Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 560264, 8 pages
doi:10.1155/2009/560264

Research Article

Common Fixed Points for Maps on Topological Vector Space Valued Cone Metric Spaces

Ismat Beg,[1] Akbar Azam,[2] and Muhammad Arshad[3]

[1] *Centre for Advanced Studies in Mathematics, Lahore University of Management Sciences, Lahore Cantt 54792, Pakistan*
[2] *Mathematics Department, COMSATS Institute of Information Technology, Islamabad 44000, Pakistan*
[3] *Mathematics Department, International Islamic University, Islamabad 44000, Pakistan*

Correspondence should be addressed to Ismat Beg, ibeg@lums.edu.pk

Received 31 August 2009; Revised 30 November 2009; Accepted 13 December 2009

Recommended by Yuri Latushkin

We introduced a notion of topological vector space valued cone metric space and obtained some common fixed point results. Our results generalize some recent results in the literature.

1. Introduction

Huang and Zhang [1] generalized the notion of metric space by replacing the set of real numbers by ordered Banach space, deffined a cone metric space, and established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, several other authors [2–5] studied the existence of common fixed point of mappings satisfying a contractive type condition in normal cone metric spaces. Afterwards, Rezapour and Hamlbarani [6] studied fixed point theorems of contractive type mappings by omitting the assumption of normality in cone metric spaces (see also [7–14]). In this paper we obtain common fixed points for a pair of self-mappings satisfying a generalized contractive type condition without the assumption of normality in a class of topological vector space valued cone metric spaces which is bigger than that introduced by Huang and Zhang [1].

Let (E, τ) be always a topological vector space and P a subset of E. Then, P is called a cone whenever

(i) P is closed, nonempty and $P \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b,

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a topological vector space valued cone metric space.

Note that Huang and Zhang [1] notion of cone metric space is a special case of our notion of topological vector space valued cone metric space.

Example 1.2. Let $X = [0, 1]$, and let E be the set of all real valued functions on X which also have continuous derivatives on X, then E is a vector space over $\mathbb{R}$ under the following operations:

$$(f + g)(t) = f(t) + g(t), \qquad (\alpha f)(t) = \alpha f(t), \tag{1.1}$$

for all $f, g \in E, \alpha \in \mathbb{R}$. Let τ be the strongest vector (locally convex) topology on E, then (X, τ) is a topological vector space which is not normable and is not even metrizable (see [15]). Define $d : X \times X \to E$ as follows:

$$\begin{gathered} (d(x, y))(t) = |x - y| e^t, \\ P = \{x \in E : x(t) \geqslant 0 \ \forall t \in X\}. \end{gathered} \tag{1.2}$$

Then (X, d) is a topological vector space valued cone metric space.

Example 1.2 shows that this category of cone metric spaces is larger than that considered in [1–8] .

Definition 1.3. Let (X, d) be a topological vector space valued cone metric space, and let $x \in X$ and $\{x_n\}_{n \geq 1}$ be a sequence in X. Then

(i) $\{x_n\}_{n \geq 1}$ converges to x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

(ii) $\{x_n\}_{n \geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete topological vector space valued cone metric space if every Cauchy sequence is convergent.

2. Fixed Point

In this section, we shall give some results which generalize [6, Theorems 2.3, 2.6, 2.7, and 2.8] (and so [1, Theorems 1, 3, and 4]).

Theorem 2.1. *Let (X, d) be a complete topological vector space valued cone metric space and let the self-mappings $S, T : X \to X$ satisfy*

$$d(Sx, Ty) \leq kd(x, y) + l(d(x, Ty) + d(y, Sx)), \tag{2.1}$$

for all $x, y \in X$, where $k, l \in [0, 1)$ with $k + 2l < 1$. Then S and T have a unique common fixed point.

Proof. For $x_0 \in X$ and $n \geq 0$, define $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$. Then,

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leqslant kd(x_{2n}, x_{2n+1}) + l[d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})] \\ &= kd(x_{2n}, x_{2n+1}) + l[d(x_{2n}, Tx_{2n+1})] \\ &\leqslant kd(x_{2n}, x_{2n+1}) + l[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\ &= [k + l]d(x_{2n}, x_{2n+1}) + ld(x_{2n+1}, x_{2n+2}). \end{aligned} \tag{2.2}$$

It implies that $d(x_{2n+1}, x_{2n+2}) \leqslant [(k + l)/(1 - l)]d(x_{2n}, x_{2n+1})$. Similarly,

$$\begin{aligned} d(x_{2n+2}, x_{2n+3}) &= d(Sx_{2n+2}, Tx_{2n+1}) \\ &\leqslant kd(x_{2n+2}, x_{2n+1}) + l[d(x_{2n+2}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n+2})] \\ &\leqslant kd(x_{2n+2}, x_{2n+1}) + l[d(x_{2n+2}, x_{2n+3}) + d(x_{2n+1}, x_{2n+2})] \\ &= [k + l]d(x_{2n+1}, x_{2n+2}) + ld(x_{2n+2}, x_{2n+3}). \end{aligned} \tag{2.3}$$

Hence, $d(x_{2n+2}, x_{2n+3}) \leq [(k + l)/(1 - l)]d(x_{2n+1}, x_{2n+2})$. Thus,

$$d(x_n, x_{n+1}) \leqslant \lambda^n d(x_0, x_1), \tag{2.4}$$

for all $n \geq 0$, where $\lambda = ((k + l)/(1 - l)) < 1$. Now, for $n > m$ we have

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leqslant \left(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m\right) d(x_0, x_1) \\ &\leqslant \frac{\lambda^m}{1 - \lambda} d(x_0, x_1). \end{aligned} \tag{2.5}$$

Let $0 \ll c$. Take a symmetric neighborhood V of 0 such that $c + V \subseteq \operatorname{int} P$. Also, choose a natural number N_1 such that $(\lambda^m/(1 - \lambda))d(x_1, x_0) \in V$, for all $m \geq N_1$. Then, $(\lambda^m/(1 - \lambda))d(x_1, x_0) \ll c$, for all $m \geq N_1$. Thus,

$$d(x_n, x_m) \leq \frac{\lambda^m}{1 - \lambda} d(x_1, x_0) \ll c, \tag{2.6}$$

for all $n > m$. Therefore, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X, d). Since X is complete, there exists $u \in X$ such that $x_n \to u$. Choose a natural number N_2 such that $d(x_n, u) \ll [c(1-l)/2(1+l)]$ for all $n \geqslant N_2$. Thus,

$$\begin{aligned} d(u, Tu) &\leqslant d(u, x_{2n+1}) + d(x_{2n+1}, Tu) \\ &= d(u, x_{2n+1}) + d(Sx_{2n}, Tu) \\ &\leqslant d(u, x_{2n+1}) + kd(u, x_{2n}) + l[d(u, Sx_{2n}) + d(x_{2n}, Tu)] \\ &\leqslant d(u, x_{2n+1}) + kd(u, x_{2n}) + l[d(u, x_{2n+1}) + d(x_{2n}, u) + d(u, Tu)] \\ &= (1+l)d(u, x_{2n+1}) + (k+l)d(u, x_{2n}) + ld(u, Tu). \end{aligned} \tag{2.7}$$

So,

$$\begin{aligned} d(u, Tu) &\leqslant \left[\frac{1+l}{1-l}\right] d(u, x_{2n+1}) + \left[\frac{k+l}{1-l}\right] d(u, x_{2n}) \\ &\leqslant \left[\frac{1+l}{1-l}\right] d(u, x_{2n+1}) + \left[\frac{1+l}{1-l}\right] d(u, x_{2n}) \\ &= \frac{c}{2} + \frac{c}{2} = c, \end{aligned} \tag{2.8}$$

for all $n \geq N_2$. Therefore, $d(u, Tu) \ll c/i$ for all $i \geqslant 1$. Hence, $(c/i) - d(u, Tu) \in P$ for all $i \geqslant 1$. Since P is closed, $-d(u, Tu) \in P$ and so $d(u, Tu) = 0$. Hence, u is a fixed point of T. Similarly, we can show that $u = Su$. Now, we show that S and T have a unique fixed point. For this, assume that there exists another point u^* in X such that $u^* = Tu^* = Su^*$. Then,

$$\begin{aligned} d(u, u^*) &= d(Su, Tu^*) \\ &\leqslant kd(u, u^*) + l[d(u, Tu^*) + d(u^*, Su)] \\ &\leqslant kd(u, u^*) + l[d(u, u^*) + d(u^*, u)] \\ &\leqslant (k+2l)d(u, u^*). \end{aligned} \tag{2.9}$$

Since $k + 2l < 1$, $d(u, u^*) = 0$ and so $u = u^*$. □

The following corollary generalizes [6, Theorems 2.3, 2.7, and 2.8] (and so [1, Theorems 1 and 4]).

Corollary 2.2. *Let (X, d) be a complete topological vector space valued cone metric space and let the self-mapping $T : X \to X$ satisfy $d(Tx, Ty) \leqslant ad(x, y) + bd(x, Ty) + cd(y, Tx)$ for all $x, y \in X$, where $a, b, c \in [0, 1)$ with $a + b + c < 1$. Then T has a unique fixed point.*

Proof. The symmetric property of d and the above inequality imply that

$$d(Tx,Ty) \leqslant ad(x,y) + \frac{b+c}{2}\left[d(x,Ty) + d(y,Tx)\right]. \tag{2.10}$$

By substituting $S = Ta = k$ and $(b+c)/2 = l$ in Theorem 2.1, we obtain the required result. □

Theorem 2.3. *Let (X,d) be a complete topological vector space valued cone metric space and let the self-mappings $S,T : X \to X$ satisfy*

$$d(Sx,Ty) \leq kd(x,y) + l(d(x,Sx) + d(y,Ty)), \tag{2.11}$$

for all $x,y \in X$, where $k,l \in [0,1)$ with $k+2l < 1$. Then S and T have a unique common fixed point.

Proof. For $x_0 \in X$ and $n \geq 0$, define $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$. Then,

$$\begin{aligned} d(x_{2n+1},x_{2n+2}) &= d(Sx_{2n},Tx_{2n+1}) \\ &\leqslant kd(x_{2n},x_{2n+1}) + l[d(x_{2n},Sx_{2n}) + d(x_{2n+1},Tx_{2n+1})] \\ &= kd(x_{2n},x_{2n+1}) + l[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})] \\ &= [k+l]d(x_{2n},x_{2n+1}) + ld(x_{2n+1},x_{2n+2}). \end{aligned} \tag{2.12}$$

It implies that $d(x_{2n+1},x_{2n+2}) \leqslant [(k+l)/(1-l)]d(x_{2n},x_{2n+1})$. Similarly,

$$\begin{aligned} d(x_{2n+2},x_{2n+3}) &= d(Sx_{2n+2},Tx_{2n+1}) \\ &\leqslant kd(x_{2n+2},x_{2n+1}) + l[d(x_{2n+2},Sx_{2n+2}) + d(x_{2n+1},Tx_{2n+1})] \\ &= kd(x_{2n+2},x_{2n+1}) + l[d(x_{2n+2},x_{2n+3}) + d(x_{2n+1},x_{2n+2})] \\ &= [k+l]d(x_{2n+1},x_{2n+2}) + ld(x_{2n+2},x_{2n+3}). \end{aligned} \tag{2.13}$$

Hence, $d(x_{2n+2},x_{2n+3}) \leq [(k+l)/(1-l)]d(x_{2n+1},x_{2n+2})$. Thus,

$$d(x_n,x_{n+1}) \leqslant \lambda^n d(x_0,x_1), \tag{2.14}$$

for all $n \geq 0$, where $\lambda = ((k+l)/(1-l)) < 1$. Now, for $n > m$ we have

$$\begin{aligned} d(x_n,x_m) &\leqslant d(x_n,x_{n-1}) + d(x_{n-1},x_{n-2}) + \cdots + d(x_{m+1},x_m) \\ &\leqslant \left(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m\right)d(x_0,x_1) \\ &\leqslant \frac{\lambda^m}{1-\lambda}d(x_0,x_1). \end{aligned} \tag{2.15}$$

Let $0 \ll c$. Take a symmetric neighborhood V of 0 such that $c + V \subseteq \operatorname{int} P$. Also, choose a natural number N_1 such that $(\lambda^m/(1-\lambda))d(x_1, x_0) \in V$, for all $m \geq N_1$. Then, $(\lambda^m/(1-\lambda))d(x_1, x_0) \ll c$, for all $m \geq N_1$. Thus,

$$d(x_n, x_m) \leq \frac{\lambda^m}{1-\lambda} d(x_1, x_0) \ll c, \tag{2.16}$$

for all $n > m$. Therefore, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in (X, d). Since X is complete, there exists $u \in X$ such that $x_n \to u$. Choose a natural number N_2 such that $d(x_n, u) \ll [c(1-l)/2(1+l)]$ for all $n \geqslant N_2$. Thus,

$$\begin{aligned}
d(u, Tu) &\leqslant d(u, x_{2n+1}) + d(x_{2n+1}, Tu) \\
&= d(u, x_{2n+1}) + d(Sx_{2n}, Tu) \\
&\leqslant d(u, x_{2n+1}) + kd(u, x_{2n}) + l[d(u, Tu) + d(x_{2n}, Sx_{2n})] \\
&\leqslant d(u, x_{2n+1}) + kd(u, x_{2n}) + l[d(u, x_{2n+1}) + d(x_{2n}, u) + d(u, Tu)] \\
&= (1+l)d(u, x_{2n+1}) + (k+l)d(u, x_{2n}) + ld(u, Tu).
\end{aligned} \tag{2.17}$$

So,

$$\begin{aligned}
d(u, Tu) &\leqslant \left[\frac{1+l}{1-l}\right] d(u, x_{2n+1}) + \left[\frac{k+l}{1-l}\right] d(u, x_{2n}) \\
&\leqslant \left[\frac{1+l}{1-l}\right] d(u, x_{2n+1}) + \left[\frac{1+l}{1-l}\right] d(u, x_{2n}) \\
&\ll \frac{c}{2} + \frac{c}{2} = c,
\end{aligned} \tag{2.18}$$

for all $n \geq N_2$. Therefore, $d(u, Tu) \ll c/i$ for all $i \geqslant 1$. Hence, $(c/i) - d(u, Tu) \in P$ for all $i \geqslant 1$. Since P is closed, $-d(u, Tu) \in P$ and so $d(u, Tu) = 0$. Hence, u is a fixed point of T. Similarly, we can show that $u = Su$. Now, we show that S and T have a unique fixed point. For this, assume that there exists another point u^* in X such that $u^* = Tu^* = Su^*$. Then,

$$\begin{aligned}
d(u, u^*) &= d(Su, Tu^*) \\
&\leqslant kd(u, u^*) + l[d(u, u^*) + d(u^*, u)] \\
&= kd(u, u^*).
\end{aligned} \tag{2.19}$$

Since $k < 1$, $d(u, u^*) = 0$ and so $u = u^*$. □

The following corollary generalizes [6, Theorem 2.6] (and so [1, Theorem 3]).

Corollary 2.4. *Let (X, d) be a complete topological vector space valued cone metric space and let the self-mapping $T : X \to X$ satisfy $d(Tx, Ty) \leqslant ad(x, y) + bd(x, Tx) + cd(y, Ty)$ for all $x, y \in X$, where $a, b, c \in [0, 1)$ with $a + b + c < 1$. Then T has a unique fixed point.*

Proof is similar to the proof of Corollary 2.2.

Example 2.5. Let (X, d) be a topological vector space valued cone metric space of Example 1.2. Define $S, T : X \to X$ as follows:

$$S(t) = T(t) = \begin{cases} \dfrac{t}{3} & \text{if } x \neq 1, \\ \dfrac{1}{6} & \text{if } x = 1. \end{cases} \tag{2.20}$$

Then,

$$\left|Sx - Ty\right|e^t \leq k\left|x - y\right|e^t + l\left[|x - Sx|e^t + \left|y - Ty\right|e^t\right], \tag{2.21}$$

if $k = 1/6$, $l = 5/18$. Hence all conditions of Theorem 2.3 are satisfied.

Acknowledgment

The present version of the paper owes much to the precise and kind remarks of the learned referees.

References

[1] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.
[2] M. Abbas and G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416–420, 2008.
[3] D. Ilić and V. Rakočević, "Common fixed points for maps on cone metric space," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 876–882, 2008.
[4] S. Radenović, "Common fixed points under contractive conditions in cone metric spaces," *Computers & Mathematics with Applications*, vol. 58, no. 6, pp. 1273–1278, 2009.
[5] P. Vetro, "Common fixed points in cone metric spaces," *Rendiconti del Circolo Matematico di Palermo*, vol. 56, no. 3, pp. 464–468, 2007.
[6] Sh. Rezapour and R. Hamlbarani, "Some notes on the paper: "Cone metric spaces and fixed point theorems of contractive mappings"," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 719–724, 2008.
[7] M. Arshad, A. Azam, and P. Vetro, "Some common fixed point results in cone metric spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 493965, 11 pages, 2009.
[8] A. Azam, M. Arshad, and I. Beg, "Common fixed points of two maps in cone metric spaces," *Rendiconti del Circolo Matematico di Palermo*, vol. 57, no. 3, pp. 433–441, 2008.
[9] B. S. Choudhury and N. Metiya, "Fixed points of weak contractions in cone metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1589–1593, 2010.
[10] A. Azam, M. Arshad, and I. Beg, "Banach contraction principle on cone rectangular metric spaces," *Applicable Analysis and Discrete Mathematics*, vol. 3, no. 2, pp. 236–241, 2009.
[11] A. Azam and M. Arshad, "Common fixed points of generalized contractive maps in cone metric spaces," *Bulletin of the Iranian Mathematical Society*, vol. 35, no. 2, pp. 255–264, 2009.

[12] P. Raja and S. M. Vaezpour, “Some extensions of Banach’s contraction principle in complete cone metric spaces,” *Fixed Point Theory and Applications*, vol. 2008, Article ID 768294, 11 pages, 2008.
[13] S. Janković, Z. Kadelburg, S. Radenović, and B. E. Rhoades, “Assad-Kirk-type fixed point theorems for a pair of nonself mappings on cone metric spaces,” *Fixed Point Theory and Applications*, vol. 2009, Article ID 761086, 16 pages, 2009.
[14] Z. Kadelburg, S. Radenović, and B. Rosić, “Strict contractive conditions and common fixed point theorems in cone metric spaces,” *Fixed Point Theory and Applications*, vol. 2009, Article ID 173838, 14 pages, 2009.
[15] H. H. Schaefer, *Topological Vector Spaces*, vol. 3 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1971.