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Research Article

On a Class of Multitime Evolution Equations with Nonlocal Initial Conditions

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The existence and uniqueness of the strong solution for a multitime evolution equation with nonlocal initial conditions are proved. The proof is essentially based on a priori estimates and on the density of the range of the operator generated by the considered problem.

1. Introduction and motivation

Throughout this paper, H will denote a complex Hilbert space, endowed with the inner product $(\cdot,\cdot)$ and the norm $|\cdot|$, and $\mathcal{L}(H)$ denotes the Banach algebra of bounded linear operators on H.

Mathematical models for a number of natural phenomena can be formulated in terms of partial differential equations (PDEs) of the form

$$\sum_{i=1}^{m} k_i(x,t)u_{t_i} = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u + f(x,t), \qquad (E_{pp})$$

$$\frac{\partial^{s_1+s_2+\cdots+s_m}u}{\partial t_1^{s_1}\partial t_2^{s_2}\cdots\partial t_m^{s_m}} + \sum_{i=1}^{m} p_i(x,t)u_{t_i} = \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u + f(x,t), \qquad (E_{ph})$$

where $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ is the "space variable," and $t=(t_1,\dots,t_m)\in\mathbb{R}^m$ is the "time variable." The right-hand side of (E_{pp}) (resp., (E_{ph})) is assumed to be elliptic, that is

$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq a_0 \sum_{i=1}^{n} \xi_i^2$ where $a_0 > 0$ is a constant, for every $\xi \in \mathbb{R}^n$ and for all values (x,t) in some domain.

When $m = 1$, (E_{pp}) is called *parabolic*, and when $m = 1$, $s_1 = 2$, (E_{ph}) is called *hyperbolic*. When $m \geq 2$, this kind of equations is called *multitime evolution equations*, and (E_{pp}) (resp., (E_{ph})) is called *pluriparabolic* (ultraparabolic) (resp., *plurihyperbolic*). Thus, in the multitime case, there are several "time-like" variables in the equations.

The multi-time evolution equations are encountered for instance in the theory of Brownian motion (diffusion process with inertia) [1], transport theory (Fokker-Planck-type equations) [2], biology (age-structured population dynamics) [3], waves and Maxwell's equations [4], and other practical applications of mathematical physics and engineering sciences.

Plurihyperbolic equations with standard Goursat conditions, Cauchy conditions, Picard conditions, mixed conditions [5–14] are well studied with the help of the energy inequality method and Riemann functions.

Nonlocal problems for some classes of PDEs depending on one time variable have attracted much interest in recent years, and have been studied extensively by many authors, see for instance Ashyralyev et al. [15–20], Byszewski and Lakshmikantham [21], Balachandran and Park [22], Chesalin and Yurchuk [23–25], Gordeziani and Avalishvili [26], and Agarwal et al. [27]. However, the case of multi-time equations with nonlocal conditions does not seem to have been widely investigated and few results are available, see, for example, the articles by the authors Rebbani et al. [28, 29]. The study of this case is caused not only by theoretical interest, but also by practical necessity.

In this paper, we investigate a class of nonclassical problems for plurihyperbolic equations with nonlocal conditions. The multi-time PDE considered is formulated as follows.

Let $D =]0,T_1[\times]0,T_2[$ be a bounded rectangle in the plane $\mathbb{R}^2$ with coordinates $t = (t_1,t_2)$. We consider

$$\mathcal{L}_\lambda u = \frac{\partial^2 u}{\partial t_1 \partial t_2} + A(t)\left(u + \lambda \frac{\partial^2 u}{\partial t_1 \partial t_2}\right) = f(t), \quad t \in D, \tag{1.1}$$

$$\begin{aligned} l_{1\mu}u &= B_1(\mu)u\,|_{t_1=0} - B_2(\mu)u\,|_{t_1=T_1} = \varphi(t_2), \quad t_2 \in [0,T_2], \\ l_{2\mu}u &= B_1(\mu)u\,|_{t_2=0} - B_2(\mu)u\,|_{t_2=T_2} = \psi(t_1), \quad t_1 \in [0,T_1], \end{aligned} \tag{1.2}$$

where u and f are H-valued functions on D, φ (resp., ψ) is H-valued function on $[0,T_2]$ (resp., $[0,T_1]$), λ is a positive parameter, μ is a complex parameter belonging to $\mathcal{M}$, a set of arbitrary nature on which the notion of convergence of sequences is defined and $A(t)$ is an unbounded linear operator in H, with domain of definition $\mathfrak{D}(A(t))$ densely defined and independent of t.

Here we are concerned by the existence and uniqueness of the strong solution to the problem (1.1)-(1.2).

We suppose that $A(t)$ and $B_i(\mu)$, $i = 1,2$, satisfy the following conditions.

Condition ($\mathcal{A}_1$). The operator $A(t)$ is selfadjoint and strongly positive for every $t \in \overline{D}$, that is,

$$A(t) = A(t)^*, \quad (A(t)u, u) \geq c_0|u|^2, \quad \forall u \in \mathcal{D}(A(t)), \tag{1.3}$$

where c_0 is a positive constant not depending on u and t.

$B_1(\mu)$ and $B_2(\mu)$ are two families of operators belonging to the Banach space $\mathcal{L}(H)$ and $\mathcal{D}(A)$ is invariant for these families of operators $B_i(\mu)(\mathcal{D}(A(t)) \subseteq \mathcal{D}(A(t))$. Moreover, the operators $B_i(\mu)$, $i = 1, 2$, satisfy one of the following conditions.

Condition ($\mathcal{B}_1$). The operator $B_1(\mu)$ admits a bounded inverse $B_1^{-1}(\mu)$ in H such that

$$\alpha_1 = \|B_1^{-1}(\mu)B_2(\mu)\|^2_{\mathcal{L}(H)} \exp\left(3C(T_1 + T_2)\right) < 1. \tag{1.4}$$

Condition ($\mathcal{B}_2$). The operator $B_2(\mu)$ admits a bounded inverse $B_2^{-1}(\mu)$ in H such that

$$\alpha_2 = \|B_2^{-1}(\mu)B_1(\mu)\|^2_{\mathcal{L}(H)} \exp\left(3C(T_1 + T_2)\right) < 1, \tag{1.5}$$

where C is a positive constant depending on $A(t)$ and its derivatives.

The functions φ and ψ satisfy the compatibility condition:

$$B_1(\mu)\varphi(0) - B_2(\mu)\varphi(T_2) = B_1(\mu)\psi(0) - B_2(\mu)\psi(T_1). \tag{1.6}$$

Remark 1.1. (1) We note that the case where $B_1(\mu) = \mu_1$ and $B_2(\mu) = \mu_2$ are complex parameters was studied in [29].

(2) If $\lambda = 0$, $B_2(\mu) = 0$, and $B_1(\mu) = I$, we obtain the Goursat conditions, and the results of this case are contained in [7, 10].

In this paper, we continue the investigation started in [29] for a plurihyperbolic equation with nonlocal initial conditions of the nonclassical type. We prove existence and uniqueness of a strong solution of the problem ($\mathcal{P}$ = (1.1)-(1.2)).

We reformulate problem (1.1)-(1.2) as a problem of solving the operator equation

$$Lu = \mathcal{F}, \tag{$\mathcal{E}$}$$

where L is generated by (1.1) and (1.2), with domain of definition $\mathcal{D}(L)$, the operator L is considered from the Banach space $\mathbb{B}$ into the Hilbert space $\mathbb{F}$, which will be defined later. For this operator, we establish an energy inequality

$$\|u\|_{\mathbb{B}} \leq k\|Lu\|_{\mathbb{F}}, \tag{a_1}$$

and we show that the operator L has the closure $\overline{L}$.

Definition 1.2. A solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a strong generalized solution of the problem ($\mathcal{P}$).

Inequality (a_1) can be extended to $u \in \mathcal{D}(\overline{L})$, that is,

$$\|u\|_{\mathbb{B}} \leq k\|\overline{L}u\|_{\mathbb{F}}, \quad \forall u \in \mathcal{D}(\overline{L}). \tag{a_2}$$

From this inequality we obtain the uniqueness of a strong generalized solution if it exists, and the equality of sets $\mathcal{R}(\overline{L})$ and $\overline{\mathcal{R}(L)}$. Thus, to prove the existence of a strong solution of the problem $(\mathcal{P})$ for any $\mathcal{F} \in \mathbb{F}$, it remains to prove that the set $\mathcal{R}(L)$ is dense in $\mathbb{F}$.

2. Preliminaries

In this section, we give the notation and the functional necessary for the sequel.

Let us denote by $W^r = \mathcal{D}(A^r(0))$, $0 \le r \le 1$, the space W^r endowed with the inner product $(x,y)_r = (A^r(0)x, A^r(0)x)$ and the norm $|x|_r = |A^r(0)x|$ is a Hilbert space. We show that the operator $A(t)$ (resp., $A^{1/2}(t)$) is bounded from W^1 (resp., $W^{1/2}$) into H, that is, $A(t)$ (resp., $A^{1/2}(t)) \in \mathcal{L}(W^1;H)$ (resp., $\mathcal{L}(W^{1/2};H)$) (see [30]).

Proposition 2.1 [7]. *If the function $\overline{D} \ni t \mapsto A(t) \in \mathcal{L}(W^1;H)$ is continuous with respect to the topology of $\mathcal{L}(W^1;H)$, then there exist positive constants c_1 and c_2 such that*

$$\begin{gathered} c_1|u|_1 \le \left|A(t)u\right| \le c_2|u|_1, \quad \forall u \in W^1, \\ \sqrt{c_1}|u|_{1/2} \le \left|A^{1/2}(t)u\right| \le \sqrt{c_2}|u|_{1/2}, \quad \forall u \in W^{1/2}. \end{gathered} \tag{2.1}$$

Lemma 2.2. *If the function $\overline{D} \ni t \mapsto A(t) \in \mathcal{L}(W^1;H)$ admits bounded derivatives with respect to t_1 and t_2 with respect to the simple topology in $\mathcal{L}(W^1;H)$, then one has the estimates*

$$\left\|\frac{\partial A(t)^{1/2}}{\partial t_i}A(t)^{-1/2}\right\|_{\mathcal{L}(H)} \le \delta\left\|\frac{\partial A(t)}{\partial t_i}A(t)^{-1}\right\|_{\mathcal{L}(H)}, \quad i = 1,2, \tag{2.2}$$

where $\delta = \int_0^\infty \sqrt{s}/(1+s)^2 ds$. (See [30, Lemma 1.9, page 186].)

Proposition 2.3. *The operators $(\partial A(t)/\partial t_i)A(t)^{-1}$, $(\partial A(t)^{1/2}/\partial t_i)A(t)^{-1/2}$ are uniformly bounded, that is, $(\partial A(t)/\partial t_i)A(t)^{-1}$, $(\partial A(t)^{1/2}/\partial t_i)A(t)^{-1/2} \in L_\infty(D;\mathcal{L}(H))$, $i = 1,2$.*

Proof. The proof is based on the theorem of Banach-Steinhaus and the estimates (2.1) and (2.2). □

We denote by $H^{1,1}(D;W^1)$ the space obtained by completing $\mathcal{C}^\infty(\overline{D};W^1)$ with respect to the norm

$$\|u\|_{1,1}^2 = \int_D \left(\left|\frac{\partial^2 u}{\partial t_1 \partial t_2}\right|_1^2 + \left|\frac{\partial u}{\partial t_1}\right|_1^2 + \left|\frac{\partial u}{\partial t_2}\right|_1^2 + |u|_1^2\right) dt. \tag{2.3}$$

Let $H^1([0,T_i];W^{1/2})$ be the obtained space by completing $\mathcal{C}^\infty([0,T_i];W^{1/2})$, $i = 1,2$ with respect to the norms

$$\begin{aligned} \|\varphi\|_1^2 &= \int_0^{T_2} \left(|\varphi'|^2 + |\varphi|_{1/2}^2 + \lambda|\varphi'|_{1/2}^2 + \lambda|\varphi|_1^2 + \lambda^2|\varphi'|_1^2\right) dt_2, \\ \|\psi\|_1^2 &= \int_0^{T_1} \left(|\psi'|^2 + |\psi|_{1/2}^2 + \lambda|\psi'|_{1/2}^2 + \lambda|\psi|_1^2 + \lambda^2|\psi'|_1^2\right) dt_1. \end{aligned} \tag{2.4}$$

By completing the space $\mathscr{C}^\infty(\overline{D};W^1)$ with respect to the norm

$$|||u|||_1^2 = \frac{\sigma_i(\mu)}{\lambda+1}\bigg[\int_D\bigg(\lambda\left|\frac{\partial^2 u}{\partial t_1\partial t_2}\right|^2+\lambda^2\left|\frac{\partial^2 u}{\partial t_1\partial t_2}\right|_{1/2}^2+\lambda^3\left|\frac{\partial^2 u}{\partial t_1\partial t_2}\right|_1^2\bigg)dt \\ +\sup_{\tau\in D}\Big(|u(\tau_1,\cdot)\|_1^2+\|u(\cdot,\tau_2)\|_1^2\Big)\bigg], \tag{2.5}$$

where $\sigma_i(\mu)=(\alpha_i(1-\alpha_i))^2/(1+\alpha_i)^4(1+\|B_i^{-1}\|^2_{\mathscr{L}(H)})$, $i=1,2$ according to the realization of conditions $(\mathscr{B}_1)$ or $(\mathscr{B}_2)$, we obtain the space $\mathbb{E}_{\lambda,\mu}$.

Denoting by $\mathscr{V}$ the Hilbert space, we get

$$L_2(D;H)\times\mathscr{V}^1([0,T_2];W^{1/2})\times\mathscr{V}^1([0,T_1];W^{1/2}) \tag{2.6}$$

whose elements are $\mathscr{F}=(f,\varphi,\psi)$ such that the norm

$$|||\mathscr{F}|||^2=\|f\|^2+\|\varphi\|_1^2+\|\psi\|_1^2 \text{ is finite.} \tag{2.7}$$

The symbol $\|\cdot\|$ denotes the $L_2(D;H)$-norm.

$\mathscr{V}^1([0,T_2];W^{1/2})\times\mathscr{V}^1([0,T_1];W^{1/2})$ is the closed subspace of $H^1([0,T_2];W^{1/2})\times H^1([0,T_1];W^{1/2})$ composed of elements (φ,ψ) satisfying (1.6).

Let

$$C=\max(d_1,d_2),\qquad d_i=2(\delta+1)\left\|\frac{\partial A(t)}{\partial t_i}A^{-1}(t)\right\|_{\mathscr{L}(H)},\quad i=1,2, \\ \mathscr{N}=\{\mu\in\mathscr{M}:\alpha_1<1 \text{ or } \alpha_2<1\}. \tag{2.8}$$

The following technical lemmas will be needed in the analysis of the problem.

Lemma 2.4 (generalized Gronwall's lemma). (G1) *Let $v(t_1,t_2)$ and $F(t_1,t_2)$ be two nonnegative integrable functions on D such that the function $F(t_1,t_2)$ is nondecreasing with respect to the variables t_1 and t_2. Then the inequality*

$$v(t_1,t_2)\le c_3\left\{\int_0^{t_1}v(\tau_1,t_2)d\tau_1+\int_0^{t_2}v(t_1,\tau_2)d\tau_2\right\}+F(t_1,t_2),\quad c_3\ge 0, \tag{I_1}$$

gives

$$v(t_1,t_2)\le\exp(2c_3(t_1+t_2))F(t_1,t_2). \tag{I_1'}$$

(G2) *Let $v(t_1,t_2)$ and $G(t_1,t_2)$ be two nonnegative integrable functions on D such that the function $G(t_1,t_2)$ is nonincreasing with respect to the variables t_1 and t_2. Then the inequality*

$$v(t_1,t_2)\le c_4\left\{\int_{t_1}^{T_1}v(\tau_1,t_2)d\tau_1+\int_{t_2}^{T_2}v(t_1,\tau_2)d\tau_2\right\}+G(t_1,t_2),\quad c_4\ge 0, \tag{I_2}$$

yields

$$v(t_1,t_2) \leq \exp\left(2c_4(T_1+T_2-t_1-t_2)\right)G(t_1,t_2). \tag{I_2'}$$

Proof. We limit ourselves to proving the version (G1), and with the same manner we deduce the version (G2).

Inequality (I_1) can be rewritten as follows:

$$v \leq c_3\mathcal{J}v + F, \tag{a_1}$$

where $\mathcal{J}$ is the linear integral operator

$$\mathcal{J}(v)(t_1,t_2) = \int_0^{t_1} v(\tau_1,t_2)\,d\tau_1 + \int_0^{t_2} v(t_1,\tau_2)\,d\tau_2. \tag{2.9}$$

Applying the operator $\mathcal{J}$ to the inequality (a_1) and multiplying the result by c_3, we obtain

$$c_3\mathcal{J}v \leq c_3^2\mathcal{J}^2 v + c_3\mathcal{J}F, \tag{a_2}$$

which gives

$$v \leq c_3^2\mathcal{J}^2 v + c_3\mathcal{J}F + F. \tag{a_3}$$

By repeating this process n-times, we derive

$$v \leq c_3^n\mathcal{J}^{n+1}v + \sum_{k=0}^{k=n}\mathcal{J}^k F. \tag{a_4}$$

Since the function $F(t_1,t_2)$ is nonnegative and nondecreasing with respect to the variables t_1 and t_2, we can estimate $\sum_{k=0}^{k=n}\mathcal{J}^k F$ as follows:

$$\sum_{k=0}^{k=n}\mathcal{J}^k(F)(t_1,t_2) \leq \sum_{k=0}^{k=n}\frac{c_3^k(t_1+t_2)^k F(t_1,t_2)}{n!}. \tag{a_5}$$

Similarly the quantity $c_3^n\mathcal{J}^{n+1}v$ can be estimated as follows:

$$c_3^n\mathcal{J}^{n+1}(v)(t_1,t_2) \leq \frac{c_3^n 2^{n+1}(t_1+t_2)^{n+1}|v|_\infty}{(n+1)!}. \tag{a_6}$$

Combining (a_4), (a_5), and (a_6), we obtain

$$v(t_1,t_2) \leq \sum_{k=0}^{k=n}\frac{c_3^k(t_1+t_2)^k F(t_1,t_2)}{n!} + \frac{c_3^n 2^{n+1}(t_1+t_2)^{n+1}|v|_\infty}{(n+1)!}. \tag{a_7}$$

Observing that

$$\frac{c_3^n 2^{n+1}(t_1+t_2)^{n+1}|v|_\infty}{(n+1)!} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$
$$\sum_{k=0}^{k=n} \frac{c_3^k(t_1+t_2)^k F(t_1,t_2)}{n!} \longrightarrow \exp\left(c_3(t_1+t_2)\right)F(t_1,t_2) \quad \text{as } n \longrightarrow \infty, \tag{2.10}$$

then passing to the limit, as $n \to \infty$ in (a_7), we obtain the desired inequality (I_1'). □

Lemma 2.5. *Let* $|\cdot|_m$ *be the norm in* W^m $(m = 0, 1/2, 1)$, *let* g *be a function of variable* $t \in [0,T]$ *in* W^m, *and let* $h = B_1(\mu)g(0) - B_2(\mu)g(T)$. *Then, if the condition* $(\mathcal{B}_1)$ *holds, one has*

$$\frac{1}{2}(1+\alpha_1)\,|g(0)|_m^2 - ||B_1^{-1}(\mu)B_2(\mu)||_{\mathcal{L}(H)}^2\,|g(T)|_m^2 \leq \frac{(1+\alpha_1)||B_1^{-1}(\mu)||_{\mathcal{L}(H)}^2}{(1-\alpha_1)}|h|_m^2, \tag{2.11}$$

and if the condition $(\mathcal{B}_2)$ *holds, one has*

$$\frac{1}{2}(1+\alpha_2)\,|g(T)|_m^2 - ||B_2^{-1}(\mu)B_1(\mu)||_{\mathcal{L}(H)}^2\,|g(0)|_m^2 \leq \frac{(1+\alpha_2)||B_2^{-1}(\mu)||_{\mathcal{L}(H)}^2}{(1-\alpha_2)}|h|_m^2. \tag{2.12}$$

(See [23].)

Lemma 2.6 (the method of continuity). *Let* E_1, E_2 *be two Banach spaces and let* L_0, L_1 *be bounded operators from* E_1 *into* E_2. *For each* $r \in [0,1]$, *set*

$$L_r = (1-r)L_0 + rL_1 \tag{2.13}$$

and suppose that there is a constant k such that

$$\|u\|_{E_1} \leq k\|L_r u\|_{E_2} \tag{2.14}$$

for $r \in [0,1]$. *Then* L_1 *maps* E_1 *onto* E_2 *if and only if* L_0 *maps* E_1 *onto* E_2. *(See [31, Theorem 5.2, page 75].)*

We also need the ε-inequality: $2|ab| \leq \varepsilon|a|^2 + \varepsilon^{-1}|b|^2$, $\varepsilon > 0$.

3. Uniqueness and continuous dependence

For the operator $L_{\lambda,\mu} = (\mathcal{L}_\lambda, l_{1\mu}, l_{2\mu})$ acting from $\mathbb{E}_{\lambda,\mu}$ into $\mathcal{V}$ with domain of definition $\mathfrak{D}(L_{\lambda,\mu}) = H^{1,1}(D;W^1) \subset \mathbb{E}_{\lambda,\mu}$ we establish an a priori estimate and some corollaries resulting directly from this estimate.

For this purpose, we assume the following.

Condition $(\mathcal{A}_2)$. (1)

$$A(t_1,T_2) = A(t_1,0), \quad t_1 \in [0,T_1]; \tag{3.1}$$

(2)

$$A(T_1,t_2) = A(0,t_2), \quad t_2 \in [0,T_2]; \tag{3.2}$$

(3)

$$\frac{\partial A(t_1,0)}{\partial t_1} B_i(\mu)u = B_i(\mu)\frac{\partial A(t_1,0)}{\partial t_1}u, \quad i = 1,2,\ \mu \in \mathcal{N},\ u \in \mathfrak{D}(A(t)); \tag{3.3}$$

(4)

$$\frac{\partial A(0,t_2)}{\partial t_2} B_i(\mu)u = B_i(\mu)\frac{\partial A(0,t_2)}{\partial t_2}u, \quad i = 1,2,\ \mu \in \mathcal{N},\ u \in \mathfrak{D}(A(t)); \tag{3.4}$$

(5)

$$B_j(\mu)B_{3-j}(\mu)u = B_{3-j}(\mu)B_j(\mu)u, \quad (j = 1,2),\ \mu \in \mathcal{N},\ u \in \mathfrak{D}(A(t)). \tag{3.5}$$

We are now in a position to state and to prove the main results of this paper.

Theorem 3.1. *Let the function $D \ni t \mapsto A(t) \in \mathcal{L}(W^1;H)$ have bounded derivatives with respect to t_1 and t_2 with respect to the simple convergence topology of $\mathcal{L}(W^1;H)$ and let the conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{B}_1)$ or $(\mathcal{B}_2)$ be fulfilled. Then, one has*

$$|||u|||_1^2 \leq S|||L_{\lambda,\mu}u|||^2, \quad \forall u \in H^{1,1}(D;W^1), \tag{3.6}$$

where S is a positive constant independent of λ, μ, and u.

Proof. Taking the inner product of the expression $\mathcal{L}_\lambda u$ and $Mu = \partial u/\partial t_1 + \partial u/\partial t_2 + \lambda A(\partial u/\partial t_1 + \partial u/\partial t_2)$, we get

$$\frac{\partial}{\partial t_1}(F(t_2)) + \frac{\partial}{\partial t_2}(F(t_1)) = G(t), \tag{3.7}$$

where

$$F_2(t)=\left\{\left|\frac{\partial u}{\partial t_2}\right|^2+\left|A^{1/2}u\right|^2+2\lambda\left|A^{1/2}\frac{\partial u}{\partial t_2}\right|^2+\lambda|Au|^2+\lambda^2\left|A\frac{\partial u}{\partial t_2}\right|^2\right\},$$

$$F_1(t)=\left\{\left|\frac{\partial u}{\partial t_1}\right|^2+\left|A^{1/2}u\right|^2+2\lambda\left|A^{1/2}\frac{\partial u}{\partial t_1}\right|^2+\lambda|Au|^2+\lambda^2\left|A\frac{\partial u}{\partial t_1}\right|^2\right\},$$

$$\begin{aligned}G(t)={}&2\operatorname{Re}(\mathcal{L}_\lambda u,Mu)+4\lambda\operatorname{Re}\left((A^{1/2})_2\frac{\partial u}{\partial t_1},A^{1/2}\frac{\partial u}{\partial t_1}\right)+2\operatorname{Re}\left((A^{1/2})_1u,A^{1/2}u\right)\\&+2\lambda\operatorname{Re}(A_1u,Au)+2\lambda^2\operatorname{Re}\left(A_2\frac{\partial u}{\partial t_1},A\frac{\partial u}{\partial t_1}\right)+4\lambda\operatorname{Re}\left((A^{1/2})_1\frac{\partial u}{\partial t_2},A^{1/2}\frac{\partial u}{\partial t_2}\right)\\&+2\operatorname{Re}\left((A^{1/2})_2u,A^{1/2}u\right)+2\lambda\operatorname{Re}(A_2u,Au)+2\lambda^2\operatorname{Re}\left(A_1\frac{\partial u}{\partial t_2},A\frac{\partial u}{\partial t_2}\right),\end{aligned}$$

$$A_i=\frac{\partial}{\partial t_i}(A(t)),\quad (A^{1/2})_i=\frac{\partial}{\partial t_i}(A(t)^{1/2}),\quad i=1,2. \tag{3.8}$$

Integrating the identity (3.7) over $D_\tau=]0,\tau_1[\times]0,\tau_2[\subset D$, we get

$$\int_0^{\tau_1}F_1(t_1,\tau_2)dt_1+\int_0^{\tau_2}F_2(\tau_1,t_2)dt_2=\int_0^{\tau_2}\int_0^{\tau_1}G(t)dt+\int_0^{\tau_1}F_1(t_1,0)dt_1+\int_0^{\tau_2}F_2(0,t_2)dt_2. \tag{3.9}$$

By making use of (2.1), (2.2) and some elementary estimates, we derive the following inequality:

(i)

$$\begin{aligned}&\int_0^{\tau_1}F_1(t_1,\tau_2)dt_1+\int_0^{\tau_2}F_2(\tau_1,t_2)dt_2\\&\quad\le\int_0^{\tau_2}\int_0^{\tau_1}2\left|(\mathcal{L}_\lambda u,Mu)\right|dt+C\int_0^{\tau_2}\int_0^{\tau_1}(F_1(t)+F_2(t))dt\\&\quad+\int_0^{\tau_1}F_1(t_1,0)dt_1+\int_0^{\tau_2}F_2(0,t_2)dt_2.\end{aligned} \tag{3.10}$$

By making similar calculations in the rectangles $]\tau_1,T_1[\times]\tau_2,T_2[$, $]0,\tau_1[\times]\tau_2,T_2[$ end $]\tau_1,T_1[\times]0,\tau_2[$, respectively, we get

(ii)

$$\begin{aligned}
&-\int_{\tau_1}^{T_1} F_1(t_1,\tau_2)\,dt_1 - \int_{\tau_2}^{T_2} F_2(\tau_1,t_2)\,dt_2 \\
&\qquad \le \int_{\tau_2}^{T_2}\int_{\tau_1}^{T_1} 2\,|(\mathcal{L}_\lambda u, Mu)|\,dt + C\int_{\tau_2}^{T_2}\int_{\tau_1}^{T_1}(F_1(t)+F_2(t))\,dt \\
&\qquad\quad - \int_{\tau_1}^{T_1} F_1(t_1,T_2)\,dt_1 - \int_{\tau_2}^{T_2} F_2(T_1,t_2)\,dt_2,
\end{aligned} \tag{3.11}$$

(iii)

$$\begin{aligned}
&\int_{\tau_2}^{T_2} F_2(\tau_1,t_2)\,dt_2 - \int_{0}^{\tau_1} F_1(t_1,\tau_2)\,dt_1 \\
&\qquad \le \int_{\tau_2}^{T_2}\int_{0}^{\tau_1} 2\,|(\mathcal{L}_\lambda u, Mu)|\,dt + C\int_{\tau_2}^{T_2}\int_{0}^{\tau_1}(F_1(t)+F_2(t))\,dt \\
&\qquad\quad + \int_{\tau_2}^{T_2} F_2(0,t_2)\,dt_2 - \int_{0}^{\tau_1} F_1(t_1,T_2)\,dt_1,
\end{aligned} \tag{3.12}$$

(iv)

$$\begin{aligned}
&\int_{\tau_1}^{T_1} F_1(t_1,\tau_2)\,dt_1 - \int_{0}^{\tau_2} F_2(\tau_1,t_2)\,dt_2 \\
&\qquad \le \int_{0}^{\tau_2}\int_{\tau_1}^{T_1} 2\,|(\mathcal{L}_\lambda u, Mu)|\,dt + C\int_{0}^{\tau_2}\int_{\tau_1}^{T_1}(F_1(t)+F_2(t))\,dt \\
&\qquad\quad + \int_{\tau_1}^{T_1} F_1(t_1,0)\,dt_1 - \int_{0}^{\tau_2} F_2(T_1,t_2)\,dt_2.
\end{aligned} \tag{3.13}$$

In this step, we study the case where the condition $(\mathcal{B}_1)$ is realized, the case $(\mathcal{B}_2)$ is treated by the same methodology. Let the condition $(\mathcal{B}_1)$ be fulfilled.

By a straightforward application of Lemma 2.4 to (3.10), we obtain

$$\begin{aligned}
&\int_{0}^{\tau_1} F_1(t_1,\tau_2)\,dt_1 + \int_{0}^{\tau_2} F_2(\tau_1,t_2)\,dt_2 \\
&\qquad \le \exp\left(3C(T_1+T_2)\right)\left[\int_{0}^{\tau_2}\int_{0}^{\tau_1} 2\,|(\mathcal{L}_\lambda u, Mu)|\,dt + \int_{0}^{\tau_1} F_1(t_1,0)\,dt_1 + \int_{0}^{\tau_2} F_2(0,t_2)\,dt_2\right].
\end{aligned} \tag{3.14}$$

For the inequality (3.12), we can write

$$\int_{\tau_2}^{T_2} F_2(\tau_1,t_2)\,dt_2 + \int_0^{\tau_1} F_1(t_1,T_2)\,dt_1 \\ \le \int_0^{\tau_1}\int_{\tau_2}^{T_2} 2\,|\,(\mathcal{L}_\lambda u, Mu)\,|\,dt + C\int_0^{\tau_1}\int_{\tau_2}^{T_2} (F_1(t)+F_2(t))\,dt \\ + \int_0^{\tau_1} F_1(t_1,\tau_2)\,dt_1 + \int_{\tau_2}^{T_2} F_2(0,t_2)\,dt_2. \tag{3.15}$$

We fix the variable τ_2 and consider the function

$$Y(\tau_1,\tau_2) = \int_{\tau_2}^{T_2} F_2(\tau_1,t_2)\,dt_2 + \int_0^{\tau_1} F_1(t_1,T_2)\,dt_1 \tag{3.16}$$

as a function of one variable τ_1 with a parameter τ_2, and by using the classical Gronwall lemma we derive the following inequality:

$$\int_{\tau_2}^{T_2} F_2(\tau_1,t_2)\,dt_2 - \exp(CT_1)_1\int_0^{\tau_1} F_1(t_1,\tau_2)\,dt_1 \\ \le \exp(CT_1)\left[\int_{\tau_2}^{T_2}\int_0^{\tau_1} 2\,|\,(\mathcal{L}_\lambda u, Mu)\,|\,dt + C\int_{\tau_2}^{T_2}\int_0^{\tau_1} F_1(t)dt + \int_{\tau_2}^{T_2} F_2(0,t_2)\,dt_2\right] \\ - \int_0^{\tau_1} F_1(t_1,T_2)\,dt_1. \tag{3.17}$$

In a similar way, we derive from (3.13) the inequality

$$\int_{\tau_1}^{T_1} F_1(t_1,\tau_2)\,dt_1 - \exp(CT_2)\int_0^{\tau_2} F_2(\tau_1,t_2)\,dt_2 \\ \le \exp(CT_2)\left[\int_0^{\tau_2}\int_{\tau_1}^{T_1} 2\,|\,(\mathcal{L}_\lambda u, Mu)\,|\,dt + C\int_0^{\tau_2}\int_{\tau_1}^{T_1} F_2(t)dt + \int_{\tau_1}^{T_1} F_1(t_1,0)\,dt_1\right] \\ - \int_0^{\tau_2} F_2(T_1,t_2)\,dt_2. \tag{3.18}$$

Multiplying the inequalities (3.14) by $(1/4)(1+\alpha_1)^2$, (3.17) by $(1/2)(1+\alpha_1)\alpha_1\exp(CT_2)$, (3.18) by $1/2(1+\alpha_1)\alpha_1\exp(CT_1)$, and (3.11) by α_1^2 and summing up the obtained inequalities after using of some elementary estimates, we get

$$\begin{aligned}
&\frac{1}{4}(1+\alpha_1)(1-\alpha_1)\left[\int_0^{\tau_1} F_1(t_1,\tau_2)\,dt_1+\int_0^{\tau_2} F_2(\tau_1,t_2)\,dt_2\right]\\
&\quad+\frac{1}{2}(1-\alpha_1)\alpha_1\left[\int_{\tau_1}^{T_1} F_1(t_1,\tau_2)\,dt_1+\int_{\tau 2}^{T_2} F_2(\tau_1,t_2)\,dt_2\right]\\
&\quad\le\frac{1}{4}(1+\alpha_1)^2\eta\int_0^{T_2}\int_0^{T_1} 2(\mathcal{L}_\lambda,Mu)\,dt\\
&\quad+\frac{1}{2}(1+\alpha_1)\eta C\left[\int_{\tau_1}^{T_1}\int_0^{T_2} F_2(t)\,dt+\int_{\tau_2}^{T_2}\int_0^{T_1} F_1(t)\,dt\right]\\
&\quad+\frac{1}{2}(1+\alpha_1)\eta\left[\frac{1}{2}(1+\alpha_1)\int_0^{T_1} F_1(t_1,0)\,dt_1-\|B_2^{-1}(\mu)B_1(\mu)\|_{\mathcal{L}(H)}^2\int_0^{T_1} F_1(t_1,T_2)\,dt_1\right]\\
&\quad+\frac{1}{2}(1+\alpha_1)\eta\left[\frac{1}{2}(1+\alpha_1)\int_0^{T_2} F_2(0,t_2)\,dt_2-\|B_2^{-1}(\mu)B_1(\mu)\|_{\mathcal{L}(H)}^2\int_0^{T_2} F_2(T_1,t_2)\,dt_2\right],
\end{aligned}\tag{3.19}$$

where $\eta=\exp(3C(T_1+T_2))$.

For simplicity we put

$$\begin{aligned}
Q_1(\tau_1,\tau_2)&=\frac{1}{4}(1+\alpha_1)(1-\alpha_1)\left[\int_0^{\tau_1} F_1(t_1,\tau_2)\,dt_1+\int_0^{\tau_2} F_2(\tau_1,t_2)\,dt_2\right],\\
Q_2(\tau_1,\tau_2)&=\frac{1}{2}(1-\alpha_1)\alpha_1\left[\int_{\tau_1}^{T_1} F_1(t_1,\tau_2)\,dt_1+\int_{\tau 2}^{T_2} F_2(\tau_1,t_2)\,dt_2\right],\\
R&=\frac{1}{4}(1+\alpha_1)^2\eta\int_0^{T_2}\int_0^{T_1} 2(\mathcal{L}_\lambda,Mu)\,dt,\\
H(\tau_1,\tau_2)&=\frac{1}{2}(1+\alpha_1)\alpha_1\eta C\left[\int_{\tau_1}^{T_1}\int_0^{T_2} F_2(t)\,dt+\int_{\tau_2}^{T_2}\int_0^{T_1} F_1(t)\,dt\right],\\
N_1&=\frac{1}{2}(1+\alpha_1)\eta\left[\frac{1}{2}(1+\alpha_1)\int_0^{T_1} F_1(t_1,0)\,dt_1-\|B_2^{-1}(\mu)B_1(\mu)\|_{\mathcal{L}(H)}^2\int_0^{T_1} F_1(t_1,T_2)\,dt_1\right].
\end{aligned}\tag{3.20}$$

The inequality (3.19) can be rewritten as

$$Q_1+Q_2\le H+R+N_1+N_2.\tag{3.21}$$

By virtue of (2.1) and Lemma 2.5, the quantities N_1 and N_2 are dominated as follows:

$$N_1+N_2 \leq k_1\left[\|l_{1\mu}u\|_1^2+\|l_{2\mu}u\|_1^2\right]=N_3, \tag{3.22}$$

where $k_1=(1/2)((1+\alpha_1)^2/(1-\alpha_1))\|B_1(\mu)^{-1}\|^2_{\mathscr{L}(H)}\exp(3C(T_1+T_2))\max(1,c_2^2)$.

Let us consider the first case $(0<\alpha_1<1/3)$. We observe that $((1/2)(1+\alpha_1)\leq 2(1-\alpha_1))$, which yields

$$\begin{aligned}\mathfrak{Q}(\tau_1,\tau_2)&=\alpha_1(1-\alpha_1)\left[\int_0^{T_1}F_1(t_1,\tau_1)dt_1+\int_0^{T_2}F_2(\tau_1,t_2)dt_2\right]\\&\leq 2R+2N_3+4(1-\alpha_1)\alpha_1\eta C\left[\int_{\tau_1}^{T_1}\int_0^{T_2}F_2(t)dt+\int_{\tau_2}^{T_2}\int_0^{T_1}F_1(t)dt\right].\end{aligned} \tag{3.23}$$

Hence, by (G2) of Gronwall's lemma it follows that

$$\mathfrak{Q}(\tau_1,\tau_2)\leq\theta(2R+2N_3)=N_4, \tag{3.24}$$

where $\theta=\exp(8C\exp(C(T_1+T_2))(T_1+T_2))$.

By using the ε-inequality, the quantity N_4 can be estimated as follows:

$$\begin{aligned}N_4\leq{}&\theta^2(1+\alpha_1)^2\left[(\varepsilon_1^{-1}+\varepsilon_2^{-1})\|\mathscr{L}_\lambda u\|^2+\varepsilon_1\int_0^{T_2}\int_0^{T_1}F_1(t)dt+\varepsilon_2\int_0^{T_2}\int_0^{T_1}F_2(t)dt\right]\\&+\theta^2\frac{(1+\alpha_1)^2}{(1-\alpha)}\|B_1^{-1}(\mu)\|^2_{\mathscr{L}(H)}\max(1,c_2^2)\left[\|l_{1\mu}u\|_1^2+\|l_{2\mu}u\|_1^2\right],\end{aligned} \tag{3.25}$$

which implies

$$\begin{aligned}\mathfrak{Q}(\tau_1,\tau_2)\leq{}&\theta^2(1+\alpha_1)^2\left[(\varepsilon_1^{-1}+\varepsilon_2^{-1})\|\mathscr{L}_\lambda u\|^2+\varepsilon_1\int_0^{T_2}\int_0^{T_1}F_1(t)dt+\varepsilon_2\int_0^{T_2}\int_0^{T_1}F_2(t)dt\right]\\&+\theta^2\frac{(1+\alpha_1)^2}{(1-\alpha)}\|B_1^{-1}(\mu)\|^2_{\mathscr{L}(H)}\max(1,c_2^2)\left[\|l_{1\mu}u\|_1^2+\|l_{2\mu}u\|_1^2\right].\end{aligned} \tag{3.26}$$

Taking $\varepsilon_i=\alpha_1(1-\alpha_1)/2\theta^2(1+\alpha_1)^2T_{3-i}$ and integrating (3.26) with respect to τ_i from 0 to T_i, $i=1,2$, we obtain

$$\begin{aligned}&\frac{1}{2}\alpha_1(1-\alpha_1)\left[T_1\int_0^{T_2}\int_0^{T_1}F_1(t)dt+T_2\int_0^{T_2}\int_0^{T_2}F_2(t)dt\right]\\&\quad\leq\gamma_1\|\mathscr{L}_\lambda u\|^2+\gamma_2\left[\|l_{1\mu}u\|_1^2+\|l_{2\mu}u\|_1^2\right],\end{aligned} \tag{3.27}$$

where

$$\gamma_1 = \frac{2\theta^4(1+\alpha_1)^4(T_1+T_2)T_1T_2}{\alpha_1(1-\alpha_1)},$$
$$\gamma_2 = \frac{\theta^2(1+\alpha_1)^2(T_1+T_2)}{\alpha_1(1-\alpha_1)}||B_1^{-1}(\mu)||^2_{\mathcal{L}(H)}\max(1,c_2^2). \tag{3.28}$$

By combining (3.26) and (3.27), it follows that

$$\mathcal{Q}(\tau_1,\tau_2) \le \gamma_3[||\mathcal{L}_\lambda u||^2 + ||l_{1\mu}u||_1^2 + ||l_{2\mu}u||_1^2], \tag{3.29}$$

with $\gamma_3 = (2\theta^4(1+\alpha_1)^4/\alpha_1(1-\alpha_1))(T_1+T_2+1)^3(1+\|B_1^{-1}(\mu)\|^2_{\mathcal{L}(H)})\max(1,c_2^2)$. By virtue of (3.29), (2.1), we obtain

$$\sigma_1(\mu)[||u(\cdot,\tau_2)||^2 + ||u(\tau_1,\cdot)||^2] \le S_1[||\mathcal{L}_\lambda u||^2 + ||l_{1\mu}u||_1^2 + ||l_{2\mu}u||_1^2], \tag{3.30}$$

with $S_1 = 2\theta^4(T_1+T_2+1)^3(\max(1,c_2^2)/\min(1,c_1^2))$.

Multiplying (1.1) by $\sqrt{\lambda}$ and estimating with $L_2(D;H)$-norm by use of (2.1) and (3.30), we derive the following inequality

$$\min(1,c_1^2)\sigma_1(\mu)\left[\int_D\left(\lambda\left|\frac{\partial^2 u}{\partial t_1\partial t_2}\right|^2 + \lambda^2\left|\frac{\partial^2 u}{\partial t_1\partial t_2}\right|^2_{1/2} + \lambda^3\left|\frac{\partial^2 u}{\partial t_1\partial t_2}\right|^2_1\right]dt\right]$$
$$\le S_2[||\mathcal{L}_\lambda u||^2 + ||l_{1\mu}u||_1^2 + ||l_{2\mu}u||_1^2], \tag{3.31}$$

with $S_2 = 4\theta^4(T_1+T_2+1)^4\max(1,c_2^2)(1+\lambda)$.

Combining (3.30) and (3.31), we obtain

$$\frac{\sigma_1(\mu)}{1+\lambda}\left[\int_D\left(\lambda\left|\frac{\partial^2 u}{\partial t_1\partial t_2}\right|^2 + \lambda^2\left|\frac{\partial^2 u}{\partial t_1\partial t_2}\right|^2_{1/2} + \lambda^3\left|\frac{\partial^2 u}{\partial t_1\partial t_2}\right|^2_1\right)dt + (||u(\cdot,\tau_2)||^2 + ||u(\tau_1,\cdot)||^2)\right]$$
$$\le S_3[||\mathcal{L}_\lambda u||^2 + ||l_{1\mu}u||_1^2 + ||l_{2\mu}u||_1^2], \tag{3.32}$$

where $S_3 = 6\theta^4(T_1+T_2+1)^4(\max(1,c_2^2)/\min(1,c_1^2))$.

The right-hand side of (3.32) is independent of τ. Hence taking the upper bound of the left-hand side with respect to τ, we obtain the estimate (3.6) with $S = S_3$.

Now, we consider the second case ($1/3 \le \alpha_1 < 1$).

By making the change of variable $\sigma_1(\mu) = (\beta_1(\alpha_1) = (1/2)(1-\alpha_1))$ which implies that $(0 < \beta_1 \le 1/3)$. Observe that $((\beta_1(1-\beta_1))^2/(1+\beta_1)^4 \ge (\alpha_1(1-\alpha_1))^2/4(1+\alpha_1)^4)$ which involves for all $0 < \alpha_1 < 1$,

$$|||u|||_1^2 \le S|||L_{\lambda,\mu}u|||^2, \quad \forall u \in \mathcal{D}(L_{\lambda,\mu}). \tag{3.33}$$

We recall that in the case $(\mathcal{B}_2)$ we proceed with the same methodology used in the case $(\mathcal{B}_1)$ to obtain the desired estimate (3.6). The proof of Theorem 3.1 is complete. □

Now we are interested in the consequences of Theorem 3.1.

It can be proved in the standard way that the operator admits a closure.

Proposition 3.2. *If the conditions of Theorem 3.1 are satisfied, then the operator $L_{\lambda,\mu}$ admits a closure $\overline{L_{\lambda,\mu}}$ with domain of definition denoted by $\mathcal{D}(\overline{L_{\lambda,\mu}})$.*

The solution of the equation

$$\overline{L_{\lambda,\mu}}u = \mathcal{F}, \quad \mathcal{F} \in \mathcal{V}, \tag{3.34}$$

is called a strong generalized solution of problem $(\mathcal{P})$. Passing to the limit, we extend the inequality (3.6) to the strong generalized solution, we obtain

$$\big|\|u\|\big|_1^2 \leq S\big|\big|\big|\overline{L_{\lambda,\mu}}u\big|\big|\big|^2, \quad \forall u \in \mathcal{D}(\overline{L_{\lambda,\mu}}), \tag{3.35}$$

from which we deduce the following.

Corollary 3.3. *From the inequality (3.35), deduce that if the strong generalized solution exists, then it depends continuously on $\mathcal{F} = (f, \varphi, \psi)$.*

Corollary 3.4. *The set of values $\mathcal{R}(\overline{L_{\lambda,\mu}})$ of the operator $\overline{L_{\lambda,\mu}}$ is equal to the closure $\overline{\mathcal{R}(L_{\lambda,\mu})}$ of $R(L_{\lambda,\mu})$ and $(\overline{L_{\lambda,\mu}})^{-1} = \overline{L_{\lambda,\mu}^{-1}}$.*

This corollary allows us to claim that to establish the existence of the strong solution to problem $(\mathcal{P})$ it suffices to prove the density of the set $\mathcal{R}(L_{\lambda,\mu})$ in $\mathcal{V}$.

4. Solvability of the problem

To establish the density of $\mathcal{R}(L_{\lambda,\mu})$ in $\mathcal{F}$, that is, $R(L_{\lambda,\mu})^{\perp} = \{(0,0,0)\}$, we introduce the following Hilbert structure.

Let $H^{1,1}(D;H)$ be the Hilbert space obtained by completion of $\mathcal{C}^{\infty}(\overline{D};H)$ with respect to the norm

$$\|u\|_{1,1}^2 = \int_D \left(\left| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right|^2 + \left| \frac{\partial u}{\partial t_1} \right|^2 + \left| \frac{\partial u}{\partial t_2} \right|^2 + |u|^2 \right) dt. \tag{4.1}$$

Let $H^1([0,T_2];H)$ be the Hilbert space obtained by completion of the space $\mathcal{C}^{\infty}([0,T_2];H)$ with respect to the norm

$$\|\varphi\|_1^2 = \|\varphi\|^2 + \|\varphi'\|^2. \tag{4.2}$$

We construct $H^1([0,T_1];H)$ in a similar manner.

Denote by $\mathcal{W}$ the Hilbert space $L_2(D;H) \times \mathcal{W}^1([0,T_2];H) \times \mathcal{W}^1([0,T_1];H)$ that is composed of elements $\mathcal{F} = (f, \varphi, \psi)$ such that the norm

$$\big|\big|\big|\mathcal{F}\big|\big|\big|^2 = \|f\|^2 + \|\varphi\|_1^2 + \|\psi\|_1^2 \text{ is finite}, \tag{4.3}$$

where $\mathcal{W}^1([0,T_2];H) \times \mathcal{W}^1([0,T_1];H)$ is the closed subspace of $H^1([0,T_2];H) \times$

$H^1([0,T_1];H)$ composed of elements (φ,ψ) such that

$$B_2^*(\mu)\varphi(0) - B_1^*(\mu)\varphi(T_2) = B_2^*(\mu)\psi(0) - B_1^*(\mu)\psi(T_1), \tag{4.4}$$

"$*$" denotes the symbol of the adjoint.

We denote by $H_0^{1,1}(D;W^1)$ the closed subspace of $H^{1,1}(D;W^1)$ defined by

$$\begin{aligned} H_0^{1,1}(D;W^1) = \{u \in H^{1,1}(D;W^1) : B_1(\mu)u\,|_{t_1=0} - B_2(\mu)u\,|_{t_1=T_1} = 0, \\ B_1(\mu)u\,|_{t_2=0} - B_2(\mu)u\,|_{t_2=T_2} = 0\}. \end{aligned} \tag{4.5}$$

$H_0^{1,1}(D;H)$ is the closed subspace of $H^{1,1}(D;H)$ defined by

$$\begin{aligned} H_0^{1,1}(D;H) = \{u \in H^{1,1}(D;H) : B_1(\mu)u\,|_{t_1=0} - B_2(\mu)u\,|_{t_1=T_1} = 0, \\ B_1(\mu)u\,|_{t_2=0} - B_2(\mu)u\,|_{t_2=T_2} = 0\}. \end{aligned} \tag{4.6}$$

$\widehat{H}_0^{1,1}(D;H)$ is the closed subspace of $H^{1,1}(D,H)$ defined by

$$\begin{aligned} \widehat{H}_0^{1,1}(D;H) = \{u \in H^{1,1}(D;H) : B_2^*(\mu)u\,|_{t_1=0} - B_1^*(\mu)u\,|_{t_1=T_1} = 0, \\ B_2^*(\mu)u\,|_{t_2=0} - B_1^*(\mu)u\,|_{t_2=T_2} = 0\}. \end{aligned} \tag{4.7}$$

In proving the existence theorem we meet some difficulties, and to surmount these difficulties, we use the regularization technique (for more details, see [32]).

Definition 4.1. Put

$$\begin{aligned} A_\varepsilon(t) &= (I+\varepsilon A(t)), \qquad J_\varepsilon(t) = A_\varepsilon^{-1}(t) = (I+\varepsilon A(t))^{-1}, \\ R_\varepsilon(t) &= A(t)(I+\varepsilon A(t))^{-1} = \frac{1}{\varepsilon}(I-J_\varepsilon(t)), \quad \varepsilon > 0, \end{aligned} \tag{4.8}$$

and call $R_\varepsilon(t)$ the *Yosida approximation* of $A(t)$.

Some basic properties of R_ε are listed in the following proposition.

Proposition 4.2 (see [33]). *One has*

(1) J_ε, $R_\varepsilon \in \mathscr{L}(H)$, $\|J_\varepsilon\| \le 1$, $\|R_\varepsilon\| \le 1/\varepsilon$, *for all* $\varepsilon > 0$;
(2) $J_\varepsilon A u = A J_\varepsilon u$, *for all* $u \in W^1$.
(3) $|R_\varepsilon u| \le |u|_1$, *for all* $\varepsilon > 0$, *for all* $u \in W^1$;
(4) $\lim_{\varepsilon\to 0} J_\varepsilon u = u$, *for all* $u \in H$;
(5) $\lim_{\varepsilon\to 0} R_\varepsilon u = Au$, *for all* $u \in W^1$.

Let us now establish the density of the set $\mathcal{R}(L_{\lambda,\mu})$ in $\mathcal{V}$. For this purpose, we assume the following.

Condition $(\mathcal{H})$. $D \ni t \mapsto A(t) \in \mathscr{L}(D;W^1)$ admits mixed derivatives

$$\frac{\partial^2 A}{\partial t_1 \partial t_2}, \quad \frac{\partial^2 A}{\partial t_2 \partial t_1} \quad \text{with } \frac{\partial A}{\partial t_1 \partial t_2}A^{-1}, \quad \frac{\partial A}{\partial t_2 \partial t_1}A^{-1} \in L_2(D;\mathscr{L}(H)). \tag{4.9}$$

THEOREM 4.3. *Under the conditions of Theorem 3.1 and the condition $(\mathcal{H})$, the set $\mathcal{R}(L_{\lambda,\mu})$ is dense in $\mathcal{V}$.*

Proof. The idea is to prove the result in the case $\lambda = 0$, that is, $\overline{\mathcal{R}(L_{0,\mu})} = \mathcal{V}$ and by means of the method of continuity we establish the general case.

The case $\boldsymbol{\lambda} = \mathbf{0}$.

Let $\mathcal{L}_0 = \partial^2/\partial t_1 \partial t_2 + A(t)$ be the corresponding operator to $\lambda = 0$ and let $V = (v, \xi, \chi)$ be an orthogonal element to $\mathcal{R}(L_{0,\mu})$. Then we have

$$\langle L_{0,\mu}u, V\rangle_{\mathcal{V}} = \langle \mathcal{L}_0 u, v\rangle + \langle l_{1\mu}u, \xi\rangle + \langle l_{2\mu}u, \chi\rangle = 0, \quad \forall u \in H^{1,1}(D, W^1). \tag{4.10}$$

We need the following proposition.

PROPOSITION 4.4. *If for every $v \in L_2(D;H)$, one has*

$$\langle \mathcal{L}_0 u, v\rangle = \left\langle \frac{\partial^2 u}{\partial t_1 \partial_2} + A(t)u, v\right\rangle = 0, \quad \forall u \in H_0^{1,1}(D;W^1), \tag{4.11}$$

then $v = 0$.

Proof. Let $w = A_\varepsilon^{-1}v$ and $h = A_\varepsilon u$. After substitution in (4.10), we get

$$\left\langle \frac{\partial^2 h}{\partial t_1 \partial t_2} - \frac{\partial}{\partial t_1}(B_{1\varepsilon}^* h) - \frac{\partial}{\partial t_2}(B_{2\varepsilon}^* h), w\right\rangle = -\langle h, (AA_\varepsilon^{-1} + B_{0\varepsilon}A_\varepsilon^{-1})v\rangle. \tag{4.12}$$

Here, h may be considered as an arbitrary function of $H_0^{1,1}(D;H)$ and

$$\begin{aligned} B_{i\varepsilon}^*(t) &= \varepsilon \frac{\partial A(t)}{\partial t_{3-i}} A_\varepsilon^{-1}(t), \quad i = 1,2, \\ B_{0\varepsilon}^*(t) &= \varepsilon \frac{\partial^2 A(t)}{\partial t_2 \partial t_1} A_\varepsilon^{-1}(t), \qquad B_{j\varepsilon}(t) \in \mathcal{L}(H), \quad j = 0,1,2. \end{aligned} \tag{4.13}$$

Equation (4.12) leads to the study of the operators $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{L}'}$ defined by

$$\begin{aligned} \mathcal{D}(\widetilde{\mathcal{L}}) &= \hat{H}_0^{1,1}(D;H), \\ \widetilde{\mathcal{L}}u &= \frac{\partial^2 u}{\partial t_1 \partial t_2} + B_{1\varepsilon}\frac{\partial u}{\partial t_1} + B_{2\varepsilon}\frac{\partial u}{\partial t_2}, \\ D(\widetilde{\mathcal{L}'}) &= H_0^{1,1}(D;H), \\ \widetilde{\mathcal{L}'}u &= \frac{\partial^2 u}{\partial t_1 \partial t_2} - \frac{\partial}{\partial t_1}(B_{1\varepsilon}^* u) - \frac{\partial}{\partial t_2}(B_{2\varepsilon}^* u). \end{aligned} \tag{4.14}$$

We show that $\widetilde{\mathcal{L}'}$ is the adjoint of $\widetilde{\mathcal{L}}$ and we have

$$\langle \widetilde{\mathcal{L}'}v, u\rangle = \langle v, \widetilde{\mathcal{L}}u\rangle, \quad \forall u \in \hat{H}_0^{1,1}(D;H),\ \forall v \in H_0^{1,1}(D;H). \tag{4.15}$$

Equation (4.12) means that for each $\varepsilon \neq 0$, w is the weak solution to the problem

$$\begin{aligned}
\widetilde{\mathcal{L}}w &= \frac{\partial^2 w}{\partial t_1 \partial t_2} + B_{1\varepsilon}\frac{\partial}{\partial t_1}w + B_{2\varepsilon}\frac{\partial}{\partial t_2}w = -(B_{0\varepsilon}A_\varepsilon^{-1} + AA_\varepsilon^{-1})v, \\
\widetilde{l}_{1\mu}w &= B_2^*(\mu)w\,|_{t_1=0} - B_1^*(\mu)w\,|_{t_1=T_1} = 0, \\
\widetilde{l}_{2\mu}w &= B_2^*(\mu)w\,|_{t_2=0} - B_1^*(\mu)w\,|_{t_2=T_2} = 0,
\end{aligned} \tag{4.16}$$

with $v \in L_2(D;H)$, $B_{j\varepsilon} \in \mathcal{L}(H)$, $j = 0,1,2$.

Consider the operator $\widetilde{L} = (\widetilde{\mathcal{L}}, \widetilde{l}_{1\mu}, \widetilde{l}_{2\mu})$ acting from $H^{1,1}(D;H)$ into $\mathcal{W}$. For this operator, we establish the following propositions.

Proposition 4.5. *The operator $\widetilde{L}$ is isomorphism from $H^{1,1}(D;H)$ into $\mathcal{W}$.*

Proof. We must show that $\mathcal{R}(\widetilde{L}) = \mathcal{W}$ and

(i)

$$|||\widetilde{L}u|||^2 \leq K_1 \|u\|_{1,1}^2, \quad \forall u \in H^{1,1}(D;H), \tag{4.17}$$

(ii)

$$\|u\|_{1,1}^2 \leq K_2 |||\widetilde{L}u|||^2, \quad \forall u \in H^{1,1}(D;H), \tag{4.18}$$

where K_1 and K_2 are positive constants independent of u.

(i) By virtue of $B_{i\varepsilon} \in \mathcal{L}(H)$ and

$$\begin{aligned}
\|B_{i\varepsilon}\|_{\mathcal{L}(H)} &= \left\| \varepsilon \frac{\partial A}{\partial t_{3-i}} A_\varepsilon^{-1} \right\|_{\mathcal{L}(H)} = \left\| \frac{\partial A}{\partial t_{3-i}} A^{-1}(I - A_\varepsilon^{-1}) \right\|_{\mathcal{L}(H)} \\
&\leq \left\| \frac{\partial A}{\partial t_{3-i}} A^{-1} \right\|_{\mathcal{L}(H)} \|(I - A_\varepsilon^{-1})\|_{\mathcal{L}(H)} \leq C, \quad i = 1,2,
\end{aligned} \tag{4.19}$$

$|\widetilde{\mathcal{L}}u|^2$ can be an estimate as follows:

$$\begin{aligned}
|\widetilde{\mathcal{L}}u|^2 &= \left| \frac{\partial^2 u}{\partial t_1 \partial t_2} + B_{1\varepsilon}\frac{\partial u}{\partial t_1} + B_{2\varepsilon}\frac{\partial u}{\partial t_2} \right|^2 \\
&\leq \left\{ \left| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right| + \left| B_{1\varepsilon}\frac{\partial u}{\partial t_1} \right| + \left| B_{2\varepsilon}\frac{\partial u}{\partial t_2} \right| \right\}^2 \\
&\leq 4\max(1, C^2) \left\{ \left| \frac{\partial^2 u}{\partial t_1 \partial t_2} \right|^2 + \left| \frac{\partial u}{\partial t_1} \right|^2 + \left| \frac{\partial u}{\partial t_2} \right|^2 + |u|^2 \right\},
\end{aligned} \tag{4.20}$$

which implies that

$$\|\widetilde{\mathcal{L}}u\|^2 \leq 4\max(1, C^2)\|u\|_{1,1}^2, \quad \forall u \in H^{1,1}(D;H). \tag{4.21}$$

By virtue of the continuity of the operators $\widetilde{l_{1\mu}}$, $\widetilde{l_{2\mu}}$ from $H^{1,1}(D;H)$ into $H^1([0,T_2];H)$, $H^1([0,T_1];H)$, respectively, and the inequality (4.21), we obtain the estimate (i).

(ii) Following the same techniques to those used to establish the estimate (3.6) in Theorem 3.1, we establish the estimate (4.17).

From the continuity of the operator $\widetilde{L}$ and the inequality (4.18), we conclude that the operator $\widetilde{L}$ is an isomorphism from $H^{1,1}(D;H)$ into the closed subspace $\mathcal{R}(\widetilde{L}) = \widetilde{L}(H^{1,1}(D;H))$.

To prove that $\mathcal{R}(\widetilde{L_s}) = \mathcal{W}$, we proceed by the method of continuity. For this purpose, we introduce the family of operators $\{\widetilde{L}_s\}_{s\in[0,1]}$ defined by

$$\begin{aligned}
\widetilde{L_s} &= (\widetilde{\mathcal{L}_s}, \widetilde{l_{1\mu}}, \widetilde{l_{2\mu}}), \quad s \in [0,1],\\
\widetilde{\mathcal{L}_s} u &= \frac{\partial^2 u}{\partial t_1 \partial t_2} + sBu, \quad \text{with } Bu = B_{1\varepsilon}\frac{\partial u}{\partial t_1} + B_{2\varepsilon}\frac{\partial u}{\partial t_2},\\
D(\widetilde{L_s}) &= H^{1,1}(D,H).
\end{aligned} \tag{4.22}$$

Step 1. Let us first consider the case where $s = 0$. In this step, we show that the operator $\mathcal{R}(\widetilde{L}_0) = \mathcal{W}$. Before proving this result, we need to give this auxiliary result.

It is well known that if we have two linear bounded operators S_1 and S_2 such that S_1 is invertible and $\|S_1^{-1}S_2\| < 1$ or S_2 is invertible and $\|S_2^{-1}S_1\| < 1$, then the operator $S_1 - S_2$ is invertible.

By virtue of these results and by taking into account conditions $(\mathcal{B}_i)$, $i = 1,2$, and $(\mathcal{A}_2)(5)$, we deduce that the operator $B_2^*(\mu) - B_1^*(\mu)$ is invertible in $\mathcal{L}(H)$.

Now, by using the invertibility of the operator $(B_2^*(\mu) - B_1^*(\mu))$ and a simple integration, we easily show that the solution of the operator equation

$$\begin{aligned}
\widetilde{\mathcal{L}_0} u &= \frac{\partial^2 u}{\partial t_1 \partial t_2} = \widetilde{f}(t),\\
\widetilde{l_{1\mu}} u &\equiv B_2^*(\mu)u\,|_{t_1=0} - B_1^*(\mu)u\,|_{t_1=T_1} = \widetilde{\varphi}(t_2),\\
\widetilde{l_{2\mu}} u &\equiv B_2^*(\mu)u\,|_{t_2=0} - B_1^*(\mu)u\,|_{t_2=T_2} = \widetilde{\psi}(t_1)
\end{aligned} \tag{4.23}$$

is given by the formula

$$\begin{aligned}
u(t_1,t_2) = {} & (B_2^*(\mu) - B_1^*(\mu))^{-1}(\widetilde{\varphi}(t_2) + \widetilde{\psi}(t_1) - B_2^*(\mu)\widetilde{\psi}(0) + B_1^*(\mu)\widetilde{\psi}(T_1))\\
& + \int_0^{t_2}\int_0^{t_1} \widetilde{f}(\tau)d\tau + B_1^*(\mu)(B_2^*(\mu)\\
& - B_1^*(\mu))^{-1}\Big(\int_0^{t_2}\int_0^{T_1} \widetilde{f}(\tau)d\tau + \int_0^{T_2}\int_0^{t_1} \widetilde{f}(\tau)d\tau + B_1^*(\mu)(B_2^*(\mu)\\
& - B_1^*(\mu))^{-1}\int_0^{T_2}\int_0^{T_1} \widetilde{f}(\tau)d\tau\Big).
\end{aligned} \tag{4.24}$$

This shows that the operator $\widetilde{L_0}$ is surjective, and thus $\mathcal{R}(\widetilde{L_0}) = \mathcal{W}$, which ensures that $\widetilde{L_0}$ is an isomorphism from $H^{1,1}(D;H)$ into $\mathcal{W}$.

Step 2. For $s_0,\ s \in [0,1]$, we can write

$$\widetilde{L_s} = \widetilde{L_{s_0}} + (s - s_0)(\widetilde{L_1} - \widetilde{L_0}) \quad \text{with } (\widetilde{L_1} - \widetilde{L_0}) = (B, \widetilde{l_{1\mu}}, \widetilde{l_{2\mu}}). \tag{4.25}$$

It is easy to obtain the estimate

$$\|Bu\|^2 \le 2C^2 \|u\|_{1,1}^2, \quad \forall u \in H^{1,1}(D;H). \tag{4.26}$$

By virtue of the inequality (4.26) and the continuity of the operators $\widetilde{l_{1\mu}}$, $\widetilde{l_{2\mu}}$, we obtain

$$|||(\widetilde{L_1} - \widetilde{L_0})u|||^2 \le K_3 \|u\|_{1,1}^2, \quad \forall u \in H^{1,1}(D;H). \tag{4.27}$$

Now, we prove that

$$\|u\|_{1,1}^2 \le K_4 |||\widetilde{L_s}u|||^2, \quad \forall u \in H^{1,1}(D;H), \tag{4.28}$$

where K_4 is a positive constant independent of u.

Thanks to the inequality (4.18), we have

$$\forall s \in [0,1], \quad \|u\|_{1,1}^2 \le K(s) |||\widetilde{L_s}u|||^2, \quad \forall u \in H^{1,1}(D;H). \tag{4.29}$$

Putting $h(s) = \inf_{u \in H^{1,1}(D;H)}(|||\widetilde{L_s}u|||/\|u\|_{1,1})$, let us show that h is continuous on $[0,1]$.

Let $\varepsilon > 0$ and $\delta = \varepsilon/\sqrt{K_3}$. For $s_0,\ s \in [0,1]$ such that $|s_0 - s| < \delta$, we have

$$\begin{aligned} |||\widetilde{L_s}u||| - |||\widetilde{L_{s_0}}u||| &\le |||\widetilde{L_s}u - \widetilde{L_{s_0}}u||| = |s_0 - s| \, |||\widetilde{L_1}u - \widetilde{L_0}u||| \\ &\le \delta |||\widetilde{L_1}u - \widetilde{L_0}u||| \le \frac{\varepsilon}{\sqrt{K_3}} \sqrt{K_3} \|u\|_{1,1}^2 = \varepsilon \|u\|_{1,1}^2, \end{aligned} \tag{4.30}$$

which implies

$$\frac{|||\widetilde{L_{s_0}}u|||}{\|u\|_{1,1}} - \varepsilon \le \frac{|||\widetilde{L_s}u|||}{\|u\|_{1,1}} \le \frac{|||\widetilde{L_{s_0}}u|||}{\|u\|_{1,1}} + \varepsilon. \tag{4.31}$$

By passing to the inf on $H^{1,1}(D;H)$ in (4.31), we obtain $|h(s) - h(s_0)| \le \varepsilon$. Thus the function h is continuous and reaches its lower bound. Denoting this lower bound by $1/\sqrt{K_4}$, we obtain (4.28).

The equation $\widetilde{L_s}u = F$ can be rewritten under the following form:

$$\widetilde{L_s}u = \widetilde{L_{s_0}}u + (s - s_0)(\widetilde{L_1} - \widetilde{L_0})u = F. \tag{4.32}$$

We suppose that $\mathcal{R}(\widetilde{L_{s_0}}) = \mathcal{W}$, and we prove that $\mathcal{R}(\widetilde{L_s}) = \mathcal{W}$ for s near to s_0.

Equation (4.32) is equivalent to

$$u + (s - s_0)(\widetilde{L_{s_0}})^{-1}(\widetilde{L_1} - \widetilde{L_0})u = (\widetilde{L_{s_0}})^{-1}F. \tag{4.33}$$

From (4.28) and (4.27), we have

$$\begin{aligned} ||(\widetilde{L_{s_0}})^{-1}F||_{1,1} &\le \sqrt{K_4}|||F|||, \\ ||(\widetilde{L_{s_0}})^{-1}(\widetilde{L_1}-\widetilde{L_0})u||_{1,1} &\le \sqrt{K_4}|||(\widetilde{L_1}-\widetilde{L_0})u||| \le \sqrt{K_4}\sqrt{K_3}\|u\|_{1,1} = K_5\|u\|_{1,1}. \end{aligned} \tag{4.34}$$

Denoting by

$$\mathcal{T} = (s-s_0)(\widetilde{L_{s_0}})^{-1}(\widetilde{L_1}-\widetilde{L_0}), \qquad g = (\widetilde{L_{s_0}})^{-1}F, \tag{4.35}$$

then (4.33) becomes

$$u+\mathcal{T}u = g. \tag{4.36}$$

Let $s \in [0,1]$ such that $|s_0-s| \le \rho < 1/K_5$, then

$$\|\mathcal{T}\| = \sup_{\|u\|_{1,1}\le 1} \|\mathcal{T}u\|_{1,1} = |s-s_0|\,||(\widetilde{L_{s_0}})^{-1}(\widetilde{L_1}-\widetilde{L_0})u||_{1,1} \le |s-s_0|K_5 < 1. \tag{4.37}$$

Hence the operator $(I+\mathcal{T})$ is invertible, and the solution of (4.36) is given by the Neumann series

$$u = \sum_{n=0}^{\infty}(-1)^n\mathcal{T}^n g. \tag{4.38}$$

This shows that $\mathcal{R}(\widetilde{L}_s) = \mathcal{W}$, for all $s : |s_0-s| \le \rho < 1/K_5$.

If we take $s_0 = 0$, we obtain $\mathcal{R}(\widetilde{L}_s) = \mathcal{W}$, for all $s : 0 < s \le \rho$.

Now, if we put $s_0 = \rho$ and by the same procedure, we obtain $\mathcal{R}(\widetilde{L}_s) = \mathcal{W}$, for all $s : 0 < s \le 2\rho$. Proceeding step by step in this way, we establish that $\mathcal{R}(\widetilde{L}_s) = \mathcal{W}$, for every $s \in [0,1]$. For the case $s = 1$, we have $\mathcal{R}(\widetilde{L_1}) = \mathcal{R}(\widetilde{L}) = \mathcal{W}$. This proves Proposition 4.5. □

Proposition 4.6. *The operator $\widetilde{L} = \widetilde{\mathcal{L}}$ is closed.*

Proof. Let $(u_n) \subset \mathcal{D}(\widetilde{L}) = \hat{H}_0^{1,1}(D,H)$ such that

$$u_n \longrightarrow u \quad \text{in } L_2(D;H), \quad \widetilde{L}u_n \longrightarrow f \quad \text{in } L_2(D;H), \quad n \longrightarrow \infty. \tag{4.39}$$

From (4.18) we deduce that (u_n) is a Cauchy sequence in $H^{1,1}(D;H)$, then $u_n \to v$ in $H^{1,1}(D;H)$. Since $\hat{H}_0^{1,1}(D;H)$ is a closed subspace of $H^{1,1}(D,H)$, then $v \in \hat{H}_0^{1,1}(D,H)$. The convergence $u_n \to u$ in $H^{1,1}(D;H)$ implies the convergence $u_n \to v$ in $L_2(D;H)$, since we have supposed that $u_n \to u$ in $L_2(D;H)$, then $u = v$, and the boundedness of the operator $\widetilde{L}$ from $H^{1,1}(D,H)$ into $L_2(D;H)$ gives $\widetilde{L}u = f$. This completes the proof. □

Now we give some basic properties of the operator $\widetilde{L}' = \widetilde{\mathcal{L}}'$.

It follows from the above propositions that the operator $\widetilde{L}' = \widetilde{\mathcal{L}}'$ is continuous from $H_0^{1,1}(D;H)$ into $L_2(D;H)$.

Moreover, from the properties of the operators with closed range, it follows that

$$\begin{aligned}\mathcal{N}(\widetilde{L}') &= \mathcal{R}(\widetilde{L})^{\perp} = L_2(D;H)^{\perp} = \{0\},\\ \mathcal{R}(\widetilde{L}') &= \overline{\mathcal{R}(\widetilde{L}')} = \mathcal{N}(\widetilde{L})^{\perp} = \{0\}^{\perp} = L_2(D;H).\end{aligned} \tag{4.40}$$

Hence $\widetilde{L}'$ is an isomorphism from $H_0^{1,1}(D;H)$ into $L_2(D;H)$ and it is closed in the topology of $L_2(D;H)$.

Definition 4.7. Denote by $\widehat{\mathcal{L}} = (\widetilde{\mathcal{L}}')^*$ the weak extension of the operator $\widetilde{\mathcal{L}}$ defined by

$$\langle \widetilde{\mathcal{L}}' u, v\rangle = \langle u, \widehat{\mathcal{L}} v\rangle = \langle u, f\rangle, \quad \forall u \in H_0^{1,1}(D,H), \qquad \widehat{\mathcal{L}} v = f \in L_2(D,H). \tag{4.41}$$

Proposition 4.8. *The weak extension $\widehat{\mathcal{L}}$ of the operator $\widetilde{\mathcal{L}}$ coincides with its strong extension $(\widehat{\mathcal{L}})' = \widetilde{\mathcal{L}}'$.*

Proof. We must show that

$$\mathcal{D}(\widetilde{\mathcal{L}}) = \mathcal{D}(\widehat{\mathcal{L}}), \quad \widetilde{\mathcal{L}} u = \widehat{\mathcal{L}} u, \quad \forall u \in \mathcal{D}(\widehat{\mathcal{L}}). \tag{4.42}$$

It is clear that $\mathcal{D}(\widetilde{\mathcal{L}}) \subset \mathcal{D}(\mathcal{L})$.

By virtue of the Banach theorem for operators with closed range, we deduce that the operator $(\widehat{\mathcal{L}})^{-1}$ is defined on the closed subspace $\mathcal{R}(\widehat{\mathcal{L}}) = \mathcal{N}(\widetilde{\mathcal{L}}')^{\perp}$ and it is continuous.

We have

(i)

$$\mathcal{N}(\widehat{\mathcal{L}}) = \mathcal{R}(\widetilde{\mathcal{L}}')^{\perp} = \{0\}, \tag{4.43}$$

(ii)

$$\mathcal{N}(\widetilde{\mathcal{L}}') = \{0\}, \tag{4.44}$$

(iii)

$$\mathcal{R}(\widehat{\mathcal{L}}) = L_2(D,H). \tag{4.45}$$

From (ii) it follows that for all $f \in L_2(D,H)$ there exists a solution to the equation $\widehat{\mathcal{L}} u = f$. Let v be the solution of the equation $\widetilde{\mathcal{L}} u = f$ for a fixed f, and let us show that $u = v$.

From (4.41) and (4.15), we have

$$\begin{aligned}\langle z, \widehat{\mathcal{L}} u\rangle &= \langle \widetilde{\mathcal{L}}' z, u\rangle = \langle z, f\rangle, \quad \forall z \in H_0^{1,1}(D;H),\\ \langle z, \widetilde{\mathcal{L}} v\rangle &= \langle \widetilde{\mathcal{L}}' z, v\rangle = \langle z, f\rangle, \quad \forall z \in H_0^{1,1}(D;H),\end{aligned} \tag{4.46}$$

therefore $\langle \widetilde{\mathcal{L}}' z, v - u\rangle = 0$, for all $z \in H_0^{1,1}(D;H)$, which means that $w = v - u$ is the weak solution of the homogeneous equation $\widetilde{L} u = 0$. According to uniqueness of the weak solution, we obtain $u = v$. Consequently $u = v \in H_0^{1,1}(D;H)$ and $\widetilde{\mathcal{L}} u = \widehat{\mathcal{L}} u = f$. This completes the proof of Proposition 4.8. □

From Proposition 4.8, we deduce that the weak solution to problem (4.16) coincides with its strong solution. Hence $w \in H^{1,1}(D;H) \cap L_2(D,W^1)$ and satisfies the problem (4.16) in the strong sense, that is,

$$\begin{gathered}\frac{\partial^2 w}{\partial t_1 \partial t_2} + B_{1\varepsilon}\frac{\partial w}{\partial t_1} + B_{2\varepsilon}\frac{\partial w}{\partial t_2} + B_{0\varepsilon} w + Aw = 0, \\ B_2^*(\mu)w\mid_{t_1=0} = B_1^*(\mu)w\mid_{t_1=T_1}, \\ B_2^*(\mu)w\mid_{t_2=0} = B_1^*(\mu)w\mid_{t_2=T_2}.\end{gathered} \tag{4.47}$$

Problem (4.47) is equivalent to

$$\begin{gathered}\mathfrak{D}(\mathcal{L}) = \widehat{H}_0^{1,1}(D;H), \\ \mathcal{L}w = \frac{\partial^2 w}{\partial t_1 \partial t_2} + B_{1\varepsilon}\frac{\partial w}{\partial t_1} + B_{2\varepsilon}\frac{\partial w}{\partial t_2} + Aw = -B_{0\varepsilon} w = f.\end{gathered} \tag{4.48}$$

By similar calculations to those used to establish Theorem 3.1, we show the following.

Proposition 4.9. *Under the assumptions of Theorem 3.1, one has the estimate*

$$\left\|A^{1/2}w\right\|^2 \le K_6\left\|B_{0\varepsilon}w\right\|^2, \quad \forall w \in \widehat{H}_0^{1,1}(D;H). \tag{4.49}$$

From (4.49) and $(\mathcal{A}_1)$, it follows that

$$\|w\|^2 \le \frac{1}{c_0}\left\|A^{1/2}w\right\|^2 \le \frac{K_6}{c_0}\left\|B_{0\varepsilon}w\right\|^2. \tag{4.50}$$

Replacing w by $A_\varepsilon^{-1}v$ in (4.50), we obtain

$$\left\|A_\varepsilon^{-1}v\right\|^2 \le \frac{K_6}{c_0}\left\|B_{0\varepsilon}A_\varepsilon^{-1}v\right\|^2. \tag{4.51}$$

We have

$$\begin{aligned}\left\|B_{0\varepsilon}A_\varepsilon^{-1}v\right\| &= \left\|\left(\varepsilon\frac{\partial^2 A}{\partial t_2 \partial t_1}A_\varepsilon^{-1}\right)^* A_\varepsilon^{-1}v\right\| \\ &= \left\|\left(I - A_\varepsilon^{-1}\right)\left(\frac{\partial^2 A}{\partial t_2 \partial t_1}A^{-1}\right)^* A_\varepsilon^{-1}v\right\| \\ &\le \left\{\left\|\left(I - A_\varepsilon^{-1}\right)\left(\frac{\partial^2 A}{\partial t_2 \partial t_1}A^{-1}\right)^* \left(A_\varepsilon^{-1}v - v\right)\right\|\right. \\ &\quad \left. + \left\|\left(I - A_\varepsilon^{-1}\right)\left(\frac{\partial^2 A}{\partial t_2 \partial t_1}A^{-1}\right)^* v\right\|\right\} \longrightarrow 0, \quad \varepsilon \longrightarrow 0.\end{aligned} \tag{4.52}$$

Passing to the limit in (4.51), when $\varepsilon \to 0$ and applying the properties of A_ε^{-1}, we obtain $v = 0$. This completes the proof of Proposition 4.2. □

Let us go back to (4.10), by virtue of Proposition 4.4, we obtain $\langle l_{1\mu}u,\xi\rangle_0 + \langle l_{2\mu}u,\chi\rangle_0 = 0$. Since $l_{1\mu}$, $l_{2\mu}$ are independent and the ranges of the operators $l_{1\mu}$, $l_{2\mu}$ are dense in the corresponding spaces, we obtain $\xi = \chi = 0$. Hence $V = (0,0,0)$, and therefore, $\overline{\mathcal{R}(L_{\lambda,\mu})} = \mathcal{V}$ for $\lambda = 0$.

We consider now the case ˇ $\neq$ **0**. We need the following lemma.

Lemma 4.10. *The operator $(L_{1,\mu} - L_{0,\mu})$ is bounded, and*

$$|||(L_{1,\mu} - L_{0,\mu})u||| \leq k|||u|||_1, \tag{4.53}$$

where the constant k does not depend on u. The proof results from the continuity of $B \equiv A\partial^2/\partial t_1\partial t_2$, l_{1_μ} and l_{2_μ} in the corresponding spaces.

The equation $\overline{L_{\lambda,\mu}}u = F$ can be written as

$$\left(\overline{L_{\lambda_0,\mu}} + (\lambda - \lambda_0)\overline{(L_{1,\mu} - L_{0,\mu})}\right)u = F, \tag{4.54}$$

which is equivalent to the equation

$$u + (\lambda - \lambda_0)\left(\overline{L_{\lambda_0,\mu}}\right)^{-1}\overline{(L_{1,\mu} - L_{0,\mu})}u = \left(\overline{L_{\lambda_0,\mu}}\right)^{-1}F. \tag{4.55}$$

It follows from (3.35) and (4.53) that

$$\begin{gathered} |||\left(\overline{L_{\lambda_0,\mu}}\right)^{-1}F|||_1 \leq \sqrt{S}|||F|||, \\ |||\left(\overline{L_{\lambda_0,\mu}}\right)^{-1}\overline{(L_{1,\mu} - L_{0,\mu})}u|||_1 \leq \sqrt{S}|||\overline{(L_{1,\mu} - L_{0,\mu})}u||| \leq m|||u|||_1, \end{gathered} \tag{4.56}$$

where $m = k\sqrt{S}$.

Let $|\lambda - \lambda_0| \leq \rho < 1/m$. Putting $\Lambda = (\lambda - \lambda_0)(\overline{L_{\lambda_0,\mu}})\overline{(L_{1,\mu} - L_{0,\mu})}$ and $N = (\overline{L_{\lambda_0,\mu}})^{-1}F$, (4.55) can be written as $u + \Lambda u = N$.

Observe that $\|\Lambda\| = \sup_{u\in D(\overline{L_{\lambda,\mu}})}(\|\Lambda u\|_1/\|u\|_1) < 1$. The Neumann series $u = \sum_{n=0}^{\infty}(-\Lambda)^n N$ is then a solution to (4.55). We have thus proved that if $\mathcal{R}(\overline{L_{\lambda_0,\mu}}) = \mathcal{V}$ and $|\lambda - \lambda_0| \leq \rho < 1/m$, then $\mathcal{R}(\overline{L_{\lambda,\mu}}) = \mathcal{V}$. Proceeding step by step in this way, we establish that $\mathcal{R}(\overline{L_{\lambda,\mu}}) = \mathcal{V}$ for any $\lambda \geq 0$. The proof of Theorem 4.3 is achieved. □

Theorem 4.11. *For every element $\mathcal{F} = (f,\varphi,\psi) \in \mathcal{V}$ there exists a unique strong generalized solution $u = (\overline{L_{\lambda,\mu}})^{-1}\mathcal{F} = (\overline{L_{\lambda,\mu}^{-1}})\mathcal{F}$ to problem (1.1)-(1.2) satisfying the estimate*

$$|||u|||_1^2 \leq S|||L_{\lambda,\mu}u|||^2, \quad \forall u \in H^{1,1}(D;W^1), \tag{4.57}$$

where S is a positive constant independent of λ, μ, and u.

References

[1] A. M. Il'in and R. Z. Khas'minskiĭ, "On equations of Brownian motion," *Theory of Probability and Its Applications*, vol. 9, no. 3, pp. 421–444, 1964.

[2] H. Risken, *The Fokker-Planck Equation. Methods of Solution and Applications*, vol. 18 of *Springer Series in Synergetics*, Springer, Berlin, Germany, 1984.

[3] M. Iannelli, *Mathematical Theory of Age-Structured Population Dynamics*, Giardini Editori e Stampatori, Pisa, Italy, 1995.

[4] P. Hillion, "The Goursat problem for the homogeneous wave equation," *Journal of Mathematical Physics*, vol. 31, no. 8, pp. 1939–1941, 1990.

[5] N. I. Brich and N. I. Yurchuk, "Some new boundary value problems for a class of partial differential equations—part I," *Differentsial'nye Uravneniya*, vol. 4, pp. 1081–1101, 1968 (Russian), [English translation: Differential Equations, pp. 770–775].

[6] N. I. Brich and N. I. Yurchuk, "A mixed problem for certain pluri-parabolic differential equations," *Differentsial'nye Uravneniya*, vol. 6, pp. 1624–1630, 1970 (Russian), [English translation: Differential Equations, pp. 1234–1239].

[7] N. I. Brich and N. I. Yurchuk, "Goursat problem for abstract linear differential equation of second order," *Differentsial'nye Uravneniya*, vol. 7, no. 7, pp. 1001–1030, 1971 (Russian), [English translation: Differential Equations, pp. 770–779].

[8] A. Friedman, "The Cauchy problem in several time variables," *Journal of Mathematics and Mechanics*, vol. 11, pp. 859–889, 1962.

[9] A. Friedman and W. Littman, "Partially characteristic boundary problems for hyperbolic equations," *Journal of Mathematics and Mechanics*, vol. 12, pp. 213–224, 1963.

[10] H. O. Fattorini, "The abstract Goursat problem," *Pacific Journal of Mathematics*, vol. 37, no. 1, pp. 51–83, 1971.

[11] W. J. Roth, "Goursat problems for $u_{rs} = Lu$," *Indiana University Mathematics Journal*, vol. 22, no. 8, pp. 779–788, 1973.

[12] N. I. Yurchuk, "The Goursat problem for second order hyperbolic equations of special kind," *Differentsial'nye Uravneniya*, vol. 4, pp. 1333–1345, 1968 (Russian), [English translation: Differential Equations, pp. 694–700].

[13] N. I. Yurchuk, "A partialy characteristic boundary value problem for a particular type of partial differential equation. I," *Differentsial'nye Uravneniya*, vol. 4, pp. 2258–2267, 1968 (Russian), [English translation: Differential Equations, pp. 1167–1172].

[14] N. I. Yurchuk, "A partially characteristic mixed problem with Goursat initial conditions for linear equations with two-dimensional time," *Differentsial'nye Uravneniya*, vol. 5, pp. 898–910, 1969 (Russian), [English translation: Differential Equations, pp. 652–661].

[15] A. Ashyralyev and A. Yurtsever, "On a nonlocal boundary value problem for semilinear hyperbolic-parabolic equations," *Nonlinear Analysis*, vol. 47, no. 5, pp. 3585–3592, 2001.

[16] A. Ashyralyev, A. Hanalyev, and P. E. Sobolevskii, "Coercive solvability of the nonlocal boundary value problem for parabolic differential equations," *Abstract and Applied Analysis*, vol. 6, no. 1, pp. 53–61, 2001.

[17] A. Ashyralyev, "On well-posedness of the nonlocal boundary value problems for elliptic equations," *Numerical Functional Analysis and Optimization*, vol. 24, no. 1-2, pp. 1–15, 2003.

[18] A. Ashyralyev and I. Karatay, "On the second order of accuracy difference schemes of the nonlocal boundary value problem for parabolic equations," *Functional Differential Equations*, vol. 10, no. 1-2, pp. 45–63, 2003.

[19] A. Ashyralyev and N. Aggez, "A note on the difference schemes of the nonlocal boundary value problems for hyperbolic equations," *Numerical Functional Analysis and Optimization*, vol. 25, no. 5-6, pp. 439–462, 2004.

[20] A. Ashyralyev, "Nonlocal boundary-value problems for abstract parabolic equations: well-posedness in Bochner spaces," *Journal of Evolution Equations*, vol. 6, no. 1, pp. 1–28, 2006.

[21] L. Byszewski and V. Lakshmikantham, "Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space," *Applicable Analysis*, vol. 40, no. 1, pp. 11–19, 1991.

[22] K. Balachandran and J. Y. Park, "Existence of solutions of second order nonlinear differential equations with nonlocal conditions in Banach spaces," *Indian Journal of Pure and Applied Mathematics*, vol. 32, no. 12, pp. 1883–1891, 2001.

[23] V. I. Chesalin and N. I. Yurchuk, "Nonlocal boundary value problems for abstract Liav equations," *Izvestiya Akademii Nauk BSSR. Seriya Fiziko-Matematicheskikh Nauk*, no. 6, pp. 30–35, 1973 (Russian).

[24] V. I. Chesalin, "A problem with nonlocal boundary conditions for certain abstract hyperbolic equations," *Differentsial'nye Uravneniya*, vol. 15, no. 11, pp. 2104–2106, 1979 (Russian).

[25] V. I. Chesalin, "A problem with nonlocal boundary conditions for abstract hyperbolic equations," *Vestnik Belorusskogo Gosudarstvennogo Universiteta. Seriya 1. Fizika, Matematika, Informatika*, no. 2, pp. 57–60, 1998 (Russian).

[26] D. G. Gordeziani and G. A. Avalishvili, "Time-nonlocal problems for Schrödinger-type equations. I. Problems in abstract spaces," *Differential Equations*, vol. 41, no. 5, pp. 703–711, 2005.

[27] R. P. Agarwal, M. Bohner, and V. B. Shakhmurov, "Linear and nonlinear nonlocal boundary value problems for differential-operator equations," *Applicable Analysis*, vol. 85, no. 6-7, pp. 701–716, 2006.

[28] F. Rebbani and F. Zouyed, "Boundary value problem for an abstract differential equation with nonlocal boundary conditions," *Maghreb Mathematical Review*, vol. 8, no. 1-2, pp. 141–150, 1999.

[29] F. Rebbani, N. Boussetila, and F. Zouyed, "Boundary value problem for a partial differential equation with nonlocal boundary conditions," *Proceedings of Institute of Mathematics of National Academy of Sciences of Belarus*, vol. 10, pp. 122–125, 2001.

[30] S. G. Krein, *Linear Differential Equation in Banach Space*, 'Nauka', Moscow, Russia, 1972, English-Translation, American Mathematical Society, 1976.

[31] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer, Berlin, Germany, 1998.

[32] V. I. Korzyuk, "The method of energy inequalities and of averaging operators," *Vestnik Belorusskogo Gosudarstvennogo Universiteta. Seriya 1. Fizika, Matematika, Informatika*, no. 3, pp. 55–71, 1996 (Russian).

[33] H. Brezis, *Analyse Fonctionnelle. Théorie et Applications*, Masson, Paris, France, 1993.

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