Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2009, Article ID 975901, 8 pages
doi:10.1155/2009/975901

Research Article

Multiresolution Analysis and Haar Wavelets on the Laguerre Hypergroup

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Received 26 September 2008; Accepted 7 April 2009

Recommended by Alexander P. Seyranian

Let $\mathbb{H}^n$ be the Heisenberg group. The fundamental manifold of the radial function space for $\mathbb{H}^n$ can be denoted by $[0,+\infty)\times\mathbb{R}$, which is just the Laguerre hypergroup. In this paper the multiresolution analysis on the Laguerre hypergroup $\mathbb{K} = [0,+\infty) \times \mathbb{R}$ is defined. Moreover the properties of Haar wavelet bases for $L_a^2(\mathbb{K})$ are investigated.

1. Introduction

In the past decade research on the multiresolution analysis has made considerable progress due to its wide applications. For the basic theory of multiresolution we refer readers to the work in [1, 2]. Recently, we find that a lot of authors try to extend the theory of wavelets on the Euclidean space to nilpotent Lie groups (see [3–6]).

In this paper we will give the definition of acceptable dilations on the Laguerre hypergroup. The multiresolution analysis on the Laguerre hypergroup $\mathbb{K} = [0,+\infty) \times \mathbb{R}$ is also defined. Moreover the properties of Haar wavelet bases for $L_a^2(\mathbb{K})$ are investigated. We will prove the results analogous to those on $\mathbb{R}^n$ in [2], on $\mathbb{H}^n$ in [6], and on $\mathbb{H}^1 \times \mathbb{H}^1 \times \cdots \times \mathbb{H}^1$ in [7].

Let $dm_a(x,t)$ be the positive measure defined on $\mathbb{K}$, for $a \geq 0$, by

$$dm_a(x,t) = \frac{1}{\pi\Gamma(a+1)} x^{2a+1} dx dt; \tag{1.1}$$

and $L_a^2(\mathbb{K})$ denotes the space of all measurable functions on $\mathbb{K}$ such that

$$\|f\|_{L_a^2}^2 = \int_{\mathbb{K}} |f(x,t)|^2 dm_a(x,t) < \infty. \tag{1.2}$$

The generalized translation operator $T_{(x,t)}^a$ on $\mathbb{K}$ is defined by

$$T_{(x,t)}^a f(y,s) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f\left((x^2+y^2+2xy\cos\theta)^{1/2},\ t+s+xy\sin\theta\right)d\theta, & \text{if } a=0, \\ \dfrac{a}{\pi}\displaystyle\int_0^1\int_0^{2\pi} f\left((x^2+y^2+2xy\rho\cos\theta)^{1/2}, t+s+xy\rho\sin\theta\right)\rho(1-\rho^2)^{a-1}d\theta d\rho, & \text{if } a>0, \end{cases} \tag{1.3}$$

for all $(x,t)\in\mathbb{K}$, $f\in L_a^2(\mathbb{K})$. It is said to be the Fourier transform of a function $f\in L_a^2(\mathbb{K})$ defined as follows:

$$\widehat{f}(\lambda,m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t)f(x,t)dm_a(x,t), \tag{1.4}$$

where $\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}_m^a(|\lambda|x^2)$, and the Laguerre function $\mathcal{L}_m^a$ is defined on $\mathbb{R}^+$ by $\mathcal{L}_m^a(x) = e^{-x/2}(L_m^a(x)/L_m^a(0))$, and L_m^a is the Laguerre polynomial of degree m and order a. We know that for a pair of functions f and g, the generalized convolution product on the Laguerre hypergroup is defined by

$$f*g(x,t) = \int_{\mathbb{K}} T_{(x,t)}^a f(y,s)g(y,-s)dm_a(y,s), \quad \forall\ (x,t)\in\mathbb{K}. \tag{1.5}$$

Further if f and g are in $L^1(\mathbb{K})$, then we have

$$\widehat{f*g} = \widehat{f}\cdot\widehat{g}. \tag{1.6}$$

The functional analysis and Fourier analysis on $\mathbb{K}$ and its dual have been extensively studied in [8, 9].

Let $\Gamma = \{(m,n) : m\in\mathbb{N}, n\in\mathbb{Z}\}$ be a discrete subspace of $\mathbb{K}$. An automorphism D is said to be an acceptable dilation for Γ if it satisfies the following properties:

(1) D leaves Γ invariant, that is, $D\Gamma\subseteq\Gamma$,

(2) all the eigenvalues, λ_i, of D satisfy $|\lambda_i|>1$.

The acceptable dilation D on $L_a^2(\mathbb{K})$ is defined by $Df(x,t) = f(D(x,y))$, for all $f\in L_a^2(\mathbb{K})$. Let δ_r $(r>0)$ be the dilation on the Laguerre hypergroup. Hence for all $(x,y)\in\mathbb{K}$, $\delta_r(x,y) = (rx, r^2y)$. Clearly, for every $r\in\mathbb{N}$ and $r\geq 2$, δ_r is just an acceptable dilation on the Laguerre hypergroup. Now we give the definition of multiresolution analysis on the Laguerre hypergroup.

Definition 1.1 ((MRA($\mathbb{K}$), Γ, D)). A multiresolution analysis on $\mathbb{K}$ is an increasing sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L_a^2(\mathbb{K})$ satisfying the following conditions:

(1) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_a^2(\mathbb{K})$;

(2) $f \in V_j \Leftrightarrow Df \in V_{j+1}$;

(3) $f \in V_0 \Leftrightarrow T_\gamma^a f \in V_0$, for all $\gamma \in \Gamma$;

(4) there exists a scaling function $\phi \in V_0$ such that $\{T_\gamma^a \phi\}_{\gamma\in\Gamma}$ forms an orthonormal basis of V_0.

From the above definition it is clear that $\{DT_\gamma^a \phi\}_{\gamma\in\Gamma}$ is an orthonormal basis of V_1. It follows from $V_0 \subseteq V_1$ and $\phi \in V_0 \subseteq V_1$ that there exists a sequence $\{h(\gamma)\}_{\gamma\in\Gamma}$ such that

$$\phi = \sum_{\gamma\in\Gamma} h(\gamma) DT_\gamma^a \phi. \tag{1.7}$$

The solution of (1.7) is often called a refinable function or a scaling function and $\{h(\gamma)\}_{\gamma\in\Gamma}$ is called a refinement sequence.

2. Acceptable Dilations on the Laguerre Hypergroup

In this section we will investigate the acceptable dilations on the Laguerre hypergroup. From the previous argument, we know that the acceptable dilations on the Laguerre hypergroup must satisfy three conditions:

(1) they must be a automorphism of Laguerre hypergroup;

(2) they must leave Γ invariant;

(3) the modulus of their eigenvalues must be more than 1.

Theorem 2.1. *The acceptable dilations on $\mathbb{K}$ must be the form*

$$D = \begin{pmatrix} k_1 & 0 \\ k_2 & k_3 \end{pmatrix}, \tag{2.1}$$

where k_1, k_2, $k_3 \in \mathbb{Z}$ and $k_1 > 1$, $|k_3| > 1$.

Proof. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be the acceptable dilations on $\mathbb{K}$, where $a, b, c, d \in \mathbb{R}$. From the condition (1), we can obtain

$$D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by & cx + dy \end{pmatrix} \in \mathbb{K}, \quad \forall (x, t) \in \mathbb{K}, \tag{2.2}$$

which implies that $ax + by \geq 0$ for all $x \geq 0$ and $y \in \mathbb{R}$. This yields $b = 0$ and $a \geq 0$. From $D\Gamma \subseteq \Gamma$, we get $a, c, d \in \mathbb{Z}$. By using the condition (3) we can obtain that $a > 1$ and $|d| > 1$. This concludes the proof of the theorem. □

3. Multiresolution Analysis on the Laguerre Hypergroup

In this section, we only consider the dilation δ_r, where $r \in \mathbb{N}$ and $r > 1$. For simplicity we denote it by $\delta_r = \alpha$. In order to obtain the main theorem, we need to give some lemmas to characterize the properties of the multiresolution analysis on $\mathbb{K}$.

Lemma 3.1. *Suppose* $V_j \subseteq V_{j+1}$ $(j \in \mathbb{Z})$ *where* $V_j \subset L_a^2(\mathbb{K})$ *and* $\{V_j\}_{j\in\mathbb{Z}}$ *satisfies (2) and (4) of the definition of multiresolution analysis on the Laguerre hypergroup. The characteristic function* χ_Q *of the set* Q *is a scaling function of multiresolution analysis. Then* $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Proof. Let $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1 \neq \gamma_2$. By using the property (4) of the Definition 1.1 we can obtain

$$\langle T_{\gamma_1}^a \chi_Q, T_{\gamma_2}^a \chi_Q \rangle = 0, \tag{3.1}$$

which implies that $\int_{\mathbb{K}} T_{\gamma_1}^a \chi_Q T_{\gamma_2}^a \chi_Q dm_a = 0$. From (1.3), we know that $T_\gamma^a \chi_Q \geq 0$ and there exists a constant $C > 0$ such that $|T_\gamma^a \chi_Q| \leq C$ for all $a \geq 0$ and $\gamma \in \mathbb{K}$. This yields $T_{\gamma_1}^a \chi_Q T_{\gamma_2}^a \chi_Q = 0$, which implies that $T_{\gamma_1}^a \chi_Q$ and $T_{\gamma_2}^a \chi_Q$ cannot be nonzero at the same time.

Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$. Then $f \in V_{-j}$ for any $j \in \mathbb{Z}$ which implies that $\alpha^j f \in V_0$. Thus there exists a sequence $\{b_j(\gamma)\}_{\gamma\in\Gamma}$ such that $\alpha^j f = \sum_{\gamma\in\Gamma} b_j(\gamma) T_\gamma^a \chi_Q$. This yields

$$\left|\alpha^j f\right| = \left|\sum_{\gamma\in\Gamma} b_j(\gamma) T_\gamma^a \chi_Q\right| \leq \sup_{\gamma\in\Gamma}|b_j(\gamma)| \sum_{r\in\Gamma}\left|T_\gamma^a \chi_Q\right| \leq \|\{b_j(\gamma)\}\|_{l^2} \sum_{\gamma\in\Gamma}\left|T_\gamma^a \chi_Q\right|, \tag{3.2}$$

which implies that $|\alpha^j f| \leq C\|\{b_j(\gamma)\}\|_{l^2}$. Then we can see that

$$|f(P)| = \left|\alpha^j\left(\alpha^{-j} f(P)\right)\right| = \left|\alpha^j f\left(\alpha^{-j} P\right)\right| \leq C\|\{b_j(\gamma)\}\|_{l^2(\Gamma)} = C\left\|\alpha^j f\right\|_{L_a^2}. \tag{3.3}$$

Notice that

$$\begin{aligned}
\left\|\alpha^j f\right\|_{L_a^2} &= \left(\int_K |\alpha^j f(P)|^2 dm_a\right)^{1/2} \\
&= \left(\int_K \left|f\left(\alpha^j P\right)\right|^2 dm_a\right)^{1/2} \\
&= \left(\int_{-\infty}^{+\infty}\int_0^{+\infty} \left|f\left(r^j x, r^{2j} y\right)\right|^2 \frac{x^{2a+1}}{\pi\Gamma(a+1)} dx dy\right)^{1/2} \\
&= r^{-(a+2j)} \|f\|_{L_a^2}.
\end{aligned} \tag{3.4}$$

If we let j tend to infinity, then we can obtain $f = 0$. This implies that $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. The desired result is thus obtained. □

Lemma 3.2. *Suppose* $V_j \subseteq V_{j+1}$ $(j \in \mathbb{Z})$, *where* $V_j \subset L_a^2(\mathbb{K})$ *and* $\{V_j\}_{j\in\mathbb{Z}}$ *satisfies (2), (3), and (4) of the definition of multiresolution analysis on the Laguerre hypergroup. If the scaling function* ϕ *in (4) is in* $L_a^1(\mathbb{K})$ *and* $\int_{\mathbb{K}} \phi dm_a \neq 0$, *then* $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_a^2(\mathbb{K})$.

Proof. Let $P=(x,t)\in\mathbb{K}$ and $a>0$. Then we have

$$
\begin{aligned}
&T^a_{\alpha P}f(y,s)\\
&\quad=T^a_{(rx,r^2t)}f(y,s)\\
&\quad=\frac{a}{\pi}\int_0^{2\pi}\int_0^1 f\left(\left(r^2x^2+y^2+2rxyr'\cos\theta\right)^{1/2},s+r^2t+rxyr'\sin\theta\right)r\prime(1-r'^2)^{a-1}dr'd\theta\\
&\quad=\frac{a}{\pi}\int_0^{2\pi}\int_0^1 f\left(r\left(x^2+\frac{y}{r^2}^2+2x\frac{y}{r}r'\cos\theta\right)^{1/2},r^2\left(\frac{s}{r^2}+t+x\frac{y}{r}r'\sin\theta\right)\right)r'(1-r'^2)^{a-1}dr'd\theta\\
&\quad=T^a_{(x,t)}\delta_r f\left(\frac{y}{r},\frac{s}{r^2}\right)\\
&\quad=\alpha^{-1}T^a_{(x,t)}\alpha f,
\end{aligned}
\tag{3.5}
$$

which implies $T^a_{\alpha P}=\alpha^{-1}T^a_P\alpha$. For $a=0$, we can get the same result. It is easy to see that $T^a_{\alpha^l P}=\alpha^{-l}T^a_P\alpha^l$, for all $l\in\Gamma$ and $a\ge 0$.

Let $\varphi\in\bigcup_{j\in\mathbb{Z}}V_j$. Then there exists a $j_0\in\mathbb{Z}$ such that $\varphi\in V_{j_0}$. For any $l\in\mathbb{Z}$, let $j>-l$ and $j\ge j_0$. Using $V_j\subseteq V_{j+1}$, we immediately obtain $\varphi\in V_j$. Then there exists a sequence $\{a_j(\gamma)\}_{\gamma\in\Gamma}\in l^2(\Gamma)$ such that $\varphi=\sum_{\gamma\in\Gamma}a_j(\gamma)\alpha^jT^a_\gamma\phi$, which implies

$$
T^a_{\alpha^l(P)}\varphi=\sum_{\gamma\in\Gamma}a_j(\gamma)T^a_{\alpha^l(P)}\alpha^jT^a_\gamma\phi=\sum_{\gamma\in\Gamma}a_j(\gamma)\alpha^jT^a_{\alpha^{l+j}(P)}T^a_\gamma\phi. \tag{3.6}
$$

Notice that $P\in\Gamma$, $l+j>0$, and $l+j\in\mathbb{Z}$. Thus we can see that $\alpha^{l+j}(P)\in\Gamma$ and $\alpha^jT^a_{\alpha^{l+j}(P)}T^a_\gamma\phi\in V_j$, which implies $T^a_{\alpha^l(P)}\varphi\in V_j\subseteq\overline{\bigcup_{j\in\mathbb{Z}}V_j}$, for all $l\in\mathbb{Z}$ and $P\in\Gamma$.

Let $\psi\in\overline{\bigcup_{j\in\mathbb{Z}}V_j}$. Then for any $\varepsilon>0$, there exists a $\varphi\in\bigcup_{j\in\mathbb{Z}}V_j$ such that $\|\varphi-\psi\|_{L^2_a}<\varepsilon$. It follows from $\|T^a_{\alpha^l(P)}\varphi-T^a_{\alpha^l(P)}\psi\|_{L^2_a}=\|T^a_{\alpha^l(P)}(\varphi-\psi)\|_{L^2_a}\le\|\varphi-\psi\|_{L^2_a}<\varepsilon$ and $T^a_{\alpha^l(P)}\varphi\in\overline{\bigcup_{j\in\mathbb{Z}}V_j}$ that $T^a_{\alpha^l(P)}\psi\in\overline{\bigcup_{j\in\mathbb{Z}}V_j}$, for all $l\in\mathbb{Z}$ and $P\in\Gamma$.

For any $g\in\mathbb{K}$, there must exist an element $P\in\Gamma$ and $l\in\mathbb{Z}$ such that $|\alpha^l(P)-g|$ is arbitrarily small, which implies that $\|T^a_{\alpha^l(P)}\psi-T^a_g\psi\|_2<\varepsilon$ for any arbitrarily small $\varepsilon>0$. This yields $T^a_g\psi\in\overline{\bigcup_{j\in\mathbb{Z}}V_j}$, for all $\psi\in\overline{\bigcup_{j\in\mathbb{Z}}V_j}$ and $g\in\mathbb{K}$.

Note $\widehat{\phi}(\lambda,m)=\int_{\mathbb{K}}\varphi_{-\lambda,m}(x,t)\phi(x,t)dm_a$ and $\varphi_{\lambda,m}(x,t)=e^{i\lambda t}\mathcal{L}^a_m(|\lambda|x^2)$, $\phi\in L^1_a(\mathbb{K})$. This shows that $\widehat{\phi}(\lambda,m)\to\int_{\mathbb{K}}\phi dm_a$, *when* $\lambda\to 0$. Since $\int_{\mathbb{K}}\phi dm_a\neq 0$, there exists some $\varepsilon>0$ such that $\widehat{\phi}(\lambda,m)\neq 0$ for all $|\lambda|<\varepsilon$. Let $W=\overline{\bigcup_{j\in\mathbb{Z}}V_j}$ and $\psi\in W^{\perp}$. Then $\langle\varphi,\psi\rangle=0$ for all $\varphi\in W$, which implies that for all $g\in\mathbb{K}$,

$$
0=\langle T^a_g\varphi,\psi\rangle=\int_K T^a_g\varphi(x,y)\psi(x,y)dm_a(x,y)=\int_K T^a_g\varphi(x,y)\widetilde{\psi}(x,-y)dm_a(x,y)=\varphi*\widetilde{\psi}(g), \tag{3.7}
$$

where $\widetilde{\varphi}(x,y)=\varphi(x,-y)$. Then $\widehat{\varphi*\widetilde{\varphi}}(\lambda,m)=\widehat{\varphi}(\lambda,m)\widehat{\widetilde{\varphi}}(\lambda,m)=0$. Notice that

$$\begin{aligned}
\widehat{\alpha f}(\lambda,m) &= \int_{\mathbb{K}}\varphi_{-\lambda,m}(x,t)f\left(rx,r^2t\right)dm_a \\
&= \frac{1}{r^{2a+4}}\int_{\mathbb{K}}\varphi_{-\lambda,m}\left(\frac{x'}{r},\frac{t'}{r^2}\right)f(x',t')dm_a, \\
\varphi_{\lambda,m}\left(\frac{x}{r},\frac{t}{r^2}\right) &= e^{i\lambda(t/r^2)}\mathcal{L}_m^a\left(|\lambda|\left(\frac{x}{r}\right)^2\right) \\
&= e^{i(\lambda/r^2)t}\mathcal{L}_m^a\left(\left|\frac{1}{r^2}\lambda\right|x^2\right) \\
&= \varphi_{\lambda/r^2,m}(x,t).
\end{aligned}\tag{3.8}$$

Thus we can see that $\widehat{\alpha f}(\lambda,m)=(1/r^{2a+4})\widehat{f}(\lambda/r^2,m)$, which implies $\widehat{\alpha^j f}(\lambda,m)=(1/r^{j(2a+4)})\widehat{f}(\lambda/r^{2j},m)$. Let $\varphi=r^{j(2a+4)}\alpha^j\phi$. Then $\varphi\in W$ and $\widehat{\varphi}=\widehat{\phi}(\lambda/r^{2j},m)$. This yields

$$\widehat{\phi}\left(\frac{\lambda}{r^{2j}},m\right)\widehat{\widetilde{\varphi}}(\lambda,m)=0.\tag{3.9}$$

Taking into account the fact that $\widehat{\phi}(\lambda/r^{2j},m)\neq 0$ when $|\lambda|<r^{2j}\varepsilon$, we see $\widehat{\widetilde{\varphi}}(\lambda,m)=0$ when $|\lambda|<r^{2j}\varepsilon$. Let j tend to infinity, then $\widehat{\widetilde{\varphi}}=0$ for all $\lambda\in\mathbb{R}$ which implies $\varphi=0$. Then $\overline{\bigcup_{j\in\mathbb{Z}}V_j}=L_a^2(\mathbb{K})$. We complete the proof of this theorem. □

Theorem 3.3. *Suppose $\phi=\chi_Q$ is a scaling function for a multiresolution analysis associated with (Γ,α), where χ_Q is the characteristic function of a measurable set Q . Then Q satisfies the following properties:*

(1) $T_{\gamma_1}^a\chi_Q T_{\gamma_2}^a\chi_Q=0$, *for a.e.* $x\in\mathbb{K}$, $\gamma_1\neq\gamma_2$ *and* $\gamma_1,\gamma_2\in\Gamma$;

(2) $\chi_Q=\sum_{\gamma\in\Gamma}\beta(\gamma)\alpha T_\gamma^a\chi_Q$;

(3) $|Q|=1$;

(4) $T_{\gamma_1}^a T_{\gamma_2}^a\chi_Q$ *can be represented by the sequence* $\{T_\gamma^a\chi_Q\}_{\gamma\in\Gamma}$ *where* $\gamma_1,\gamma_2\in\Gamma$.

Conversely, the characteristic function of a bounded measurable set Q that satisfies properties (1), (2), (3), and (4) is the scaling function of a multiresolution analysis associated with (Γ,α).

Proof. Suppose $\phi=\chi_Q$ is a scaling function for a multiresolution analysis associated with (Γ,α). Then $\langle T_{\gamma_1}^a\chi_Q,T_{\gamma_2}^a\chi_Q\rangle=0$ for all $\gamma_1\neq\gamma_2$ and $\gamma_1,\gamma_2\in\Gamma$, which implies

$$\int_{\mathbb{K}}T_{\gamma_1}^a\chi_Q T_{\gamma_2}^a\chi_Q dm_a=0.\tag{3.10}$$

Notice that $T_{\gamma_1}^a\chi_Q\geq 0$ and $T_{\gamma_2}^a\chi_Q\geq 0$. Thus we can obtain that $T_{\gamma_1}^a\chi_Q T_{\gamma_2}^a\chi_Q=0$, almost every $x\in\mathbb{K}$. By (1.7), we know that the second property is satisfied. Because of $\|\chi_Q\|_{L_a^2}=1$, we can see that $|Q|=1$. Let $V_0\in(\mathrm{MRA}(\mathbb{K}),\Gamma,\alpha)$. Then $T_{\gamma_2}^a\chi_Q\in V_0$. This implies $T_{\gamma_1}^a T_{\gamma_2}^a\chi_Q\in V_0$. Therefore, $T_{\gamma_1}^a T_{\gamma_2}^a\chi_Q$ can be represented by $\{T_\gamma^a\chi_Q\}$.

To see the converse, let

$$V_0 = \left\{ f \in L_a^2(\mathbb{K}) : f = \sum_{\gamma \in \Gamma} c(\gamma) T_\gamma^a \chi_Q \right\}, \qquad V_j = \alpha^j V_0. \tag{3.11}$$

Then $\{V_j\}_{j\in\mathbb{Z}}$ is a family of closed subspace of $L_a^2(\mathbb{K})$. Let $f \in V_0$. Then

$$\begin{aligned} f &= \sum_{\gamma\in\Gamma} c(\gamma) T_\gamma^a \chi_Q \\ &= \sum_{\gamma\in\Gamma} c(\gamma) T_\gamma^a \sum_{\gamma_1\in\Gamma} \beta(\gamma_1) \alpha T_{\gamma_1}^a \chi_Q \\ &= \sum_{\gamma,\gamma_1\in\Gamma} c(\gamma)\beta(\gamma_1) T_\gamma^a \alpha T_{\gamma_1}^a \chi_Q \\ &= \sum_{\gamma,\gamma_1\in\Gamma} c(\gamma)\beta(\gamma_1) \alpha T_{\alpha(\gamma)}^a T_{\gamma_1}^a \chi_Q. \end{aligned} \tag{3.12}$$

Since $\alpha(\gamma) \in \Gamma$, we can see that $T_{\alpha(\gamma)}^a T_{\gamma_1}^a \chi_Q \in V_0$, which implies $f \in V_1$. This yields $V_0 \subseteq V_1$. Then we can also get $V_j \subseteq V_{j+1}$. Notice that $f = \sum_{\gamma\in\Gamma} c(\gamma) T_\gamma^a \chi_Q$. Thus we can see that $T_{\gamma_1}^a f = \sum_{\gamma\in\Gamma} c(\gamma) T_{\gamma_1}^a T_\gamma^a \chi_Q$, for all $\gamma_1 \in \Gamma$. Because $T_{\gamma_1}^a T_{\gamma_2}^a \chi_Q$ can be represented by the sequence $\{T_\gamma^a \chi_Q\}$, thus $T_{\gamma_1}^a f \in V_0$.

In order to show that $\{V_j\}_{j\in\mathbb{Z}}$ is a multiresolution analysis associated with (Γ, α), it suffices to show that $\bigcap_{j\in\mathbb{Z}} V_j = 0$ and $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_a^2(\mathbb{K})$. Further, it follows easily from Lemmas 3.1 and 3.2 that

$$\bigcap_{j\in\mathbb{Z}} V_j = 0, \qquad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_a^2(\mathbb{K}). \tag{3.13}$$

Our result is proved. □

In this paper orthonormal Haar wavelet bases for $L_a^2(\mathbb{K})$ are not constructed. But we believe that orthonormal Haar wavelet bases for $L_a^2(\mathbb{K})$ can be constructed just as that in [2, 6, 7]. The details will appear elsewhere.

Acknowledgments

The second author is supported by the National Natural Science Foundation of China (no. 10671041) and the Doctoral Program Foundation of the Ministry of Education of China (no. 200810780002). The authors would be grateful to the referee for his/her invaluable suggestions.

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