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Research Article

Oscillation Theorems for Second-Order Quasilinear Neutral Functional Differential Equations

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New oscillation criteria are established for the second-order nonlinear neutral functional differential equations of the form $(r(t)|z'(t)|^{\alpha-1}z'(t))' + f(t, x[\sigma(t)]) = 0$, $t \geq t_0$, where $z(t) = x(t) + p(t)x(\tau(t))$, $p \in C^1([t_0,\infty),[0,\infty))$, and $\alpha \geq 1$. Our results improve and extend some known results in the literature. Some examples are also provided to show the importance of these results.

1. Introduction

This paper is concerned with the oscillation problem of the second-order nonlinear functional differential equation of the following form:

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + f(t, x[\sigma(t)]) = 0, \quad t \geq t_0, \tag{1.1}$$

where $\alpha \geq 1$ is a constant, $z(t) = x(t) + p(t)x[\tau(t)]$.

Throughout this paper, we will assume the following hypotheses:

(A_1) $r \in C^1([t_0,\infty),\mathbb{R})$, $r(t) > 0$ for $t \geq t_0$,

(A_2) $p \in C^1([t_0,\infty),[0,\infty))$,

(A_3) $\tau \in C^2([t_0,\infty),\mathbb{R})$, $\tau'(t) > 0$, $\lim_{t\to\infty}\tau(t) = \infty$,

(A_4) $\sigma \in C([t_0,\infty),\mathbb{R})$, $\lim_{t\to\infty}\sigma(t) = \infty$, $\tau \circ \sigma = \sigma \circ \tau$;

(A_5) $f(t,u) \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, and there exists a function $q \in C([t_0,\infty),[0,\infty))$ such that

$$f(t,u)\ \operatorname{sgn}\ u \geq q(t)|u|^{\alpha}, \quad \text{for } u \neq 0,\ t \geq t_0. \tag{1.2}$$

By a solution of (1.1), we mean a function $x \in C([T_x,\infty),\mathbb{R})$ for some $T_x \geq t_0$ which has the property that $r(t)|z'(t)|^{\alpha-1}z'(t) \in C^1([T_x,\infty),\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)$. As is customary, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$; otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all of its nonconstant solutions are oscillatory.

We note that neutral delay differential equations find numerous applications in electric networks. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines which rise in high-speed computers where the lossless transmission lines are used to interconnect switching circuits; see [1]. Therefore, there is constant interest in obtaining new sufficient conditions for the oscillation or nonoscillation of the solutions of varietal types of the second-order equations, see, e.g., papers [2–17].

Known oscillation criteria require various restrictions on the coefficients of the studied neutral differential equations.

Agarwal et al. [2], Chern et al. [3], Džurina and Stavroulakis [4], Kusano et al. [5, 6], Mirzov [7], and Sun and Meng [8] observed some similar properties between

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + q(t)|x[\sigma(t)]|^{\alpha-1}x[\sigma(t)] = 0 \tag{1.3}$$

and the corresponding linear equation

$$\left(r(t)x'(t)\right)' + q(t)x(t) = 0. \tag{1.4}$$

Liu and Bai [10], Xu and Meng [11, 12], and Dong [13] established some oscillation criteria for (1.3) with neutral term under the assumption that

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}\mathrm{d}t = \infty. \tag{1.5}$$

Han et al. [14] examined the oscillation of second-order linear neutral differential equation

$$\left(r(t)\left[x(t)+p(t)x(\tau(t))\right]'\right)' + q(t)x[\sigma(t)] = 0, \quad t \geq t_0, \tag{1.6}$$

where $\tau'(t) = \tau_0 > 0,\ 0 \leq p(t) \leq p_0 < \infty$, and obtained some oscillation criteria for (1.6) when

$$\int_{t_0}^{\infty} \frac{1}{r(t)}\mathrm{d}t = \infty. \tag{1.7}$$

Han et al. [15] studied the oscillation of (1.6) under the case $0 \le p(t) \le 1$ and

$$\int_{t_0}^{\infty} \frac{1}{r(t)} \mathrm{d}t < \infty. \tag{1.8}$$

Tripathy [16] considered the nonlinear dynamic equation of the form

$$\left(r(t)\left[\left(x(t)+p(t)x(t-\tau)\right)^{\Delta}\right]^{\gamma}\right)^{\Delta} + q(t)|x(t-\delta)|^{\gamma} \operatorname{sgn} x(t-\delta) = 0, \tag{1.9}$$

where $0 \le p(t) \le p_0 < \infty$, γ is a the ratios of two positive odd integers, and obtained some oscillation criteria under the following conditions:

$$\int_{t_0}^{\infty} \left(\frac{1}{r(t)}\right)^{\gamma} \mathrm{d}t = \infty. \tag{1.10}$$

Džurina [17] was concerned with the oscillation behavior of the solutions of the second-order neutral differential equations as follows

$$\left(a(t)\left[\left(x(t)+p(t)x(\tau(t))\right)'\right]^{\gamma}\right)' + q(t)x^{\beta}(\sigma(t)) = 0, \tag{1.11}$$

where $0 \le p(t) \le p_0 < \infty$, γ is a the ratios of two positive odd integers, and obtained some new results under the following conditions

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\gamma} \mathrm{d}t = \infty. \tag{1.12}$$

Our purpose of this paper is to establish some new oscillation criteria for (1.1), and we will also consider the cases (1.5) and

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)} \mathrm{d}t < \infty. \tag{1.13}$$

To the best of my knowledge, there is no result for the oscillation of (1.1) under the conditions both $0 \le p(t) \le p_0 < \infty$ and (1.13).

In this paper, we will use a new inequality to establish some oscillation criteria for (1.1) for the first time. Some examples will be given to show the importance of these results. In Sections 3 and 4, for the sake of convenience, we denote that

$$Q(t) := \min\{q(t), q[\tau(t)]\}, \qquad d_+(t) := \max\{0, d(t)\}, \qquad \xi(t) := \frac{\alpha p'[\sigma(t)]\sigma'(t)}{p[\sigma(t)]} - \frac{\tau''(t)}{\tau'(t)},$$
$$\zeta(t) := \frac{(\rho'(t))_+}{\rho(t)} + \xi(t), \qquad \varphi(t) := \left(\frac{\rho'_+(t)}{\rho(t)}\right)^{\alpha+1} + \frac{p^{\alpha}[\sigma(t)](\zeta_+(t))^{\alpha+1}}{\tau'(t)}, \qquad \delta(t) := \int_{\eta(t)}^{\infty} \frac{\mathrm{d}s}{r^{1/\alpha}(s)}. \tag{1.14}$$

2. Lemma

In this section, we give the following lemma, which we will use in the proofs of our main results.

Lemma 2.1. *Assume that* $\alpha \geq 1,\ a, b \in \mathbb{R}$. *If* $a \geq 0,\ b \geq 0$, *then one has*

$$a^\alpha + b^\alpha \geq \frac{1}{2^{\alpha-1}}(a+b)^\alpha. \tag{2.1}$$

Proof. (i) Suppose that $a = 0$ or $b = 0$. Then we have (2.1). (ii) Suppose that $a > 0,\ b > 0$. Define the function g by $g(x) = x^\alpha,\ x \in (0, \infty)$. Then $g''(x) = \alpha(\alpha-1)x^{\alpha-2} \geq 0$ for $x > 0$. Thus, g is a convex function. By the definition of convex function, for $\lambda = 1/2,\ a, b \in (0, \infty)$, we have

$$g\left(\frac{a+b}{2}\right) \leq \frac{g(a)+g(b)}{2}, \tag{2.2}$$

that is,

$$a^\alpha + b^\alpha \geq \frac{1}{2^{\alpha-1}}(a+b)^\alpha. \tag{2.3}$$

This completes the proof. □

3. Oscillation Criteria for the Case (1.5)

In this section, we will establish some oscillation criteria for (1.1) under the case (1.5).

Theorem 3.1. *Suppose that* (1.5) *holds,* $\sigma \in C([t_0, \infty), \mathbb{R}),\ \sigma'(t) > 0,\ \sigma(t) \leq t$, *and* $\sigma(t) \leq \tau(t)$ *for* $t \geq t_0$. *Furthermore, assume that there exists a function* $\rho \in C([t_0, \infty), (0, \infty))$ *such that*

$$\limsup_{t \to \infty} \int_{t_0}^{t} \rho(s) \left\{ \frac{Q(s)}{2^{\alpha-1}} - \frac{r[\sigma(s)]\varphi(s)}{(\alpha+1)^{\alpha+1}(\sigma'(s))^\alpha} \right\} ds = \infty. \tag{3.1}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0,\ x[\tau(t)] > 0$ and $x[\sigma(t)] > 0$ for all $t \geq t_1$. By applying (1.1), for all sufficiently large t, we obtain that

$$\begin{aligned} &\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + q(t)x^\alpha[\sigma(t)] + q[\tau(t)]p^\alpha[\sigma(t)]x^\alpha(\sigma[\tau(t)]) \\ &\quad + \frac{p^\alpha[\sigma(t)]}{\tau'(t)}\left(r[\tau(t)]\left|z'[\tau(t)]\right|^{\alpha-1}z'[\tau(t)]\right)' \leq 0. \end{aligned} \tag{3.2}$$

Using (2.1) and the definition of z, we conclude that

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + \frac{1}{2^{\alpha-1}}Q(t)z^{\alpha}[\sigma(t)] + \frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\left(r[\tau(t)]\left|z'[\tau(t)]\right|^{\alpha-1}z'[\tau(t)]\right)' \leq 0. \quad (3.3)$$

In view of (1.1), we obtain that

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' \leq -q(t)x^{\alpha}[\sigma(t)] \leq 0, \quad t \geq t_1. \quad (3.4)$$

Thus, $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function. Now we have two possible cases for $z'(t)$: (i) $z'(t) < 0$ eventually and (ii) $z'(t) > 0$ eventually.

(i) Suppose that $z'(t) < 0$ for $t \geq t_2 \geq t_1$. Then, from (3.4), we get

$$r(t)\left|z'(t)\right|^{\alpha-1}z'(t) \leq r(t_2)\left|z'(t_2)\right|^{\alpha-1}z'(t_2), \quad t \geq t_2, \quad (3.5)$$

which implies that

$$z(t) \leq z(t_2) - r^{1/\alpha}(t_2)\left|z'(t_2)\right|\int_{t_2}^{t} r^{-1/\alpha}(s)\mathrm{d}s. \quad (3.6)$$

Letting $t \to \infty$, by (1.5), we find $z(t) \to -\infty$, which is a contradiction.

(ii) Suppose that $z'(t) > 0$ for $t \geq t_2 \geq t_1$. We define a Riccati substitution

$$\omega(t) = \rho(t)\frac{r(t)(z'(t))^{\alpha}}{(z[\sigma(t)])^{\alpha}}, \quad t \geq t_2. \quad (3.7)$$

Then $\omega(t) > 0$. From (3.4), we have

$$z'[\sigma(t)] \geq \left(\frac{r(t)}{r[\sigma(t)]}\right)^{1/\alpha}z'(t). \quad (3.8)$$

Differentiating (3.7), we find that

$$\begin{aligned}\omega'(t) = {} & \rho'(t)\frac{r(t)(z'(t))^{\alpha}}{(z[\sigma(t)])^{\alpha}} + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}} \\ & - \alpha\rho(t)\frac{r(t)(z'(t))^{\alpha}z^{\alpha-1}[\sigma(t)]z'[\sigma(t)]\sigma'(t)}{(z[\sigma(t)])^{2\alpha}}.\end{aligned} \quad (3.9)$$

Therefore, by (3.7), (3.8), and (3.9), we see that

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}} - \frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\omega^{(\alpha+1)/\alpha}(t). \quad (3.10)$$

Similarly, we introduce a Riccati substitution

$$\upsilon(t)=\rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[\sigma(t)])^{\alpha}},\quad t\geq t_2. \tag{3.11}$$

Then $\upsilon(t)>0$. From (3.4), we have

$$z'[\sigma(t)]\geq\left(\frac{r[\tau(t)]}{r[\sigma(t)]}\right)^{1/\alpha}z'[\tau(t)]. \tag{3.12}$$

Differentiating (3.11), we find that

$$\begin{aligned}\upsilon'(t)=\rho'(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[\sigma(t)])^{\alpha}}+\rho(t)\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}}\\-\alpha\rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}z^{\alpha-1}[\sigma(t)]z'[\sigma(t)]\sigma'(t)}{(z[\sigma(t)])^{2\alpha}}.\end{aligned} \tag{3.13}$$

Therefore, by (3.11), (3.12), and (3.13), we see that

$$\upsilon'(t)\leq\frac{\rho'(t)}{\rho(t)}\upsilon(t)+\rho(t)\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}}-\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\upsilon^{(\alpha+1)/\alpha}(t). \tag{3.14}$$

Thus, from (3.10) and (3.14), we have

$$\begin{aligned}\omega'(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\upsilon'(t)\leq\rho(t)\left\{\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}}+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\sigma(t)])^{\alpha}}\right\}+\frac{\rho'(t)}{\rho(t)}\omega(t)\\-\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\omega^{(\alpha+1)/\alpha}(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\rho'(t)}{\rho(t)}\upsilon(t)\\-\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned} \tag{3.15}$$

It is follows from (3.3) that

$$\begin{aligned}\omega'(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\upsilon'(t)\leq&-\frac{1}{2^{\alpha-1}}\rho(t)Q(t)+\frac{\rho'_{+}(t)}{\rho(t)}\omega(t)\\&-\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\omega^{(\alpha+1)/\alpha}(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\rho'_{+}(t)}{\rho(t)}\upsilon(t)\\&-\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned} \tag{3.16}$$

Integrating the above inequality from t_2 to t, we obtain that

$$\begin{aligned}
&\omega(t)-\omega(t_2)+\frac{p^\alpha[\sigma(t)]}{\tau'(t)}v(t)-\frac{p^\alpha[\sigma(t_2)]}{\tau'(t_2)}v(t_2)\\
&\quad\le-\int_{t_2}^{t}\frac{1}{2^{\alpha-1}}\rho(s)Q(s)\mathrm{d}s\\
&\qquad+\int_{t_2}^{t}\left[\frac{\rho'_+(s)}{\rho(s)}\omega(s)-\frac{\alpha\sigma'(s)}{\rho^{1/\alpha}(s)r^{1/\alpha}[\sigma(s)]}\omega^{(\alpha+1)/\alpha}(s)\right]\mathrm{d}s\\
&\qquad+\int_{t_2}^{t}\frac{p^\alpha[\sigma(s)]}{\tau'(s)}\left\{\left[\frac{\rho'_+(s)}{\rho(s)}+\xi(s)\right]_+v(s)-\frac{\alpha\sigma'(s)}{\rho^{1/\alpha}(s)r^{1/\alpha}[\sigma(s)]}v^{(\alpha+1)/\alpha}(s)\right\}\mathrm{d}s.
\end{aligned}\tag{3.17}$$

Define

$$A:=\left[\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\right]^{\alpha/(\alpha+1)}\omega(t),\qquad B:=\left[\frac{\rho'_+(t)}{\rho(t)}\frac{\alpha}{\alpha+1}\left[\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}.\tag{3.18}$$

Using the following inequality:

$$\frac{\alpha+1}{\alpha}AB^{1/\alpha}-A^{(\alpha+1)/\alpha}\le\frac{1}{\alpha}B^{(\alpha+1)/\alpha},\quad\text{for } A\ge0,\ B\ge0 \text{ are constants},\tag{3.19}$$

we have

$$\frac{\rho'_+(t)}{\rho(t)}\omega(t)-\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\omega^{(\alpha+1)/\alpha}(t)\le\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[\sigma(t)]\left(\rho'_+(t)\right)^{\alpha+1}}{\left(\rho(t)\sigma'(t)\right)^{\alpha}}.\tag{3.20}$$

On the other hand, define

$$A:=\left[\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\right]^{\alpha/(\alpha+1)}v(t),\qquad B:=\left[\zeta_+(t)\frac{\alpha}{\alpha+1}\left[\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}.\tag{3.21}$$

So we have

$$\zeta_+(t)v(t)-\frac{\alpha\sigma'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\sigma(t)]}v^{(\alpha+1)/\alpha}(t)\le\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[\sigma(t)](\zeta_+(t))^{\alpha+1}\rho(t)}{(\sigma'(t))^{\alpha}}.\tag{3.22}$$

Thus, from (3.17), we get

$$\omega(t)-\omega(t_2)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}v(t)-\frac{p^{\alpha}[\sigma(t_2)]}{\tau'(t_2)}v(t_2)$$
$$\leq -\int_{t_2}^{t}\rho(s)\left\{\frac{Q(s)}{2^{\alpha-1}}-\frac{r[\sigma(s)]}{(\alpha+1)^{\alpha+1}(\sigma'(s))^{\alpha}}\left[\left(\frac{\rho'_{+}(s)}{\rho(s)}\right)^{\alpha+1}+\frac{p^{\alpha}[\sigma(s)](\zeta_{+}(s))^{\alpha+1}}{\tau'(s)}\right]\right\}ds, \tag{3.23}$$

which contradicts (3.1). This completes the proof. □

When $p(t)\leq p_0<\infty,\ \tau'(t)\geq\tau_0>0$, where p_0,τ_0 are constants, we obtain the following result.

Theorem 3.2. *Suppose that* (1.5) *holds,* $p(t)\leq p_0<\infty,\ \tau'(t)\geq\tau_0>0,\ \sigma\in C([t_0,\infty),\mathbb{R}),\ \sigma'(t)>0,\ \sigma(t)\leq t,\ \sigma(t)\leq\tau(t)$ *for* $t\geq t_0$. *Further, assume that there exists a function* $\rho\in C([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{r[\sigma(s)](\rho'_{+}(s))^{\alpha+1}}{(\rho(s)\sigma'(s))^{\alpha}}\right]ds=\infty. \tag{3.24}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1\geq t_0$ such that $x(t)>0,\ x[\tau(t)]>0$ and $x[\sigma(t)]>0$ for all $t\geq t_1$. Using (1.1), for all sufficiently large t, we obtain that

$$\left(r(t)|z'(t)|^{\alpha-1}z'(t)\right)'+q(t)x^{\alpha}[\sigma(t)]+p_0^{\alpha}q[\tau(t)]x^{\alpha}(\sigma[\tau(t)])$$
$$+\frac{p_0^{\alpha}}{\tau_0}\left(r[\tau(t)]|z'[\tau(t)]|^{\alpha-1}z'[\tau(t)]\right)'\leq 0. \tag{3.25}$$

By applying (2.1) and the definition of z, we conclude that

$$\left(r(t)|z'(t)|^{\alpha-1}z'(t)\right)'+\frac{1}{2^{\alpha-1}}Q(t)z^{\alpha}[\sigma(t)]$$
$$+\frac{p_0^{\alpha}}{\tau_0}\left(r[\tau(t)]|z'[\tau(t)]|^{\alpha-1}z'[\tau(t)]\right)'\leq 0. \tag{3.26}$$

The remainder of the proof is similar to that of Theorem 3.1 and hence is omitted. □

Theorem 3.3. *Suppose that* (1.5) *holds,* $\tau(t)\leq t,\ \sigma(t)\geq\tau(t)$ *for* $t\geq t_0$. *Furthermore, assume that there exists a function* $\rho\in C([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\rho(s)\left\{\frac{Q(s)}{2^{\alpha-1}}-\frac{r[\tau(s)]\varphi(s)}{(\alpha+1)^{\alpha+1}(\tau'(s))^{\alpha}}\right\}ds=\infty. \tag{3.27}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x[\tau(t)] > 0$ and $x[\sigma(t)] > 0$ for all $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we get (3.3) and (3.4). In view of (3.4), $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function. Now we have two possible cases for $z'(t)$: (i) $z'(t) < 0$ eventually and (ii) $z'(t) > 0$ eventually.

(i) Suppose that $z'(t) < 0$ for $t \geq t_2 \geq t_1$. Then, similar to the proof of case (i) of Theorem 3.1, we obtain a contradiction.

(ii) Suppose that $z'(t) > 0$ for $t \geq t_2 \geq t_1$. We define a Riccati substitution

$$\omega(t) = \rho(t)\frac{r(t)(z'(t))^{\alpha}}{(z[\tau(t)])^{\alpha}}, \quad t \geq t_2. \tag{3.28}$$

Then $\omega(t) > 0$. From (3.4), we have

$$z'[\tau(t)] \geq \left(\frac{r(t)}{r[\tau(t)]}\right)^{1/\alpha} z'(t). \tag{3.29}$$

Differentiating (3.28), we find that

$$\omega'(t) = \rho'(t)\frac{r(t)(z'(t))^{\alpha}}{(z[\tau(t)])^{\alpha}} + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}} - \alpha\rho(t)\frac{r(t)(z'(t))^{\alpha}z^{\alpha-1}[\tau(t)]z'[\tau(t)]\tau'(t)}{(z[\tau(t)])^{2\alpha}}. \tag{3.30}$$

Therefore, by (3.28), (3.29), and (3.30), we see that

$$\omega'(t) \leq \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}} - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t). \tag{3.31}$$

Similarly, we introduce a Riccati substitution

$$\upsilon(t) = \rho(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[\tau(t)])^{\alpha}}, \quad t \geq t_2. \tag{3.32}$$

Then $\upsilon(t) > 0$. Differentiating (3.32), we find that

$$\begin{aligned}\upsilon'(t) = \rho'(t)\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}}{(z[\tau(t)])^{\alpha}} &+ \rho(t)\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}} \\ - \alpha\rho(t)&\frac{r[\tau(t)](z'[\tau(t)])^{\alpha}z^{\alpha-1}[\tau(t)]z'[\tau(t)]\tau'(t)}{(z[\tau(t)])^{2\alpha}}.\end{aligned} \tag{3.33}$$

Therefore, by (3.32) and (3.33), we see that

$$\upsilon'(t) = \frac{\rho'(t)}{\rho(t)}\upsilon(t) + \rho(t)\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}} - \frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t). \tag{3.34}$$

Thus, from (3.31) and (3.33), we have

$$\begin{aligned}\omega'(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\upsilon'(t)\le \rho(t)\left\{\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}}+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\left(r[\tau(t)](z'[\tau(t)])^{\alpha}\right)'}{(z[\tau(t)])^{\alpha}}\right\}+\frac{\rho'(t)}{\rho(t)}\omega(t)\\ -\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\rho'(t)}{\rho(t)}\upsilon(t)\\ -\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned} \tag{3.35}$$

It follows from (3.3) that

$$\begin{aligned}\omega'(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\upsilon'(t)\le & -\frac{1}{2^{\alpha-1}}\rho(t)Q(t)+\frac{\rho'_{+}(t)}{\rho(t)}\omega(t)\\ & -\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\rho'_{+}(t)}{\rho(t)}\upsilon(t)\\ & -\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned} \tag{3.36}$$

Integrating the above inequality from t_2 to t, we obtain that

$$\begin{aligned}&\omega(t)-\omega(t_2)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\upsilon(t)-\frac{p^{\alpha}[\sigma(t_2)]}{\tau'(t_2)}\upsilon(t_2)\\ &\quad\le -\int_{t_2}^{t}\frac{1}{2^{\alpha-1}}\rho(s)Q(s)\mathrm{d}s\\ &\qquad+\int_{t_2}^{t}\left[\frac{\rho'_{+}(s)}{\rho(s)}\omega(s)-\frac{\alpha\tau'(s)}{\rho^{1/\alpha}(s)r^{1/\alpha}[\tau(s)]}\omega^{(\alpha+1)/\alpha}(s)\right]\mathrm{d}s\\ &\qquad+\int_{t_2}^{t}\frac{p^{\alpha}[\sigma(s)]}{\tau'(s)}\left\{\left[\frac{\rho'_{+}(s)}{\rho(s)}+\xi(s)\right]_{+}\upsilon(s)-\frac{\alpha\tau'(s)}{\rho^{1/\alpha}(s)r^{1/\alpha}[\tau(s)]}\upsilon^{(\alpha+1)/\alpha}(s)\right\}\mathrm{d}s.\end{aligned} \tag{3.37}$$

Define

$$A:=\left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{\alpha/(\alpha+1)}\omega(t),\qquad B:=\left[\frac{\rho'_{+}(t)}{\rho(t)}\frac{\alpha}{\alpha+1}\left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}. \tag{3.38}$$

Using (3.19), we have

$$\frac{\rho'_+(t)}{\rho(t)}\omega(t)-\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\omega^{(\alpha+1)/\alpha}(t)\le\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[\tau(t)]\left(\rho'_+(t)\right)^{\alpha+1}}{\left(\rho(t)\tau'(t)\right)^{\alpha}}. \tag{3.39}$$

On the other hand, define

$$A:=\left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{\alpha/(\alpha+1)}\upsilon(t),\qquad B:=\left[\zeta_+(t)\frac{\alpha}{\alpha+1}\left[\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\right]^{-\alpha/(\alpha+1)}\right]^{\alpha}. \tag{3.40}$$

So we have

$$\zeta_+(t)\upsilon(t)-\frac{\alpha\tau'(t)}{\rho^{1/\alpha}(t)r^{1/\alpha}[\tau(t)]}\upsilon^{(\alpha+1)/\alpha}(t)\le\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r[\tau(t)](\zeta_+(t))^{\alpha+1}\rho(t)}{(\tau'(t))^{\alpha}}. \tag{3.41}$$

Thus, from (3.37), we get

$$\begin{aligned}&\omega(t)-\omega(t_2)+\frac{p^{\alpha}[\sigma(t)]}{\tau'(t)}\upsilon(t)-\frac{p^{\alpha}[\sigma(t_2)]}{\tau'(t_2)}\upsilon(t_2)\\&\quad\le-\int_{t_2}^{t}\rho(s)\left\{\frac{Q(s)}{2^{\alpha-1}}-\frac{r[\tau(s)]}{(\alpha+1)^{\alpha+1}(\tau'(s))^{\alpha}}\left[\left(\frac{\rho'_+(s)}{\rho(s)}\right)^{\alpha+1}+\frac{p^{\alpha}[\sigma(s)](\zeta_+(s))^{\alpha+1}}{\tau'(s)}\right]\right\}ds,\end{aligned} \tag{3.42}$$

which contradicts (3.27). This completes the proof. □

When $p(t)\le p_0<\infty,\ \tau'(t)\ge\tau_0>0$, where p_0,τ_0 are constants, we obtain the following result.

Theorem 3.4. *Suppose that* (1.5) *holds,* $p(t)\le p_0<\infty,\ \tau'(t)\ge\tau_0>0,\ \tau(t)\le t,\ \sigma(t)\ge\tau(t)$ *for* $t\ge t_0$. *Furthermore, assume that there exists a function* $\rho\in C([t_0,\infty),(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{r[\tau(s)]\left(\rho'_+(s)\right)^{\alpha+1}}{\left(\tau_0\rho(s)\right)^{\alpha}}\right]ds=\infty. \tag{3.43}$$

Then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1\ge t_0$ such that $x(t)>0,\ \ x[\tau(t)]>0$ and $x[\sigma(t)]>0$ for all $t\ge t_1$. Using (1.1) and the definition of z, we obtain (3.26) for all sufficiently large t. The remainder of the proof is similar to that of Theorem 3.3 and hence is omitted. □

4. Oscillation Criteria for the Case (1.13)

In this section, we will establish some oscillation criteria for (1.1) under the case (1.13).

In the following, we assume that p_0,τ_0 are constants.

Theorem 4.1. *Suppose that* (1.13) *holds,* $p(t) \le p_0 < \infty$, $\tau'(t) \ge \tau_0 > 0$, $\sigma(t) \le t$, $\sigma \in C([t_0,\infty),\mathbb{R})$, $\sigma'(t) > 0$, $\sigma(t) \le \tau(t)$ *for* $t \ge t_0$. *Further, assume that there exists a function* $\rho \in C([t_0,\infty),(0,\infty))$ *such that* (3.24) *holds. If there exists a function* $\eta \in C^1([t_0,\infty),\mathbb{R})$, $\eta(t) \ge t$, $\eta'(t) > 0$ *for* $t \ge t_0$ *such that*

$$\limsup_{t'\to\infty}\int_{t_0}^{t'}\left[\frac{Q(s)}{2^{\alpha-1}}\delta^{\alpha}(s) - \left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\eta'(s)}{\delta(s)r^{1/\alpha}[\eta(s)]}\right]ds = \infty, \tag{4.1}$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x[\tau(t)] > 0$ and $x[\sigma(t)] > 0$ for all $t \ge t_1$. Proceeding as in the proof of Theorem 3.2, we get (3.26). In view of (1.1), we have (3.4). Thus, $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function. Now we have two possible cases for $z'(t)$: (i) $z'(t) < 0$ eventually and (ii) $z'(t) > 0$ eventually.

(i) Suppose that $z'(t) > 0$ for $t \ge t_2 \ge t_1$. Then, by Theorem 3.2, we obtain a contradiction with (3.24).

(ii) Suppose that $z'(t) < 0$ for $t \ge t_2 \ge t_1$. We define the function u by

$$u(t) = -\frac{r(t)(-z'(t))^{\alpha}}{z^{\alpha}[\eta(t)]}, \quad t \ge t_2. \tag{4.2}$$

Then $u(t) < 0$. Noting that $r(t)(-z'(t))^{\alpha}$ is increasing, we get

$$r^{1/\alpha}(s)z'(s) \le r^{1/\alpha}(t)z'(t), \quad s \ge t \ge t_2. \tag{4.3}$$

Dividing the above inequality by $r^{1/\alpha}(s)$, and integrating it from $\eta(t)$ to t', we obtain that

$$z(t') \le z[\eta(t)] + r^{1/\alpha}(t)z'(t)\int_{\eta(t)}^{t'}\frac{ds}{r^{1/\alpha}(s)}. \tag{4.4}$$

Letting $t' \to \infty$, we have

$$0 \le z[\eta(t)] + r^{1/\alpha}(t)z'(t)\delta(t), \tag{4.5}$$

that is,

$$-\delta(t)\frac{r^{1/\alpha}(t)z'(t)}{z[\eta(t)]} \le 1. \tag{4.6}$$

Hence, by (4.2), we get

$$-\delta^{\alpha}(t)u(t) \le 1. \tag{4.7}$$

Similarly, we define the function v by

$$v(t) = -\frac{r[\tau(t)](-z'[\tau(t)])^{\alpha}}{z^{\alpha}[\eta(t)]}, \quad t \geq t_2. \tag{4.8}$$

Then $v(t) < 0$. Noting that $r(t)(-z'(t))^{\alpha}$ is increasing, we get the following:

$$r(t)(-z'(t))^{\alpha} \geq r[\tau(t)](-z'[\tau(t)])^{\alpha}. \tag{4.9}$$

Thus $0 < -v(t) \leq -u(t)$. So by (4.7), we see that

$$-\delta^{\alpha}(t)v(t) \leq 1. \tag{4.10}$$

Differentiating (4.2), we obtain that

$$u'(t) = \frac{(-r(t)(-z'(t))^{\alpha})' z^{\alpha}[\eta(t)] + \alpha r(t)(-z'(t))^{\alpha} z^{\alpha-1}[\eta(t)] z'[\eta(t)] \eta'(t)}{z^{2\alpha}[\eta(t)]}, \tag{4.11}$$

by (3.4), and we have $z'[\eta(t)] \leq (r(t)/r[\eta(t)])^{1/\alpha} z'(t)$, so

$$u'(t) \leq \frac{(-r(t)(-z'(t))^{\alpha})'}{z^{\alpha}[\eta(t)]} - \alpha \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-u(t))^{(\alpha+1)/\alpha}. \tag{4.12}$$

Similarly, we see that

$$v'(t) \leq \frac{(-r[\tau(t)](-z'[\tau(t)])^{\alpha})'}{z^{\alpha}[\eta(t)]} - \alpha \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-v(t))^{(\alpha+1)/\alpha}. \tag{4.13}$$

Therefore, by (4.12) and (4.13), we get the following:

$$\begin{aligned} u'(t) + \frac{p_0^{\alpha}}{\tau_0} v'(t) \leq{}& \frac{(-r(t)(-z'(t))^{\alpha})'}{z^{\alpha}[\eta(t)]} + \frac{p_0^{\alpha}}{\tau_0} \frac{(-r[\tau(t)](-z'[\tau(t)])^{\alpha})'}{z^{\alpha}[\eta(t)]} \\ & - \alpha \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-u(t))^{(\alpha+1)/\alpha} - \frac{\alpha p_0^{\alpha}}{\tau_0} \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-v(t))^{(\alpha+1)/\alpha}. \end{aligned} \tag{4.14}$$

Using (3.26) and (4.14), we obtain that

$$u'(t) + \frac{p_0^{\alpha}}{\tau_0} v'(t) \leq -\frac{Q(t)}{2^{\alpha-1}} - \alpha \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-u(t))^{(\alpha+1)/\alpha} - \frac{\alpha p_0^{\alpha}}{\tau_0} \frac{\eta'(t)}{r^{1/\alpha}[\eta(t)]}(-v(t))^{(\alpha+1)/\alpha}. \tag{4.15}$$

Multiplying (4.15) by $\delta^{\alpha}(t)$, and integrating it from t_2 to t', we have

$$\begin{aligned} &u(t')\delta^{\alpha}(t') - u(t_2)\delta^{\alpha}(t_2) + \alpha\int_{t_2}^{t'} \frac{\delta^{\alpha-1}(s)\eta'(s)u(s)}{r^{1/\alpha}[\eta(s)]}ds + \alpha\int_{t_2}^{t'} \frac{\eta'(s)\delta^{\alpha}(s)}{r^{1/\alpha}[\eta(s)]}(-u(s))^{(\alpha+1)/\alpha}ds \\ &\quad + \frac{p_0^{\alpha}}{\tau_0}v(t')\delta^{\alpha}(t') - \frac{p_0^{\alpha}}{\tau_0}v(t_2)\delta^{\alpha}(t_2) + \frac{\alpha p_0^{\alpha}}{\tau_0}\int_{t_2}^{t'} \frac{\delta^{\alpha-1}(s)\eta'(s)v(s)}{r^{1/\alpha}[\eta(s)]}ds \\ &\quad + \frac{\alpha p_0^{\alpha}}{\tau_0}\int_{t_2}^{t'} \frac{\eta'(s)\delta^{\alpha}(s)}{r^{1/\alpha}[\eta(s)]}(-v(s))^{(\alpha+1)/\alpha}ds + \int_{t_2}^{t'} \frac{Q(s)}{2^{\alpha-1}}\delta^{\alpha}(s)ds \le 0. \end{aligned} \tag{4.16}$$

Using (3.19), (4.7), and (4.10), we find that

$$\begin{aligned} \int_{t_2}^{t'} \left[\frac{Q(s)}{2^{\alpha-1}}\delta^{\alpha}(s) - \left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\eta'(s)}{\delta(s)r^{1/\alpha}[\eta(s)]}\right]ds &\le u(t_2)\delta^{\alpha}(t_2) + \frac{p_0^{\alpha}}{\tau_0}v(t_2)\delta^{\alpha}(t_2) \\ &\quad + 1 + \frac{p_0^{\alpha}}{\tau_0}. \end{aligned} \tag{4.17}$$

Letting $t' \to \infty$, we obtain a contradiction with (4.1). This completes the proof. □

From Theorems 3.4 and 4.1, we have the following result.

Theorem 4.2. *Suppose that* (1.13) *holds,* $p(t) \le p_0 < \infty$, $\tau'(t) \ge \tau_0 > 0$, $\tau(t) \le t$, $\sigma(t) \ge \tau(t)$ *for* $t \ge t_0$. *Further, assume that there exists a function* $\rho \in C([t_0,\infty),(0,\infty))$ *such that* (3.43) *holds. If there exists a function* $\eta \in C^1([t_0,\infty),\mathbb{R})$, $\eta(t) \ge t$, $\eta'(t) > 0$, $\sigma(t) \le \eta(t)$ *for* $t \ge t_0$ *such that* (4.1) *holds, then* (1.1) *is oscillatory.*

5. Examples

In this section, we will give some examples to illustrate the main results.

Example 5.1. Study the second-order neutral differential equation

$$\left[\left|(x(t)+tx(t-\lambda_1))'\right|^{\alpha-1}(x(t)+tx(t-\lambda_1))'\right]' + \beta|x(t-\lambda_2)|^{\alpha-1}x(t-\lambda_2) = 0, \quad t \ge t_0, \tag{5.1}$$

where $\alpha \ge 1$, $0 < \lambda_1 \le \lambda_2 < 1$, $\beta > 0$ are constants.

Let $r(t) = 1$, $p(t) = t$, $\rho(t) = 1$. It is easy to see that all the conditions of Theorem 3.1 hold. Hence, (5.1) is oscillatory.

Example 5.2. Consider the second-order quasilinear neutral differential equation

$$\left[\left|\left(x(t)+p(t)x(\lambda_1 t)\right)'\right|^{\alpha-1}\left(x(t)+p(t)x(\lambda_1 t)\right)'\right]' + \frac{\beta}{t^{\alpha+1}}|x(\lambda_2 t)|^{\alpha-1}x(\lambda_2 t) = 0, \quad t \ge t_0, \tag{5.2}$$

where $\alpha \ge 1$, $\beta > 0$ are constants, $\lambda_1, \lambda_2 \in (0,1), \lambda_2 \le \lambda_1$.

Let $r(t) = 1,\ 0 \le p(t) \le p_0 < \infty,\ q(t) = \beta/t^{\alpha+1},\ \tau(t) = \lambda_1 t,\ \sigma(t) = \lambda_2 t$, and $\rho(t) = t^\alpha$. Then, we have

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^{t}&\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^\alpha}{\tau_0}\right)\frac{r[\sigma(s)]\left((\rho'(s)_+)\right)^{\alpha+1}}{\left(\rho(s)\sigma'(s)\right)^\alpha}\right]ds\\&=\left[\frac{\beta}{2^{\alpha-1}}-\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}\lambda_2^\alpha}\left(1+\frac{p_0^\alpha}{\lambda_1}\right)\right]\limsup_{t\to\infty}\int_{t_0}^{t}\frac{ds}{s}=\infty,\end{aligned}\tag{5.3}$$

if $\beta > 2^{\alpha-1}\alpha^{\alpha+1}(1+p_0^\alpha/\lambda_1)/[(\alpha+1)^{\alpha+1}\lambda_2^\alpha]$. Hence, by Theorem 3.2, (5.2) is oscillatory if

$$\beta > \frac{2^{\alpha-1}\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}\lambda_2^\alpha}\left(1+\frac{p_0^\alpha}{\lambda_1}\right).\tag{5.4}$$

Example 5.3. Investigate the second-order neutral differential equation

$$\left[x(t)+\frac{(t-1)(t-4\pi)}{t}x(t-4\pi)\right]''+(t-2\pi)x(t-2\pi)=0,\quad t\ge t_0.\tag{5.5}$$

Let $r(t) = 1,\ p(t) = (t-1)(t-4\pi)/t,\ q(t) = t-2\pi$, and $\rho(t) = 1$. It is easy to see that all the conditions of Theorem 3.3 hold. Hence, (5.5) is oscillatory, for example, $x(t) = \sin t/t$ is a solution of (5.5).

Example 5.4. Discuss the second-order quasilinear neutral differential equation

$$\left[\left|\left(x(t)+p(t)x(\lambda_1 t)\right)'\right|^{\alpha-1}\left(x(t)+p(t)x(\lambda_1 t)\right)'\right]'+\frac{\beta}{t^{\alpha+1}}|x(\lambda_2 t)|^{\alpha-1}x(\lambda_2 t)=0,\quad t\ge t_0,\tag{5.6}$$

where $\alpha \ge 1,\ \beta > 0$ are constants, $\lambda_1 \in (0,1), \lambda_2 \in [\lambda_1,\infty)$.

Let $r(t) = 1,\ 0 \le p(t) \le p_0 < \infty,\ q(t) = \beta/t^{\alpha+1},\ \tau(t) = \lambda_1 t,\ \sigma(t) = \lambda_2 t,\ \rho(t) = t^\alpha$. Then, we have

$$\begin{aligned}\limsup_{t\to\infty}\int_{t_0}^{t}&\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^\alpha}{\tau_0}\right)\frac{r[\tau(s)]\left(\rho'(s)_+\right)^{\alpha+1}}{\tau_0\left(\rho(s)\right)^\alpha}\right]ds\\&=\left[\frac{\beta}{2^{\alpha-1}}-\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}\lambda_1^\alpha}\left(1+\frac{p_0^\alpha}{\lambda_1}\right)\right]\limsup_{t\to\infty}\int_{t_0}^{t}\frac{ds}{s}=\infty\end{aligned}\tag{5.7}$$

if $\beta > 2^{\alpha-1}\alpha^{\alpha+1}(1+p_0^\alpha/\lambda_1)/[(\alpha+1)^{\alpha+1}\lambda_1^\alpha]$. Hence, by Theorem 3.4, (5.6) is oscillatory if

$$\beta > \frac{2^{\alpha-1}\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}\lambda_1^\alpha}\left(1+\frac{p_0^\alpha}{\lambda_1}\right).\tag{5.8}$$

Example 5.5. Examine the second-order quasilinear neutral differential equation

$$\left[t^{2\alpha}\left|(x(t)+p(t)x(\lambda_1 t))'\right|^{\alpha-1}(x(t)+p(t)x(\lambda_1 t))'\right]'+\beta t^{\alpha-1}|x(\lambda_2 t)|^{\alpha-1}x(\lambda_2 t)=0,\quad t\geq t_0, \tag{5.9}$$

where $\alpha \geq 1$, $\beta > 0$ are constants, $\lambda_1, \lambda_2 \in (0,1)$, $\lambda_2 \leq \lambda_1$.

Let $r(t) = t^{2\alpha}$, $0 \leq p(t) \leq p_0 < \infty$, $q(t) = \beta t^{\alpha-1}$, $\tau(t) = \lambda_1 t$, $\sigma(t) = \lambda_2 t$, and $\rho(t) = 1$. Then, $Q(t) = \beta\lambda_1^{\alpha-1}t^{\alpha-1}$. It is easy to see that (3.24) holds. On the other hand, taking $\eta(t) = t$, then $\delta(t) = 1/t$. Therefore, one has

$$\begin{aligned}\limsup_{t'\to\infty}&\int_{t_0}^{t'}\left[\frac{Q(s)}{2^{\alpha-1}}\delta^{\alpha}(s)-\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\eta'(s)}{\delta(s)r^{1/\alpha}[\eta(s)]}\right]ds\\&=\left[\frac{\beta}{2^{\alpha-1}}\lambda_1^{\alpha-1}-\frac{1}{2}\left(1+\frac{p_0^{\alpha}}{\lambda_1}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\right]\limsup_{t'\to\infty}\int_{t_0}^{t'}\frac{ds}{s}=\infty\end{aligned} \tag{5.10}$$

if $\beta > 2^{\alpha-2}(1+p_0^{\alpha}/\lambda_1)(\alpha/\alpha+1)^{\alpha+1}/\lambda_1^{\alpha-1}$. Thus, by Theorem 4.1, (5.9) oscillates if

$$\beta > \frac{2^{\alpha-2}}{\lambda_1^{\alpha-1}}\left(1+\frac{p_0^{\alpha}}{\lambda_1}\right)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}. \tag{5.11}$$

6. Conclusions

Inequality technique plays an important role in studying the oscillatory behavior of differential equations; in this paper, we establish a new inequality (2.1); by using (2.1) and Riccati substitution, we establish some new oscillation criteria for (1.1). Theorem 3.1 can be applied to the case $\tau(t) \geq t$. Specially, taking $\alpha = 1$, our results include and improve the results in [15]; for example, and Theorem 4.1 includes [15, Theorem 3.1], Theorem 4.2 includes [15, Theorem 3.2]. The method can be applied on the second-order Emden-Fowler neutral differential equations

$$\left[r(t)(x(t)+p(t)x(\tau(t)))'\right]'+q(t)|x(\delta(t))|^{\alpha-1}x(\delta(t))=0,\quad t\geq t_0, \tag{6.1}$$

where $\alpha \geq 1$. It would be interesting to find another method to investigate (1.1) when $\tau \circ \sigma \not\equiv \sigma \circ \tau$.

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