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Research Article

Convergence Theorems on a New Iteration Process for Two Asymptotically Nonexpansive Nonself-Mappings with Errors in Banach Spaces

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We introduce a new two-step iterative scheme for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space. Weak and strong convergence theorems are established for this iterative scheme in a uniformly convex Banach space. The results presented extend and improve the corresponding results of Chidume et al. (2003), Wang (2006), Shahzad (2005), and Thianwan (2008).

1. Introduction

Let E be a real normed space and K be a nonempty subset of E. A mapping $T : K \to K$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$. A mapping $T : K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $\|T^n x - T^n y\| \le k_n \|x - y\|$ for all $x, y \in K$ and $n \ge 1$. T is called uniformly L-Lipschitzian if there exists a real number $L > 0$ such that $\|T^n x - T^n y\| \le L\|x - y\|$ for all $x, y \in K$ and $n \ge 1$. It is easy to see that if T is an asymptotically nonexpansive, then it is uniformly L-Lipschitzian with the uniform Lipschitz constant $L = \sup\{k_n : n \ge 1\}$.

Iterative techniques for nonexpansive and asymptotically nonexpansive mappings in Banach spaces including Mann type and Ishikawa type iteration processes have been studied extensively by various authors; see [1–8]. However, if the domain of T, $D(T)$, is a proper subset of E (and this is the case in several applications), and T maps $D(T)$ into E, then the iteration processes of Mann type and Ishikawa type studied by the authors mentioned above, and their modifications introduced may fail to be well defined.

A subset K of E is said to be a retract of E if there exists a continuous map $P : E \to K$ such that $Px = x$, for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P : E \to K$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all $y \in R(P)$, the range of P.

The concept of asymptotically nonexpansive nonself-mappings was firstly introduced by Chidume et al. [4] as the generalization of asymptotically nonexpansive self-mappings. The asymptotically nonexpansive nonself-mapping is defined as follows.

Definition 1.1 (see [4]). Let K be a nonempty subset of real normed linear space E. Let $P : E \to K$ be the nonexpansive retraction of E onto K. A nonself mapping $T : K \to$ E is called asymptotically nonexpansive if there exists sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ $(n \to \infty)$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le k_n\|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \ge 1$. T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \le L\|x - y\| \tag{1.2}$$

for all $x, y \in K$ and $n \ge 1$.

In [4], they study the following iterative sequence:

$$x_{n+1} = P\Big((1 - \alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n\Big), \quad x_1 \in K,\ n \ge 1 \tag{1.3}$$

to approximate some fixed point of T under suitable conditions. In [9], Wang generalized the iteration process (1.3) as follows:

$$\begin{aligned} x_{n+1} &= P\Big((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n\Big),\\ y_n &= P\Big((1 - \alpha'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}x_n\Big), \quad x_1 \in K,\ n \ge 1, \end{aligned} \tag{1.4}$$

where $T_1, T_2 : K \to E$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\alpha'_n\}$ are sequences in $[0, 1]$. He studied the strong and weak convergence of the iterative scheme (1.4) under proper conditions. Meanwhile, the results of [9] generalized the results of [4].

In [10], Shahzad studied the following iterative sequence:

$$x_{n+1} = P\big((1 - \alpha_n)x_n + \alpha_n TP\big[(1 - \beta_n)x_n + \beta_n Tx_n\big]\big), x_1 \in K, \quad n \ge 1, \tag{1.5}$$

where $T : K \to E$ is a nonexpansive nonself-mapping and K is a nonempty closed convex nonexpansive retract of a real uniformly convex Banach space E with P, nonexpansive retraction.

Recently, Thianwan [11] generalized the iteration process (1.5) as follows:

$$x_{n+1} = P\big((1-\alpha_n-\gamma_n)x_n + \alpha_n TP\big((1-\beta_n)y_n + \beta_n T y_n\big) + \gamma_n u_n\big),$$
$$y_n = P\big((1-\alpha'_n-\gamma'_n)x_n + \alpha'_n TP\big((1-\beta'_n)x_n + \beta'_n T x_n\big) + \gamma'_n v_n\big), \quad x_1 \in K,\ n \geq 1, \tag{1.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$,$\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are appropriate sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K. He proved weak and strong convergence theorems for nonexpansive nonself-mappings in uniformly convex Banach spaces.

The purpose of this paper, motivated by the Wang [9], Thianwan [11] and some others, is to construct an iterative scheme for approximating a fixed point of asymptotically nonexpansive nonself-mappings (provided that such a fixed point exists) and to prove some strong and weak convergence theorems for such maps.

Let E be a normed space, K a nonempty convex subset of E, $P : E \to K$ the nonexpansive retraction of E onto K, and $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings. Then, for given $x_1 \in K$ and $n \geq 1$, we define the sequence $\{x_n\}$ by the iterative scheme:

$$x_{n+1} = P\Big((1-\alpha_n-\gamma_n)x_n + \alpha_n T_1(PT_1)^{n-1} P\Big((1-\beta_n)y_n + \beta_n T_1(PT_1)^{n-1} y_n\Big) + \gamma_n u_n\Big),$$
$$y_n = P\Big((1-\alpha'_n-\gamma'_n)x_n + \alpha'_n T_2(PT_2)^{n-1} P\Big((1-\beta'_n)x_n + \beta'_n T_2(PT_2)^{n-1} x_n\Big) + \gamma'_n v_n\Big), \tag{1.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are appropriate sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n = 1 = \alpha'_n+\beta'_n+\gamma'_n$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K. Clearly, the iterative scheme (1.7) is generalized by the iterative schemes (1.4) and (1.6).

Now, we recall the well-known concepts and results.

Let E be a Banach space with dimension $E \geq 2$. The modulus of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : \|x\| = \|y\| = 1,\ \varepsilon = \|x-y\|\right\}. \tag{1.8}$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

A Banach space E is said to satisfy Opial's condition [12] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\| \tag{1.9}$$

for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$converges weakly to x.

The mapping $T : K \to E$ with $F(T) \neq \emptyset$ is said to satisfy condition (A) [13] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$\|x - Tx\| \geq f(d(x, F(T))) \tag{1.10}$$

for all $x \in K$, where $d(x,F(T)) = \inf\{\|x-p\| : p \in F(T)\}$; (see [13, page 337]) for an example of nonexpansive mappings satisfying condition (A).

Two mappings $T_1, T_2 : K \to E$ are said to satisfy condition (A') [14] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$\frac{1}{2}(\|x - T_1x\| + \|x - T_2x\|) \geq f(d(x, F(T))) \tag{1.11}$$

for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T) = F(T_1) \cap F(T_2)\}$.

Note that condition (A') reduces to condition (A) when $T_1 = T_2$ and hence is more general than the demicompactness of T_1 and T_2 [13]. A mapping $T : K \to K$ is called: (1) demicompact if any bounded sequence $\{x_n\}$ in K such that $\{x_n - Tx_n\}$ converges has a convergent subsequence, (2) semicompact (or hemicompact) if any bounded sequence $\{x_n\}$ in K such that $\{x_n - Tx_n\} \to 0$ as $n \to \infty$ has a convergent subsequence. Every demicompact mapping is semicompact but the converse is not true in general.

Senter and Dotson [13] have approximated fixed points of a nonexpansive mapping T by Mann iterates, whereas Maiti and Ghosh [14] and Tan and Xu [5] have approximated the fixed points using Ishikawa iterates under the condition (A) of Senter and Dotson [13]. Tan and Xu [5] pointed out that condition (A) is weaker than the compactness of K. Khan and Takahashi [6] have studied the two mappings case for asymptotically nonexpansive mappings under the assumption that the domain of the mappings is compact. We shall use condition (A') instead of compactness of K to study the strong convergence of $\{x_n\}$ defined in (1.7).

In the sequel, we need the following usefull known lemmas to prove our main results.

Lemma 1.2 (see [5]). *Let $\{a_n\}$, $\{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 1. \tag{1.12}$$

If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then

(i) $\lim_{n\to\infty} a_n$ *exists;*

(ii) *In particular, if $\{a_n\}$ has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 1.3 (see [2]). *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \qquad \limsup_{n\to\infty} \|y_n\| \leq r, \qquad \lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = r \tag{1.13}$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.4 (see [4]). *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E, and $T : K \to E$ be a nonexpansive mapping. Then, $(I - T)$ is demiclosed at zero, that is, if $x_n \rightharpoonup x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set fixed point of T.*

2. Main Results

We shall make use of the following lemmas.

Lemma 2.1. *Let E be a normed space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings of E with sequences $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty}(l_n-1)<\infty$, respectively and $F(T_1)\cap F(T_2) := \{x \in K : T_1x = T_2x = x\} \neq \emptyset$. Suppose that $\{u_n\}$, $\{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\gamma'_n<\infty$. Starting from an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.7)*. Then, $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p \in F(T_1)\cap F(T_2)$.*

Proof. Let $p \in F(T_1)\cap F(T_2)$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K, we have

$$r = \max\left\{\sup_{n\geq 1}\|u_n-p\|, \sup_{n\geq 1}\|v_n-p\|\right\}. \tag{2.1}$$

Set $\sigma_n = (1-\beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n$ and $\delta_n = (1-\beta'_n)x_n + \beta'_n T_2(PT_2)^{n-1}x_n$. Firstly, we note that

$$\begin{aligned}
\|\sigma_n - p\| &= \left\|(1-\beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n - p\right\| \\
&\leq \beta_n\left\|T_1(PT_1)^{n-1}y_n - p\right\| + (1-\beta_n)\|y_n-p\| \\
&\leq \beta_n k_n\|y_n-p\| + (1-\beta_n)\|y_n-p\| \\
&\leq k_n\|y_n-p\|,
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\|\delta_n - p\| &= \left\|(1-\beta'_n)x_n + \beta'_n T_2(PT_2)^{n-1}x_n - p\right\| \\
&\leq \beta'_n\left\|T_2(PT_2)^{n-1}x_n - p\right\| + (1-\beta'_n)\|x_n-p\| \\
&\leq \beta'_n l_n\|x_n-p\| + (1-\beta'_n)\|x_n-p\| \\
&\leq l_n\|x_n-p\|.
\end{aligned} \tag{2.3}$$

From (1.7) and (2.3), we have

$$\begin{aligned}
\|y_n - p\| &= \left\|P\Big((1-\alpha'_n-\gamma'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}P\delta_n + \gamma'_n v_n\Big) - p\right\| \\
&\leq \left\|(1-\alpha'_n-\gamma'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}P\delta_n + \gamma'_n v_n - p\right\| \\
&\leq \alpha'_n\left\|T_2(PT_2)^{n-1}P\delta_n - p\right\| + (1-\alpha'_n-\gamma'_n)\|x_n-p\| + \gamma'_n\|v_n-p\| \\
&\leq \alpha'_n l_n\|\delta_n-p\| + (1-\alpha'_n-\gamma'_n)\|x_n-p\| + \gamma'_n\|v_n-p\| \\
&\leq \alpha'_n l_n^2\|x_n-p\| + (1-\alpha'_n-\gamma'_n)\|x_n-p\| + \gamma'_n r \\
&\leq l_n^2\|x_n-p\| + \gamma'_n r.
\end{aligned} \tag{2.4}$$

Substituting (2.4) into (2.2), we obtain

$$\|\sigma_n - p\| \le k_n\|y_n - p\| \le k_n l_n^2\|x_n - p\| + k_n\gamma_n' r. \tag{2.5}$$

It follows from (1.7) and (2.5) that

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\|P\left((1-\alpha_n-\gamma_n)x_n + \alpha_n T_1(PT_1)^{n-1}P\sigma_n + \gamma_n u_n\right) - p\right\| \\
&\le \left\|(1-\alpha_n-\gamma_n)x_n + \alpha_n T_1(PT_1)^{n-1}P\sigma_n + \gamma_n u_n - p\right\| \\
&\le \alpha_n\left\|T_1(PT_1)^{n-1}P\sigma_n - p\right\| + (1-\alpha_n-\gamma_n)\|x_n - p\| + \gamma_n\|u_n - p\| \\
&\le \alpha_n k_n\|\sigma_n - p\| + (1-\alpha_n-\gamma_n)\|x_n - p\| + \gamma_n\|u_n - p\| \\
&\le \alpha_n\left(k_n^2 l_n^2\|x_n - p\| + k_n^2\gamma_n' r\right) + (1-\alpha_n-\gamma_n)\|x_n - p\| + \gamma_n r \\
&\le k_n^2 l_n^2\|x_n - p\| + k_n^2\gamma_n' r + \gamma_n r \\
&= \left(1 + \left(l_n^2 - 1\right)\left(k_n^2 - 1\right) + \left(l_n^2 - 1\right) + \left(k_n^2 - 1\right)\right)\|x_n - p\| + \left(k_n^2\gamma_n' + \gamma_n\right)r.
\end{aligned} \tag{2.6}$$

Note that $\sum_{n=1}^{\infty} k_n - 1 < \infty$ and $\sum_{n=1}^{\infty} l_n - 1 < \infty$ are equivalent to $\sum_{n=1}^{\infty} k_n^2 - 1 < \infty$ and $\sum_{n=1}^{\infty} l_n^2 - 1 < \infty$, respectively. Since $\sum_{n=1}^{\infty}\gamma_n < \infty$ and $\sum_{n=1}^{\infty}\gamma_n' < \infty$, we have $\sum_{n=1}^{\infty}(k_n^2\gamma_n' + \gamma_n)r < \infty$. We obtained from (2.6) and Lemma 1.2 that $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$. This completes the proof. □

Lemma 2.2. *Let E be a normed space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be nonself uniformly L_1-Lipschitzian, L_2-Lipschitzian, respectively. Suppose that $\{u_n\}$, $\{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n' < \infty$. Starting from an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.7) *and set $C_n = \|x_n - T_1(PT_1)^{n-1}x_n\|$, $C_n' = \|x_n - T_2(PT_2)^{n-1}x_n\|$ for all $n \ge 1$. If $\lim_{n\to\infty}C_n = \lim_{n\to\infty}C_n' = 0$, then*

$$\lim_{n\to\infty}\|x_n - T_1x_n\| = \lim_{n\to\infty}\|x_n - T_2x_n\| = 0. \tag{2.7}$$

Proof. Since $\{u_n\}$, $\{v_n\}$ are bounded, it follows from Lemma 2.1 that $\{u_n - x_n\}$ and $\{v_n - x_n\}$ are all bounded. We set

$$\begin{aligned}
r_1 &= \sup\{\|u_n - x_n\| : n \ge 1\}, \qquad r_2 = \sup\{\|v_n - x_n\| : n \ge 1\}, \\
r_3 &= \sup\{\|u_{n-1} - x_{n-1}\| : n \ge 1\}, \qquad r = \max\{r_i : i = 1,2,3\}.
\end{aligned} \tag{2.8}$$

Let $\sigma_n = (1-\beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n$ and $\delta_n = (1-\beta'_n)x_n + \beta'_n T_2(PT_2)^{n-1}x_n$. Then, we have

$$\begin{aligned}
\|\sigma_n - x_n\| &= \left\|(1-\beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n - x_n\right\| \\
&\leq \beta_n \left\|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\right\| \\
&\quad + \beta_n \left\|T_1(PT_1)^{n-1}x_n - x_n\right\| + (1-\beta_n)\|y_n - x_n\| \\
&\leq (L_1+1)\|y_n - x_n\| + C_n,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\|\delta_n - x_n\| &= \left\|(1-\beta'_n)x_n + \beta'_n T_2(PT_2)^{n-1}x_n - x_n\right\| \\
&\leq \beta'_n \left\|T_2(PT_2)^{n-1}x_n - x_n\right\| \\
&\leq C'_n.
\end{aligned} \tag{2.10}$$

We find the following from (1.7) and (2.10):

$$\begin{aligned}
\|y_n - x_n\| &= \left\|P\left((1-\alpha'_n-\gamma'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}P\delta_n + \gamma'_n v_n\right) - x_n\right\| \\
&\leq \left\|(1-\alpha'_n-\gamma'_n)x_n + \alpha'_n T_2(PT_2)^{n-1}P\delta_n + \gamma'_n v_n - x_n\right\| \\
&\leq \alpha'_n \left\|T_2(PT_2)^{n-1}P\delta_n - T_2(PT_2)^{n-1}x_n\right\| \\
&\quad + \alpha'_n \left\|T_2(PT_2)^{n-1}x_n - x_n\right\| + \gamma'_n\|v_n - x_n\| \\
&\leq L_2\|\delta_n - x_n\| + C'_n + \gamma'_n r \\
&\leq L_2 C'_n + C'_n + \gamma'_n r \\
&= (L_2+1)C'_n + \gamma'_n r.
\end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.9), we get

$$\|\sigma_n - x_n\| \leq (L_1+1)(L_2+1)C'_n + (L_1+1)\gamma'_n r + C_n. \tag{2.12}$$

It follows from (1.7) and (2.12) that

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le \left\| P\Big((1-\alpha_n-\gamma_n)x_n+\alpha_n T_1(PT_1)^{n-1}P\sigma_n+\gamma_n u_n \Big) - x_n \right\| \\
&\le \left\| T_1(PT_1)^{n-1}P\sigma_n - x_n \right\| + \gamma_n \|u_n - x_n\| \\
&\le \left\| T_1(PT_1)^{n-1}P\sigma_n - T_1(PT_1)^{n-1}x_n \right\| + \left\| T_1(PT_1)^{n-1}x_n - x_n \right\| + \gamma_n r \\
&\le L_1\|\sigma_n - x_n\| + C_n + \gamma_n r \\
&\le L_1\big((L_1+1)(L_2+1)C'_n + (L_1+1)\gamma'_n r + C_n \big) + C_n + \gamma_n r \\
&= (L_1+1)C_n + L_1(L_1+1)(L_2+1)C'_n + L_1(L_1+1)\gamma'_n r + \gamma_n r.
\end{aligned} \tag{2.13}$$

Using (2.11) and (2.13), we obtain

$$\begin{aligned}
\|\sigma_{n-1}-x_n\| &= \left\| (1-\beta_{n-1})y_{n-1} + \beta_{n-1}T_1(PT_1)^{n-2}y_{n-1} - x_n \right\| \\
&\le \beta_{n-1}\left\| T_1(PT_1)^{n-2}y_{n-1} - T_1(PT_1)^{n-2}x_{n-1} \right\| + \beta_{n-1}\left\| T_1(PT_1)^{n-2}x_{n-1} - x_{n-1} \right\| \\
&\quad + \beta_{n-1}\|x_n - x_{n-1}\| + (1-\beta_{n-1})\left\| y_{n-1} - x_n \right\| \\
&\le L_1\left\| y_{n-1} - x_{n-1} \right\| + C_{n-1} + \|x_n - x_{n-1}\| \\
&\quad + \left\| y_{n-1} - x_{n-1} \right\| + \|x_n - x_{n-1}\| \\
&\le (L_1+1)\left[(L_2+1)C'_{n-1} + \gamma'_{n-1} r \right] \\
&\quad + 2\begin{bmatrix} (L_1+1)C_{n-1} + L_1(L_1+1)(L_2+1)C'_{n-1} \\ +L_1(L_1+1)\gamma'_{n-1} r + \gamma_{n-1} r \end{bmatrix} + C_{n-1} \\
&= (2L_1+3)C_{n-1} + (2L_1+1)(L_1+1)(L_2+1)C'_{n-1} \\
&\quad + (2L_1+1)(L_1+1)\gamma'_{n-1} r + 2\gamma_{n-1} r.
\end{aligned} \tag{2.14}$$

Combine (2.13) with (2.14) yields that

$$\begin{aligned}
\left\|x_n-(PT_1)^{n-1}x_n\right\| &= \left\|x_n-T_1(PT_1)^{n-2}x_n\right\| \\
&\le \left\|(1-\alpha_{n-1}-\gamma_{n-1})x_{n-1}+\alpha_{n-1}T_1(PT_1)^{n-2}P\sigma_{n-1}+\gamma_{n-1}u_{n-1}-T_1(PT_1)^{n-2}x_n\right\| \\
&\le \alpha_{n-1}\left\|T_1(PT_1)^{n-2}P\sigma_{n-1}-T_1(PT_1)^{n-2}x_n\right\| \\
&\quad + (1-\alpha_{n-1})\left\|x_{n-1}-T_1(PT_1)^{n-2}x_n\right\| + \gamma_{n-1}\|u_{n-1}-x_{n-1}\| \\
&\le \left\|T_1(PT_1)^{n-2}P\sigma_{n-1}-T_1(PT_1)^{n-2}x_n\right\| \\
&\quad + \left\|x_{n-1}-T_1(PT_1)^{n-2}x_n\right\| + \gamma_{n-1}r \\
&\le L_1\|\sigma_{n-1}-x_n\| + \left\|x_{n-1}-T_1(PT_1)^{n-2}x_{n-1}\right\| \\
&\quad + \left\|T_1(PT_1)^{n-2}x_n-T_1(PT_1)^{n-2}x_{n-1}\right\| + \gamma_{n-1}r \\
&\le L_1\begin{bmatrix}(2L_1+3)C_{n-1}+(2L_1+1)(L_1+1)(L_2+1)C'_{n-1} \\ +(2L_1+1)(L_1+1)\gamma'_{n-1}r+2\gamma_{n-1}r\end{bmatrix} \\
&\quad + C_{n-1}+(L_1+1)C_{n-1}+L_1(L_1+1)(L_2+1)C'_{n-1} \\
&\quad + L_1(L_1+1)\gamma'_{n-1}r+2\gamma_{n-1}r \\
&= 2(L_1+1)^2C_{n-1}+2L_1(L_1+1)^2(L_2+1)C'_{n-1} \\
&\quad + 2L_1(L_1+1)^2\gamma'_{n-1}r+2(L_1+1)\gamma_{n-1}r,
\end{aligned} \tag{2.15}$$

from which it follows that

$$\begin{aligned}
\|x_n-T_1x_n\| &= \left\|x_n-T_1(PT_1)^{n-1}x_n+T_1(PT_1)^{n-1}x_n-T_1x_n\right\| \\
&\le \left\|x_n-T_1(PT_1)^{n-1}x_n\right\| + \left\|T_1(PT_1)^{n-1}x_n-T_1x_n\right\| \\
&\le C_n+L_1\left\|(PT_1)^{n-1}x_n-x_n\right\| \\
&\le C_n+2L_1(L_1+1)^2C_{n-1}+2L_1^2(L_1+1)^2(L_2+1)C'_{n-1} \\
&\quad + 2L_1^2(L_1+1)^2\gamma'_{n-1}r+2L_1(L_1+1)\gamma_{n-1}r.
\end{aligned} \tag{2.16}$$

It follows from $\lim_{n\to\infty} C_n = \lim_{n\to\infty} C'_n = 0$ that $\lim_{n\to\infty}\|x_n - T_1x_n\| = 0$. Similarly, we can show that $\lim_{n\to\infty}\|x_n - T_2x_n\| = 0$. This completes the proof. □

Lemma 2.3. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings of E with sequences $\{k_n\}$, $\{l_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$, $\sum_{n=1}^{\infty}(l_n-1) < \infty$, respectively, and $F(T_1)\cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are appropriate sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n = 1 = \alpha'_n+\beta'_n+\gamma'_n$, and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma'_n < \infty$. Moreover, $0 < a \le \alpha_n, \alpha'_n, \beta_n, \beta'_n \le b < 1$ for all $n \ge 1$ and some $a, b \in (0,1)$. Starting from an arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by the recursion* (1.7). *Then,*

$$\lim_{n\to\infty}\|x_n - T_1x_n\| = \lim_{n\to\infty}\|x_n - T_2x_n\| = 0. \tag{2.17}$$

Proof. Let $\sigma_n = (1-\beta_n)y_n+\beta_nT_1(PT_1)^{n-1}y_n$ and $\delta_n = (1-\beta'_n)x_n+\beta'_nT_2(PT_2)^{n-1}x_n$. By Lemma 2.1, we see that $\lim_{n\to\infty}\|x_n - p\|$ exists. Assume that $\lim_{n\to\infty}\|x_n - p\| = c$. If $c = 0$, then by the continuity of T_1 and T_2 the conclusion follows. Now, suppose $c > 0$. Taking lim sup on both sides in the inequalities (2.2), (2.3), and (2.4), we have

$$\limsup_{n\to\infty}\left\|\sigma_n - p\right\| \le c, \qquad \limsup_{n\to\infty}\left\|\delta_n - p\right\| \le c, \qquad \limsup_{n\to\infty}\left\|y_n - p\right\| \le c, \tag{2.18}$$

respectively. Next, we consider

$$\begin{aligned}\left\|T_1(PT_1)^{n-1}P\sigma_n - p + \gamma_n(u_n - x_n)\right\| &\le \left\|T_1(PT_1)^{n-1}P\sigma_n - p\right\| + \gamma_n\|u_n - x_n\| \\ &\le k_n\left\|\sigma_n - p\right\| + \gamma_n r.\end{aligned} \tag{2.19}$$

Taking lim sup on both sides in the above inequality and using (2.18), we get

$$\limsup_{n\to\infty}\left\|T_1(PT_1)^{n-1}P\sigma_n - p + \gamma_n(u_n - x_n)\right\| \le c. \tag{2.20}$$

Observe that

$$\left\|x_n - p + \gamma_n(u_n - x_n)\right\| \le \left\|x_n - p\right\| + \gamma_n\|u_n - x_n\| \le \left\|x_n - p\right\| + \gamma_n r, \tag{2.21}$$

which implies that

$$\limsup_{n\to\infty}\left\|x_n - p + \gamma_n(u_n - x_n)\right\| \le c. \tag{2.22}$$

$\limsup_{n\to\infty}\|x_{n+1} - p\| = c$ means that

$$\liminf_{n\to\infty}\left\|\alpha_n\left(T_1(PT_1)^{n-1}P\sigma_n - p + \gamma_n(u_n - x_n)\right) + (1-\alpha_n)\left(x_n - p + \gamma_n(u_n - x_n)\right)\right\| \ge c. \tag{2.23}$$

On the other hand, by using (2.23) and (2.5), we have

$$\begin{aligned}
&\left\| \alpha_n \left(T_1 (PT_1)^{n-1} P\sigma_n - p + \gamma_n (u_n - x_n) \right) + (1-\alpha_n)\left(x_n - p + \gamma_n (u_n - x_n)\right) \right\| \\
&\quad \le \alpha_n \left\| T_1 (PT_1)^{n-1} P\sigma_n - p \right\| + (1-\alpha_n)\|x_n - p\| + \gamma_n \|u_n - x_n\| \\
&\quad \le \alpha_n k_n \|\sigma_n - p\| + (1-\alpha_n)\|x_n - p\| + \gamma_n \|u_n - x_n\| \\
&\quad \le \alpha_n k_n \left(k_n l_n^2 \|x_n - p\| + k_n \gamma_n' r \right) + (1-\alpha_n)\|x_n - p\| + \gamma_n r \\
&\quad \le k_n^2 l_n^2 \|x_n - p\| + k_n^2 \gamma_n' r + \gamma_n r.
\end{aligned} \tag{2.24}$$

Therefore, we have

$$\limsup_{n\to\infty} \left\| \alpha_n \left(T_1 (PT_1)^{n-1} P\sigma_n - p + \gamma_n (u_n - x_n) \right) + (1-\alpha_n)\left(x_n - p + \gamma_n (u_n - x_n)\right) \right\| \le c. \tag{2.25}$$

Combining (2.23) with (2.25), we obtain

$$\lim_{n\to\infty} \left\| \alpha_n \left(T_1 (PT_1)^{n-1} P\sigma_n - p + \gamma_n (u_n - x_n) \right) + (1-\alpha_n)\left(x_n - p + \gamma_n (u_n - x_n)\right) \right\| = c. \tag{2.26}$$

Hence, applying Lemma 1.3, we find

$$\lim_{n\to\infty} \left\| T_1 (PT_1)^{n-1} P\sigma_n - x_n \right\| = 0. \tag{2.27}$$

Note that

$$\|x_n - p\| \le \left\| T_1 (PT_1)^{n-1} P\sigma_n - p \right\| + \left\| T_1 (PT_1)^{n-1} P\sigma_n - x_n \right\| \le k_n \|\sigma_n - p\| \tag{2.28}$$

which yields that

$$c \le \liminf_{n\to\infty} \|\sigma_n - p\| \le \limsup_{n\to\infty} \|\sigma_n - p\| \le c. \tag{2.29}$$

That is, $\lim_{n\to\infty}\|\sigma_n - p\| = c$. This implies that

$$\liminf_{n\to\infty} \left\| \beta_n \left(T_1 (PT_1)^{n-1} y_n - p \right) + (1-\beta_n)(y_n - p) \right\| \ge c. \tag{2.30}$$

Similarly, we have

$$\begin{aligned}&\left\|\beta_n\left(T_1(PT_1)^{n-1}y_n - p\right) + (1-\beta_n)(y_n - p)\right\| \\ &\quad\le \beta_n\left\|T_1(PT_1)^{n-1}y_n - p\right\| + (1-\beta_n)\|(y_n-p)\| \le k_n\|y_n - p\|,\end{aligned} \tag{2.31}$$

$$\limsup_{n\to\infty}\left\|\beta_n\left(T_1(PT_1)^{n-1}y_n - p\right) + (1-\beta_n)(y_n - p)\right\| \le c. \tag{2.32}$$

Combining (2.30) with (2.32), we obtain

$$\lim_{n\to\infty}\left\|\beta_n\left(T_1(PT_1)^{n-1}y_n - p\right) + (1-\beta_n)(y_n - p)\right\| = c. \tag{2.33}$$

On the other hand, we have

$$\left\|T_1(PT_1)^{n-1}y_n - \mathrm{p}\right\| \le k_n\|y_n - p\|, \tag{2.34}$$

$$\limsup_{n\to\infty}\left\|T_1(PT_1)^{n-1}y_n - p\right\| \le c. \tag{2.35}$$

Hence, using (2.32), (2.33), (2.35), and Lemma 1.3, we find

$$\lim_{n\to\infty}\left\|T_1(PT_1)^{n-1}y_n - y_n\right\| = 0. \tag{2.36}$$

Note that from (2.36), we have

$$\begin{aligned}\|\sigma_n - p\| &= \left\|(1-\beta_n)y_n + \beta_n T_1(PT_1)^{n-1}y_n - p\right\| \\ &\le (1-\beta_n)\|y_n - p\| + \beta_n\left\|T_1(PT_1)^{n-1}y_n - p\right\| \\ &\le (1-\beta_n)\|y_n - p\| + \beta_n\left\|T_1(PT_1)^{n-1}y_n - y_n\right\| + \beta_n\|y_n - p\| \\ &= \|y_n - p\|\end{aligned} \tag{2.37}$$

which yields that

$$c \le \liminf_{n\to\infty}\|y_n - p\| \le \limsup_{n\to\infty}\|y_n - p\| \le c. \tag{2.38}$$

That is, $\lim_{n\to\infty}\|y_n - p\| = c$.

Again, $\lim_{n\to\infty}\|y_n - p\| = c$ means that

$$\liminf_{n\to\infty}\left\|\alpha'_n\left(T_2(PT_2)^{n-1}P\delta_n - p + \gamma'_n(v_n - x_n)\right) + (1-\alpha'_n)\left(x_n - p + \gamma'_n(v_n - x_n)\right)\right\| \ge c. \tag{2.39}$$

By using (2.39) and (2.3), we obtain

$$\begin{aligned}
&\left\|\alpha_n'\left(T_2(PT_2)^{n-1}P\delta_n - p + \gamma_n'(v_n - x_n)\right) + (1-\alpha_n')\left(x_n - p + \gamma_n'(v_n - x_n)\right)\right\| \\
&\quad\le \alpha_n'\left\|T_2(PT_2)^{n-1}P\delta_n - p\right\| + (1-\alpha_n')\|x_n - p\| + \gamma_n'\|(v_n - x_n)\| \\
&\quad\le \alpha_n' l_n\|\delta_n - p\| + (1-\alpha_n')\|x_n - p\| + \gamma_n'\|(v_n - x_n)\| \\
&\quad\le \alpha_n' l_n^2\|x_n - p\| + (1-\alpha_n')\|x_n - p\| + \gamma_n' r \\
&\quad\le l_n^2\|x_n - p\| + \gamma_n' r.
\end{aligned} \tag{2.40}$$

Therefore, we have

$$\limsup_{n\to\infty}\left\|\alpha_n'\left(T_2(PT_2)^{n-1}P\delta_n - p + \gamma_n'(v_n - x_n)\right) + (1-\alpha_n')\left(x_n - p + \gamma_n'(v_n - x_n)\right)\right\| \le c. \tag{2.41}$$

Combining (2.39) with (2.41), we obtain

$$\lim_{n\to\infty}\left\|\alpha_n'\left(T_2(PT_2)^{n-1}P\delta_n - p + \gamma_n'(v_n - x_n)\right) + (1-\alpha_n')\left(x_n - p + \gamma_n'(v_n - x_n)\right)\right\| = c. \tag{2.42}$$

On the other hand, we have

$$\begin{aligned}
\left\|T_2(PT_2)^{n-1}P\delta_n - p + \gamma_n'(v_n - x_n)\right\| &\le \left\|T_2(PT_2)^{n-1}P\delta_n - p\right\| + \gamma_n'\|v_n - x_n\| \\
&\le l_n\|\delta_n - p\| + \gamma_n' r
\end{aligned} \tag{2.43}$$

which implies that

$$\limsup_{n\to\infty}\left\|T_2(PT_2)^{n-1}P\delta_n - p + \gamma_n'(v_n - x_n)\right\| \le c. \tag{2.44}$$

Notice that

$$\|x_n - p + \gamma_n'(v_n - x_n)\| \le \|x_n - p\| + \gamma_n'\|v_n - x_n\| \le \|x_n - p\| + \gamma_n' r, \tag{2.45}$$

which implies that

$$\limsup_{n\to\infty}\|x_n - p + \gamma_n'(v_n - x_n)\| \le c. \tag{2.46}$$

Using (2.42), (2.44), (2.46), and Lemma 1.3, we find

$$\lim_{n\to\infty}\left\|T_2(PT_2)^{n-1}P\delta_n - x_n\right\| = 0. \tag{2.47}$$

Observe that

$$\|x_n - p\| \le \left\|T_2(PT_2)^{n-1}P\delta_n - x_n\right\| + \left\|T_2(PT_2)^{n-1}P\delta_n - p\right\| \le l_n\|\delta_n - p\| \tag{2.48}$$

which yields that

$$c \le \liminf_{n\to\infty}\|\delta_n - p\| \le \limsup_{n\to\infty}\|\delta_n - p\| \le c. \tag{2.49}$$

That is, $\lim_{n\to\infty}\|\delta_n - p\| = c$. This implies that

$$\liminf_{n\to\infty}\left\|\beta'_n\left(T_2(PT_2)^{n-1}x_n - p\right) + (1-\beta'_n)(x_n - p)\right\| \ge c. \tag{2.50}$$

Similarly, we have

$$\begin{aligned}&\left\|\beta'_n\left(T_2(PT_2)^{n-1}x_n - p\right) + (1-\beta'_n)(x_n - p)\right\|\\ &\quad\le \beta'_n\left\|T_2(PT_2)^{n-1}x_n - p\right\| + (1-\beta'_n)\|x_n - p\| \le l_n\|x_n - p\|,\end{aligned} \tag{2.51}$$

$$\limsup_{n\to\infty}\left\|\beta'_n\left(T_2(PT_2)^{n-1}x_n - p\right) + (1-\beta'_n)(x_n - p)\right\| \le c. \tag{2.52}$$

Combining (2.50) with (2.52), we obtain

$$\lim_{n\to\infty}\left\|\beta'_n\left(T_2(PT_2)^{n-1}x_n - p\right) + (1-\beta'_n)(x_n - p)\right\| = c. \tag{2.53}$$

On the other hand, we have

$$\begin{aligned}&\left\|T_2(PT_2)^{n-1}x_n - p\right\| \le l_n\|x_n - p\|,\\ &\limsup_{n\to\infty}\left\|T_2(PT_2)^{n-1}x_n - p\right\| \le c,\end{aligned} \tag{2.54}$$

$$\limsup_{n\to\infty}\|x_n - p\| \le c. \tag{2.55}$$

Hence, using (2.53), (2.54), (2.55), and Lemma 1.3, we find

$$\lim_{n\to\infty}\left\|T_2(PT_2)^{n-1}x_n - x_n\right\| = 0. \tag{2.56}$$

In addition, from $y_n = P((1-\alpha'_n-\gamma'_n)x_n+\alpha'_n T_2(PT_2)^{n-1}P\delta_n+\gamma'_n v_n)$ and (2.47), we have

$$\begin{aligned}\|y_n-x_n\| &= \left\|P\left((1-\alpha'_n-\gamma'_n)x_n+\alpha'_n T_2(PT_2)^{n-1}P\delta_n+\gamma'_n v_n\right)-x_n\right\| \\ &\le \alpha'_n\left\|T_2(PT_2)^{n-1}P\delta_n-x_n\right\|+\gamma'_n\|v_n-x_n\| \\ &\le \left\|T_2(PT_2)^{n-1}P\delta_n-x_n\right\|+\gamma'_n r. \\ &\longrightarrow 0,\quad (\text{as } n\longrightarrow\infty).\end{aligned} \tag{2.57}$$

Hence, from (2.36) and (2.57), we find

$$\begin{aligned}\left\|T_1(PT_1)^{n-1}x_n-x_n\right\| &\le \left\|T_1(PT_1)^{n-1}x_n-T_1(PT_1)^{n-1}y_n\right\| \\ &\quad+\left\|T_1(PT_1)^{n-1}y_n-y_n\right\|+\|y_n-x_n\| \\ &\le k_n\|y_n-x_n\|+\left\|T_1(PT_1)^{n-1}y_n-y_n\right\|+\|y_n-x_n\| \\ &\longrightarrow 0,\quad (\text{as } n\longrightarrow\infty).\end{aligned} \tag{2.58}$$

That is,

$$\lim_{n\to\infty}\left\|T_1(PT_1)^{n-1}x_n-x_n\right\|=0. \tag{2.59}$$

Since T_1 and T_2 are uniformly L_1-Lipschitzian and uniformly L_2-Lipschitzian, respectively, for some $L_1, L_2 \ge 0$, it follows from (2.56), (2.59), and Lemma 2.2 that

$$\lim_{n\to\infty}\|x_n-T_1x_n\| = \lim_{n\to\infty}\|x_n-T_2x_n\|=0. \tag{2.60}$$

This completes the proof. □

Theorem 2.4. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1)<\infty$, $\sum_{n=1}^{\infty}(l_n-1)<\infty$, respectively, and $F(T_1)\cap F(T_2)\neq\emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are appropriate sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n=1=\alpha'_n+\beta'_n+\gamma'_n$, and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\gamma'_n<\infty$. Moreover, $0<a\le\alpha_n,\alpha'_n,\beta_n,\beta'_n\le b<1$ for all $n\ge1$ and some $a,b\in(0,1)$. If one of T_1 and T_2 is completely continuous, then the sequence $\{x_n\}$ defined by the recursion* (1.7) *converges strongly to some common fixed point of T_1 and T_2.*

Proof. By Lemma 2.1, $\{x_n\}$ is bounded. In addition, by Lemma 2.3; $\lim_{n\to\infty}\|x_n-T_1x_n\| = \lim_{n\to\infty}\|x_n-T_2x_n\|=0$; then $\{T_1x_n\}$ and $\{T_2x_n\}$ are also bounded. If T_1 is completely continuous, there exists subsequence $\{T_1x_{n_j}\}$ of $\{T_1x_n\}$ such that $T_1x_{n_j}\to p$ as $j\to\infty$. It follows from Lemma 2.3 that $\lim_{j\to\infty}\|x_{nj}-T_1x_{nj}\| = \lim_{j\to\infty}\|x_{nj}-T_2x_{nj}\|=0$. So by the continuity of T_1 and Lemma 1.4, we have $\lim_{j\to\infty}\|x_{nj}-p\|=0$ and $p\in F(T_1)\cap F(T_2)$.

Furthermore, by Lemma 2.1, we get that $\lim_{n\to\infty}\|x_n - p\|$ exists. Thus $\lim_{n\to\infty}\|x_n - p\| = 0$. The proof is completed. □

The following result gives a strong convergence theorem for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space satisfying condition (A').

Theorem 2.5. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n-1) < \infty$, $\sum_{n=1}^{\infty}(l_n-1) < \infty$, respectively, and $F(T_1)\cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are appropriate sequences in $[0,1]$ satisfying $\alpha_n+\beta_n+\gamma_n = 1 = \alpha'_n+\beta'_n+\gamma'_n$, and $\{u_n\}, \{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma'_n < \infty$. Moreover, $0 < a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0,1)$. Suppose that T_1 and T_2 satisfy condition (A'). Then, the sequence $\{x_n\}$ defined by the recursion* (1.7) *converges strongly to some common fixed point of T_1 and T_2.*

Proof. By Lemma 2.1, we readily see that $\lim_{n\to\infty}\|x_n-p\|$ and so, $\lim_{n\to\infty}d(x_n, F(T_1)\cap F(T_2))$ exists for all $p \in F(T_1)\cap F(T_2)$. Also, by Lemma 2.3, $\lim_{n\to\infty}\|T_1x_n-x_n\| = \lim_{n\to\infty}\|T_2x_n-x_n\| = 0$. It follows from condition (A') that

$$\lim_{n\to\infty} f(d(x_n, F(T_1)\cap F(T_2))) \leq \lim_{n\to\infty}\left(\frac{1}{2}(\|x_n - T_1x_n\| + \|x_n - T_2x_n\|)\right) = 0. \tag{2.61}$$

That is,

$$\lim_{n\to\infty} f(d(x_n, F(T_1)\cap F(T_2))) = 0. \tag{2.62}$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$, therefore, we have

$$\lim_{n\to\infty} d(x_n, F(T_1)\cap F(T_2)) = 0. \tag{2.63}$$

Now we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and sequence $\{y_j\} \subset F$ such that $\|x_{n_j}-y_j\| < 2^{-j}$ for all integers $j \geq 1$. Using the proof method of Tan and Xu [5], we have

$$\left\|x_{n_{j+1}} - y_j\right\| \leq \left\|x_{n_j} - y_j\right\| < 2^{-j}, \tag{2.64}$$

and hence

$$\|y_{j+1} - y_j\| \leq \left\|y_{j+1} - x_{n_{j+1}}\right\| + \left\|x_{n_{j+1}} - y_j\right\| \leq 2^{-(j+1)} + 2^{-j} < 2^{-j+1}. \tag{2.65}$$

We get that $\{y_j\}$ is a Cauchy sequence in F and so it converges. Let $y_j \to y$. Since F is closed, therefore, $y \in F$ and then $x_{n_j} \to y$. As $\lim_{n\to\infty}\|x_n - p\|$ exists, $x_n \to y \in F(T_1)\cap F(T_2)$. Thereby completing the proof. □

Remark 2.6. If $\gamma_n = \gamma'_n = \beta_n = \beta'_n = 0$, then the iterative scheme (1.7) reduces to the iterative scheme (1.4) of [9]. Moreover, the condition (A') is weaker than both the compactness of K and the semicompactness of the asymptotically nonexpansive nonself-mappings $T_1, T_2 : K \to E$. Also, the condition $0 < a \le \alpha_n, \alpha'_n \le b < 1$ for all $n \ge 1$ is weaker than the condition $0 < \varepsilon \le \alpha_{\mathrm{n}}, \alpha'_n, \le 1 - \varepsilon$, for all $n \ge 1$ and some $\varepsilon \in [0, 1)$. Hence, Theorems 2.4 and 2.5 generalize Theorems 3.3 and 3.4 in [9], respectively.

In the next result, we prove the weak convergence of the iterative scheme (1.7) for two asymptotically nonexpansive nonself-mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 2.7. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \ne \emptyset$. Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are appropriate sequences in $[0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, and $\{u_n\}, \{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Moreover, $0 < a \le \alpha_n, \alpha'_n, \beta_n, \beta'_n \le b < 1$ for all $n \ge 1$ and some $a, b \in (0, 1)$. Suppose that T_1 and T_2 satisfy Opial's condition. Then, the sequence $\{x_n\}$ defined by the recursion* (1.7) *converges weakly to some common fixed point of T and T_2.*

Proof. Let $p \in \mathrm{F}(T_1) \cap F(T_2)$. By Lemma 2.1, we see that $\lim_{n\to\infty}\|x_n - p\|$ exists and $\{x_n\}$ bounded. Now we prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T_1) \cap F(T_2)$. Firstly, suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ converge weakly to p_1 and p_2, respectively. By Lemma 2.3, we have $\lim_{n\to\infty}\|x_{n_k} - T_1 x_{n_k}\| = 0$. And Lemma 1.4 guarantees that $(I - T_1)p_1 = 0$, that is., $T_1 p_1 = p_1$. Similarly, $T_2 p_1 = p_1$. Again in the same way, we can prove that $p_2 \in F(T_1) \cap F(T_2)$.

Secondly, assume $p_1 \ne p_2$, then by Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - p_1\| &= \lim_{k\to\infty}\|x_{n_k} - p_1\| < \lim_{k\to\infty}\|x_{n_k} - p_2\| \\
&= \lim_{j\to\infty}\left\|x_{n_j} - p_2\right\| < \lim_{k\to\infty}\|x_{n_k} - p_1\| \\
&= \lim_{n\to\infty}\|x_n - p_1\|,
\end{aligned} \tag{2.66}$$

which is a contradiction, hence, $p_1 = p_2$. Then, $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2. This completes the proof. □

Remark 2.8. The above Theorem generalizes Theorem 3.5 of Wang [9].

3. Case of Two Nonself-Nonexpansive Mappings

Let $T_1, T_2 : K \to E$ be two nonexpansive nonself-mappings. Then, the iterative scheme (1.7) is written as follows:

$$\begin{aligned}
x_{n+1} &= P\big((1 - \alpha_n - \gamma_n)x_n + \alpha_n T_1 P(1 - \beta_n)y_n + \beta_n T_1 y_n + \gamma_n u_n\big), \\
y_n &= P\big((1 - \alpha'_n - \gamma'_n)x_n + \alpha'_n T_2 P\big((1 - \beta'_n)x_n + \beta'_n T_2 x_n\big) + \gamma'_n v_n\big), \quad x_1 \in K, n \ge 1.
\end{aligned} \tag{3.1}$$

Nothing prevents one from proving the results of the previous section for nonexpansive nonself-mappings case. Thus, one can easily prove the following.

Theorem 3.1. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be two nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are appropriate sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Moreover, $0 < a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0,1)$. Suppose that T_1 and T_2 satisfy condition (A'). Then, the sequence $\{x_n\}$ defined by the recursion* (3.1) *converges strongly to some common fixed point of T_1 and T_2.*

Theorem 3.2. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let $T_1, T_2 : K \to E$ be two nonexpansive nonself-mappings of E with sequences $\{k_n\}, \{l_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty}(l_n - 1) < \infty$, respectively, and $F(T_1) \cap F(T_2) \neq \emptyset$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are appropriate sequences in $[0,1]$ satisfying $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$, and $\{u_n\}$, $\{v_n\}$ are bounded sequences in K such that $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma'_n < \infty$. Moreover, $0 < a \leq \alpha_n, \alpha'_n, \beta_n, \beta'_n \leq b < 1$ for all $n \geq 1$ and some $a, b \in (0,1)$. Suppose that T_1 and T_2 satisfy Opial's condition. Then, the sequence $\{x_n\}$ defined by the recursion* (3.1) *converges weakly to some common fixed point of T and T_2.*

Remark 3.3. If $T_1 = T_2 = T$ and T is a nonexpansive nonself-mapping, then the iterative scheme (3.1) reduces to the iterative scheme (1.6) of Thianwan [11]. Then, Theorems 3.1-3.2 generalize Theorems 2.4 and 2.6 in [11], respectively.

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