$$f \prec g, \quad f(z) \prec g(z), \quad (z \in U), \tag{1.5}$$

if and only if there exists the Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $(z \in U)$.

Furthermore if g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [1, page 36]).

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi\left(p(z), zp'(z), z^2 p''(z); z\right) \prec h(z), \quad (z \in U), \tag{1.6}$$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.6). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.6) is said to be the best dominant of (1.6). (Note that the best dominant is unique up to a rotation of U.)

Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0, x \in \mathbb{C} \setminus \{0\}, \\ x(x+1)(x+2)\cdots(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, \ldots\}, \ x \in \mathbb{C}. \end{cases} \tag{1.7}$$

To prove our results, we need the following equation throughout the paper:

$$\mu_{\lambda_1,\lambda_2}^{n,m+1} f(z) = (1-\lambda_1)\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_2}(z)\right] + \lambda_1 z \left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_2}(z)\right]', \quad (z \in U), \tag{1.8}$$

where $n, m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\lambda_2 \geq \lambda_1 \geq 0$, and $\phi_{\lambda_2}(z)$ is analytic function given by

$$\phi_{\lambda_2}(z) = z + \sum_{k=2}^{\infty} \frac{z^k}{1 + \lambda_2(k-1)}. \tag{1.9}$$

Here $\mu^{n,m}_{\lambda_1,\lambda_2}$ is the generalized derivative operator which we shall introduce later in the paper. Moreover, we need the following lemmas in proving our main results.

Lemma 1.1 (see [2, page 71]). *Let h be analytic, univalent, and convex in U, with $h(0) = a$, $\gamma \neq 0$ and,* Re $\gamma \geq 0$. *If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \quad (z \in U), \tag{1.10}$$

then

$$p(z) \prec q(z) \prec h(z), \quad (z \in U), \tag{1.11}$$

where $q(z) = (\gamma/nz^{\gamma/n}) \int_0^z h(t)\ t^{(\gamma/n)-1} dt, (z \in U)$.

The function q is convex and is the best (a,n)-dominant.

Lemma 1.2 (see [3]). *Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha z g'(z), \tag{1.12}$$

where $\alpha > 0$ and n is a positive integer.

If

$$p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots, \quad (z \in U), \tag{1.13}$$

is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z), \quad (z \in U), \tag{1.14}$$

then

$$p(z) \prec g(z), \tag{1.15}$$

and this result is sharp.

Lemma 1.3 (see [4]). *Let $f \in \mathcal{A}$, if*

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}, \tag{1.16}$$

then

$$\frac{2}{z}\int_0^z f(t)dt, \quad (z \in U, z \neq 0), \tag{1.17}$$

belongs to the class of convex functions.

2. Main Results

In the present paper, we will use the method of differential subordination to derive certain properties of generalised derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m} f(z)$. Note that differential subordination has been studied by various authors, and here we follow similar works done by Oros [5] and G. Oros and G. I. Oros [6].

In order to derive our new generalised derivative operator, we define the analytic function

$$F_{\lambda_1,\lambda_2}^{m}(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}} z^k, \tag{2.1}$$

where $m \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ and $\lambda_2 \geq \lambda_1 \geq 0$. Now, we introduce the new generalised derivative operator $\mu_{\lambda_1,\lambda_2}^{n,m}$ as follows.

Definition 2.1. For $f \in \mathcal{A}$ the operator $\mu_{\lambda_1,\lambda_2}^{n,m}$ is defined by $\mu_{\lambda_1,\lambda_2}^{n,m} : \mathcal{A} \to \mathcal{A}$

$$\mu_{\lambda_1,\lambda_2}^{n,m} f(z) = F_{\lambda_1,\lambda_2}^{m}(z) * R^n f(z), \quad (z \in U), \tag{2.2}$$

where $n, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda_2 \geq \lambda_1 \geq 0,$ and $R^n f(z)$ denotes the Ruscheweyh derivative operator [7], given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k) a_k z^k, \quad (n \in \mathbb{N}_0,\ z \in U), \tag{2.3}$$

where $c(n,k) = (n+1)_{k-1}/(1)_{k-1}$.

If f is given by (1.1), then we easily find from equality (2.2) that

$$\mu_{\lambda_1,\lambda_2}^{n,m} f(z) = z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^{m-1}}\ c(n,k) a_k z^k, \quad (z \in U), \tag{2.4}$$

where $n, m \in \mathbb{N}_0 = \{0,1,2\ldots\}, \lambda_2 \geq \lambda_1 \geq 0,$ and $c(n,k) = \binom{n+k-1}{n} = (n+1)_{k-1}/(1)_{k-1}$.

Remark 2.2. Special cases of this operator include the Ruscheweyh derivative operator in two cases $\mu_{0,\lambda_2}^{n,1} \equiv R^n$ and $\mu_{\lambda_1,0}^{n,0} \equiv R^n$ [7], the Salagean derivative operator $\mu_{1,0}^{0,m} \equiv S^n$ [8], the generalised Ruscheweyh derivative operator in two cases $\mu_{\lambda_1,\lambda_2}^{n,1} \equiv R_\lambda^n$ and $\mu_{\lambda_1,\lambda_2}^{n,0} \equiv R_\lambda^n$ [9], the generalised Salagean derivative operator $\mu_{\lambda_1,0}^{0,m} \equiv S_\beta^n$ introduced by Al-Oboudi [10], and the generalised Al-Shaqsi and Darus derivative operator $\mu_{\lambda_1,0}^{n,m} \equiv D_{\lambda,\beta}^n$ that can be found in [11].

Now, we remind the well-known Carlson-Shaffer operator $L(a,c)$ [12] associated with the incomplete beta function $\phi(a,c;z)$, defined by

$$\begin{gathered} L(a,c) : \mathcal{A} \to \mathcal{A}, \\ L(a,c) f(z) := \phi(a,c;z) * f(z), \quad (z \in U), \end{gathered} \tag{2.5}$$

where

$$\phi(a,c;z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(c)_{k-1}} z^k, \tag{2.6}$$

a is any real number, and $c \notin z_0^-$; $z_0^- = \{0,-1,-2,\dots\}$.

It is easily seen that

$$\begin{gathered} \mu_{\lambda_1,0}^{0,0} f(z) = \mu_{0,0}^{0,m} f(z) = \mu_{0,\lambda_2}^{0,1} f(z) = \mu_{0,1}^{1,2} f(z) = L(a,a) f(z) = f(z), \\ \mu_{\lambda_1,0}^{1,0} f(z) = \mu_{0,0}^{1,m} f(z) = \mu_{0,\lambda_2}^{1,1} f(z) = \mu_{\lambda_1,1}^{0,0} f(z) = L(2,1) f(z) = z f'(z), \end{gathered} \tag{2.7}$$

and also

$$\mu_{\lambda_1,0}^{a-1,0} f(z) = \mu_{0,\lambda_2}^{a-1,1} f(z) = \mu_{0,0}^{a-1,m} f(z) = L(a,1) f(z), \tag{2.8}$$

where $a = 1,2,3,\dots$.

Next, we give the following.

Definition 2.3. For $n,m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0$, and $0 \leq \alpha < 1$, let $R_{\lambda_1,\lambda_2}^{n,m}(\alpha)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\operatorname{Re}\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' > \alpha, \quad (z \in U). \tag{2.9}$$

Also let $K_{\lambda_1,\lambda_2}^{n,m}(\delta)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the condition

$$\operatorname{Re}\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_2}(z)\right)' > \delta, \quad (z \in U). \tag{2.10}$$

Remark 2.4. It is clear that $R_{\lambda_1,0}^{0,1}(\alpha) \equiv R(\lambda_1,\alpha)$, and the class of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\lambda_1 z f''(z) + f'(z)\right) > \alpha, \quad (z \in U) \tag{2.11}$$

is studied by Ponnusamy [13] and others.

Now we begin with the first result as follows.

Theorem 2.5. *Let*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}, \quad (z \in U), \tag{2.12}$$

be convex in U, with h(0)=1 and $0 \leq \alpha < 1$. If $n,m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0$, and the differential subordination

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m+1} f(z)\right)' \prec h(z), \quad (z \in U), \tag{2.13}$$

holds, then

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right)' \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1 z^{1/\lambda_1}} \sigma\left(\frac{1}{\lambda_1}\right), \tag{2.14}$$

where σ *is given by*

$$\sigma(x) = \int_0^z \frac{t^{x-1}}{1+t} dt, \quad (z \in U). \tag{2.15}$$

The function q *is convex and is the best dominant.*

Proof. By differentiating (1.8), with respect to z, we obtain

$$\left(\mu^{n,m+1}_{\lambda_1,\lambda_2} f(z)\right)' = \left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right]' + \lambda_1 z \left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right]''. \tag{2.16}$$

Using (2.16) in (2.13), differential subordination (2.13) becomes

$$\left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right]' + \lambda_1 z \left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right]'' \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}. \tag{2.17}$$

Let

$$\begin{aligned} p(z) = \left[\mu^{n,m}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right]' &= \left[z + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))^m}{(1+\lambda_2(k-1))^m}\ c(n,k) a_k z^k\right]' \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned} \tag{2.18}$$

Using (2.18) in (2.17), the differential subordination becomes

$$p(z) + \lambda_1 z p'(z) \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}. \tag{2.19}$$

By using Lemma 1.1, we have

$$\begin{aligned} p(z) \prec q(z) &= \frac{1}{\lambda_1 z^{1/\lambda_1}} \int_0^z h(t)\ t^{(1/\lambda_1)-1} dt \\ &= \frac{1}{\lambda_1 z^{1/\lambda_1}} \int_0^z \left(\frac{1+(2\alpha-1)t}{1+t}\right) t^{(1/\lambda_1)-1} dt \\ &= 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1 z^{1/\lambda_1}} \sigma\left(\frac{1}{\lambda_1}\right), \end{aligned} \tag{2.20}$$

where σ is given by (2.15), so we get

$$\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_2}(z)\right]' \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1 z^{1/\lambda_1}}\sigma\left(\frac{1}{\lambda_1}\right). \tag{2.21}$$

The function q is convex and is the best dominant. The proof is complete. □

Theorem 2.6. *If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$, and $0 \leq \alpha < 1$, then one has*

$$R_{\lambda_1,\lambda_2}^{n,m+1}(\alpha) \subset K_{\lambda_1,\lambda_2}^{n,m}(\delta), \tag{2.22}$$

where

$$\delta = 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1}\sigma\left(\frac{1}{\lambda_1}\right), \tag{2.23}$$

and σ is given by (2.15).

Proof. Let $f \in R_{\lambda_1,\lambda_2}^{n,m+1}(\alpha)$, then from (2.9) we have

$$\operatorname{Re}\left(\mu_{\lambda_1,\lambda_2}^{n,m+1} f(z)\right)' > \alpha, \quad (z \in U), \tag{2.24}$$

which is equivalent to

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m+1} f(z)\right)' \prec h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}. \tag{2.25}$$

Using Theorem 2.5, we have

$$\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_2}(z)\right]' \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1 z^{1/\lambda_1}}\sigma\left(\frac{1}{\lambda_1}\right). \tag{2.26}$$

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce that

$$\begin{aligned}\operatorname{Re}\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_2}(z)\right]' &> \operatorname{Re}\ q(1) = \delta = \delta(\alpha, \lambda_1) \\ &= 2\alpha - 1 + \frac{2(1-\alpha)}{\lambda_1}\sigma\left(\frac{1}{\lambda_1}\right),\end{aligned} \tag{2.27}$$

from which we deduce $R_{\lambda_1,\lambda_2}^{n,m+1}(\alpha) \subset K_{\lambda_1,\lambda_2}^{n,m}(\delta)$. This completes the proof of Theorem 2.6. □

Remark 2.7. Special case of Theorem 2.6 with $\lambda_2 = 0$ was given earlier in [11].

Theorem 2.8. *Let q be a convex function in U, with $q(0) = 1$, and let*

$$h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U). \tag{2.28}$$

If $n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0$,*and* $f \in \mathcal{A}$ *and satisfies the differential subordination*

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m+1} f(z)\right)' \prec h(z), \quad (z \in U), \tag{2.29}$$

then

$$\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_2}(z)\right]' \prec q(z), \quad (z \in U), \tag{2.30}$$

and this result is sharp.

Proof. Using (2.18) in (2.16), differential subordination (2.29) becomes

$$p(z) + \lambda_1 z p'(z) \prec h(z) = q(z) + \lambda_1 z q'(z), \quad (z \in U). \tag{2.31}$$

Using Lemma 1.2, we obtain

$$p(z) \prec q(z), \quad (z \in U). \tag{2.32}$$

Hence

$$\left[\mu_{\lambda_1,\lambda_2}^{n,m} f(z) * \phi_{\lambda_2}(z)\right]' \prec q(z), \quad (z \in U). \tag{2.33}$$

And the result is sharp. This completes the proof of the theorem. □

We give a simple application for Theorem 2.8.

Example 2.9. For $n = 0, m = 1, \lambda_2 \geq \lambda_1 \geq 0, q(z) = (1+z)/(1-z), f \in \mathcal{A}$, and $z \in U$ and applying Theorem 2.8, we have

$$h(z) = \frac{1+z}{1-z} + \lambda_1 z \left(\frac{1+z}{1-z}\right)' = \frac{1 + 2\lambda_1 z - z^2}{(1-z)^2}. \tag{2.34}$$

By using (1.8) we find that

$$\mu_{\lambda_1,\lambda_2}^{0,1} f(z) = (1-\lambda_1) f(z) + \lambda_1 z f'(z). \tag{2.35}$$

Now,

$$\mu_{\lambda_1,\lambda_2}^{0,1} f(z) * \phi_{\lambda_2}(z) = (1-\lambda_1)\left[z + \sum_{k=2}^{\infty} \frac{a_k z^k}{1+\lambda_2(k-1)}\right] + \lambda_1 \left[z + \sum_{k=2}^{\infty} \frac{a_k k z^k}{1+\lambda_2(k-1)}\right]. \tag{2.36}$$

A straightforward calculation gives the following:

$$\begin{aligned}\left[\mu^{0,1}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right]' &= 1 + \sum_{k=2}^{\infty} \frac{(1+\lambda_1(k-1))ka_k}{1+\lambda_2(k-1)} z^{k-1} \\ &= \frac{z + \sum_{k=2}^{\infty}(ka_k(1+\lambda_1(k-1))/(1+\lambda_2(k-1)))z^k}{z} \\ &= \frac{\left[f(z) * \phi_{\lambda_2}(z)\right] * \left[z + \sum_{k=2}^{\infty} k(1+\lambda_1(k-1))z^k\right]}{z}.\end{aligned} \tag{2.37}$$

Similarly, using (1.8), we find that

$$\mu^{0,2}_{\lambda_1,\lambda_2} f(z) = (1-\lambda_1)\left[\mu^{0,1}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right] + \lambda_1 z\left[\mu^{0,1}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right]', \tag{2.38}$$

then

$$\left(\mu^{0,2}_{\lambda_1,\lambda_2} f(z)\right)' = \left(\mu^{0,1}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right)' + \lambda_1 z\left(\mu^{0,1}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right)''. \tag{2.39}$$

By using (2.37) we obtain

$$\left(\mu^{0,1}_{\lambda_1,\lambda_2} f(z) * \phi_{\lambda_2}(z)\right)'' = \sum_{k=2}^{\infty} \frac{k(k-1)a_k(1+\lambda_1(k-1))}{1+\lambda_2(k-1)} z^{k-2}. \tag{2.40}$$

We get

$$\begin{aligned}\left(\mu^{0,2}_{\lambda_1,\lambda_2} f(z)\right)' &= 1 + \sum_{k=2}^{\infty} \frac{ka_k(1+\lambda_1(k-1))}{1+\lambda_2(k-1)} z^{k-1} + \lambda_1 \sum_{k=2}^{\infty} \frac{k(k-1)a_k(1+\lambda_1(k-1))}{1+\lambda_2(k-1)} z^{k-1} \\ &= 1 + \sum_{k=2}^{\infty} \frac{ka_k(1+\lambda_1(k-1))^2}{1+\lambda_2(k-1)} z^{k-1} \\ &= \frac{\left[f(z) * \phi_{\lambda_2}(z)\right] * \left[z + \sum_{k=2}^{\infty} k(1+\lambda_1(k-1))^2 z^k\right]}{z}.\end{aligned} \tag{2.41}$$

From Theorem 2.8 we deduce that

$$\frac{\left[f(z) * \phi_{\lambda_2}(z)\right] * \left[z + \sum_{k=2}^{\infty} k(1+\lambda_1(k-1))^2 z^k\right]}{z} \prec \frac{1+2\lambda_1 z - z^2}{(1-z)^2}, \tag{2.42}$$

implies that

$$\frac{\left[f(z) * \phi_{\lambda_2}(z)\right] * \left[z + \sum_{k=2}^{\infty} k(1+\lambda_1(k-1))z^k\right]}{z} \prec \frac{1+z}{1-z}, \quad (z \in U). \tag{2.43}$$

Theorem 2.10. *Let q be a convex function in U, with $q(0) = 1$ and let*

$$h(z) = q(z) + zq'(z), \quad (z \in U). \tag{2.44}$$

If $n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0$, and $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' \prec h(z), \tag{2.45}$$

then

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z), \quad (z \in U). \tag{2.46}$$

And the result is sharp.

Proof. Let

$$\begin{aligned} p(z) &= \frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \\ &= \frac{z + \sum_{k=2}^{\infty}\left((1+\lambda_1(k-1))^m/(1+\lambda_2(k-1))^{m-1}\right)c(n,k)a_k z^k}{z} \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned} \tag{2.47}$$

Differentiating (2.47), with respect to z, we obtain

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' = p(z) + zp'(z), \quad (z \in U). \tag{2.48}$$

Using (2.48), (2.45) becomes

$$p(z) + zp'(z) \prec h(z) = q(z) + zq'(z). \tag{2.49}$$

Using Lemma 1.2, we deduce that

$$p(z) \prec q(z), \quad (z \in U), \tag{2.50}$$

and using (2.47), we have

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z), \quad (z \in U). \tag{2.51}$$

This proves Theorem 2.10. □

We give a simple application for Theorem 2.10.

Example 2.11. For $n = 0, m = 1, \lambda_2 \geq \lambda_1 \geq 0, q(z) = 1/(1-z), f \in \mathcal{A}$, and $z \in U$ and applying Theorem 2.10, we have

$$h(z) = \frac{1}{1-z} + z\left(\frac{1}{1-z}\right)' = \frac{1}{(1-z)^2}. \tag{2.52}$$

From Example 2.9, we have

$$\mu_{\lambda_1,\lambda_2}^{0,1} f(z) = (1-\lambda_1) f(z) + \lambda_1 z f'(z), \tag{2.53}$$

so

$$\left(\mu_{\lambda_1,\lambda_2}^{0,1} f(z)\right)' = f'(z) + \lambda_1 z f''(z). \tag{2.54}$$

Now, from Theorem 2.10 we deduce that

$$f'(z) + \lambda_1 z f''(z) \prec \frac{1}{(1-z)^2} \tag{2.55}$$

implies that

$$\frac{(1-\lambda_1) f(z) + \lambda_1 z f'(z)}{z} \prec \frac{1}{1-z}. \tag{2.56}$$

Theorem 2.12. *Let*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1+z}, \quad (z \in U), \tag{2.57}$$

be convex in U, with $h(0) = 1$ and $0 \leq \alpha < 1$. If $n, m \in \mathbb{N}_0, \lambda_2 \geq \lambda_1 \geq 0, f \in \mathcal{A}$, and the differential subordination holds as

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' \prec h(z), \tag{2.58}$$

then

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \tag{2.59}$$

The function q is convex and is the best dominant.

Proof. Let

$$\begin{aligned} p(z) &= \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \\ &= \frac{z + \sum_{k=2}^{\infty}\left((1+\lambda_1(k-1))^m/(1+\lambda_2(k-1))^{m-1}\right)c(n,k)\ a_k z^k}{z} \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad \left(p \in \mathcal{H}[1,1],\ z \in U\right). \end{aligned} \tag{2.60}$$

Differentiating (2.60), with respect to z, we obtain

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' = p(z) + zp'(z), \quad (z \in U). \tag{2.61}$$

Using (2.61), the differential subordination (2.58) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}, \quad (z \in U). \tag{2.62}$$

From Lemma 1.1, we deduce that

$$\begin{aligned} p(z) \prec q(z) &= \frac{1}{z}\int_0^z h(t)dt \\ &= \frac{1}{z}\int_0^z \left(\frac{1+(2\alpha-1)t}{1+t}\right)dt \\ &= \frac{1}{z}\left[\int_0^z \frac{1}{1+t}dt + (2\alpha-1)\int_0^z \frac{t}{1+t}dt\right] \\ &= 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \end{aligned} \tag{2.63}$$

Using (2.60), we have

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z) = 2\alpha - 1 + \frac{2(1-\alpha)\ln(1+z)}{z}. \tag{2.64}$$

The proof is complete. □

From Theorem 2.12, we deduce the following corollary.

Corollary 2.13. *If* $f \in R^{n,m}_{\lambda_1,\lambda_2}(\alpha)$, *then*

$$\operatorname{Re}\left(\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}\right) > (2\alpha-1) + 2(1-\alpha)\ln 2, \quad (z \in U). \tag{2.65}$$

Proof. Since $f \in R^{n,m}_{\lambda_1,\lambda_2}(\alpha)$, from Definition 2.3 we have

$$\operatorname{Re}\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' > \alpha, \quad (z \in U), \tag{2.66}$$

which is equivalent to

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' \prec h(z) = \frac{1+(2\alpha-1)z}{1+z}. \tag{2.67}$$

Using Theorem 2.12, we have

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z) = (2\alpha-1) + 2(1-\alpha)\frac{\ln(1+z)}{z}. \tag{2.68}$$

Since q is convex and $q(U)$ is symmetric with respect to the real axis, we deduce that

$$\operatorname{Re}\left(\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z}\right) > \operatorname{Re} q(1) = (2\alpha-1) + 2(1-\alpha)\ln 2, \quad (z \in U). \tag{2.69}$$

□

Theorem 2.14. *Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$ which satisfy the inequality*

$$\operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad (z \in U). \tag{2.70}$$

If $n, m \in \mathbb{N}_0$, $\lambda_2 \geq \lambda_1 \geq 0$,and $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\mu^{n,m}_{\lambda_1,\lambda_2} f(z)\right)' \prec h(z), \quad (z \in U), \tag{2.71}$$

then

$$\frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)dt. \tag{2.72}$$

Proof. Let

$$\begin{aligned} p(z) &= \frac{\mu^{n,m}_{\lambda_1,\lambda_2} f(z)}{z} \\ &= \frac{z + \sum_{k=2}^{\infty}\left((1+\lambda_1(k-1))^m/(1+\lambda_2(k-1))^{m-1}\right)c(n,k)a_k z^k}{z} \\ &= 1 + p_1 z + p_2 z^2 + \cdots, \quad (p \in \mathcal{H}[1,1],\ z \in U). \end{aligned} \tag{2.73}$$

Differentiating (2.73), with respect to z, we have

$$\left(\mu_{\lambda_1,\lambda_2}^{n,m} f(z)\right)' = p(z) + zp'(z), \quad (z \in U). \tag{2.74}$$

Using (2.74), the differential subordination (2.71) becomes

$$p(z) + zp'(z) \prec h(z), \quad (z \in U). \tag{2.75}$$

From Lemma 1.1, we deduce that

$$p(z) \prec q(z) = \frac{1}{z} \int_0^z h(t)dt, \tag{2.76}$$

and using (2.73), we obtain

$$\frac{\mu_{\lambda_1,\lambda_2}^{n,m} f(z)}{z} \prec q(z) = \frac{1}{z} \int_0^z h(t)dt. \tag{2.77}$$

From Lemma 1.3, we have that the function q is convex, and from Lemma 1.1, q is the best dominant for subordination (2.71). This completes the proof of Theorem 2.14. □

3. Conclusion

We remark that several subclasses of analytic univalent functions can be derived and studied using the operator $\mu_{\lambda_1,\lambda_2}^{n,m}$.

Acknowledgment

This work is fully supported by UKM-ST-06-FRGS0107-2009, MOHE, Malaysia.

References

[1] K. S. Padmanabhan and R. Manjini, "Certain applications of differential subordination," *Publications de Institut Mathématique*, vol. 39, no. 53, pp. 107–118, 1986.

[2] S. S. Miller and P. T. Mocanu, *Differential Subordinations. Theory and Applications*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.

[3] S. S. Miller and P. T. Mocanu, "On some classes of first-order differential subordinations," *The Michigan Mathematical Journal*, vol. 32, no. 2, pp. 185–195, 1985.

[4] P. T. Mocanu, T. Bulboaca, and G. S. Salagean, *Teoria Geometrica a Functiilor Univalente*, Casa Cartii de Stiinta, Cluj-Napoca, Romania, 1999.

[5] G. I. Oros, "A class of holomorphic functions defined using a differential operator," *General Mathematics*, vol. 13, no. 4, pp. 13–18, 2005.

[6] G. Oros and G. I. Oros, "Differential superordination defined by Salagean operator," *General Mathematics*, vol. 12, no. 4, pp. 3–10, 2004.

[7] S. Ruscheweyh, "New criteria for univalent functions," *Proceedings of the American Mathematical Society*, vol. 49, pp. 109–115, 1975.

[8] G. S. Salagean, "Subclasses of univalent functions," in *Complex Analysis—5th Romanian-Finnish Seminar, Part 1 (Bucharest, 1981)*, vol. 1013 of *Lecture Notes in Mathematics*, pp. 362–372, Springer, Berlin, Germany, 1983.

[9] K. Al-Shaqsi and M. Darus, "On univalent functions with respect to k-symmetric points defined by a generalized Ruscheweyh derivatives operator," *Journal of Analysis and Applications*, vol. 7, no. 1, pp. 53–61, 2009.

[10] F. M. Al-Oboudi, "On univalent functions defined by a generalized Salagean operator," *International Journal of Mathematics and Mathematical Sciences*, no. 25–28, pp. 1429–1436, 2004.

[11] M. Darus and K. Al-Shaqsi, "Differential sandwich theorems with generalised derivative operator," *International Journal of Computational and Mathematical Sciences*, vol. 2, no. 2, pp. 75–78, 2008.

[12] B. C. Carlson and D. B. Shaffer, "Starlike and prestarlike hypergeometric functions," *SIAM Journal on Mathematical Analysis*, vol. 15, no. 4, pp. 737–745, 1984.

[13] S. Ponnusamy, "Differential subordination and starlike functions," *Complex Variables. Theory and Application*, vol. 19, no. 3, pp. 185–194, 1992.