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Research Article

A Regularized Gradient Projection Method for the Minimization Problem

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We investigate the following regularized gradient projection algorithm $x_{n+1} = P_c(I - \gamma_n(\nabla f + \alpha_n I))x_n$, $n \geq 0$. Under some different control conditions, we prove that this gradient projection algorithm strongly converges to the minimum norm solution of the minimization problem $\min_{x\in C} f(x)$.

1. Introduction

Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $f : H \to R$ be a real-valued convex function.

Consider the following constrained convex minimization problem:

$$\min_{x\in C} f(x). \tag{1.1}$$

Assume that (1.1) is consistent, that is, it has a solution and we use Ω to denote its solution set. If f is Fréchet differentiable, then $x^* \in C$ solves (1.1) if and only if $x^* \in C$ satisfies the following optimality condition:

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.2}$$

where ∇f denotes the gradient of f. Note that (1.2) can be rewritten as

$$\langle x^* - (x^* - \nabla f(x^*)), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

This shows that the minimization (1.1) is equivalent to the fixed point problem

$$P_C(x^* - \gamma \nabla f(x^*)) = x^*, \tag{1.4}$$

where $\gamma > 0$ is any constant and P_C is the nearest point projection from H onto C. By using this relationship, the gradient-projection algorithm is usually applied to solve the minimization problem (1.1). This algorithm generates a sequence $\{x_n\}$ through the recursion:

$$x_{n+1} = P_C(x_n - \gamma_n \nabla f(x_n)), \quad n \geq 0, \tag{1.5}$$

where the initial guess $x_0 \in C$ is chosen arbitrarily and $\{\gamma_n\}$ is a sequence of stepsizes which may be chosen in different ways. The gradient-projection algorithm (1.5) is a powerful tool for solving constrained convex optimization problems and has well been studied in the case of constant stepsizes $\gamma_n = \gamma$ for all n. The reader can refer to [1–9] and the references therein. It is known [3] that if f has a Lipschitz continuous and strongly monotone gradient, then the sequence $\{x_n\}$ can be strongly convergent to a minimizer of f in C. If the gradient of f is only assumed to be Lipschitz continuous, then $\{x_n\}$ can only be weakly convergent if H is infinite dimensional. In order to get the strong convergence, Xu [10] studied the following regularized method:

$$x_{n+1} = P_C(I - \gamma_n(\nabla f + \alpha_n I))x_n, \quad n \geq 0, \tag{1.6}$$

where the sequences $\{\theta_n\} \subset (0,1)$ and $\{\gamma_n\} \subset (0,\infty)$ satisfy the following conditions:

(1) $0 < \gamma_n \leq \alpha_n/(L + \alpha_n)^2$ for all n;

(2) $\lim_{n\to\infty}\alpha_n = 0$;

(3) $\sum_{n=0}^{\infty} \alpha_n \gamma_n = \infty$;

(4) $\lim_{n\to\infty}(|\gamma_n - \gamma_{n-1}| + |\alpha_n\gamma_n - \alpha_{n-1}\gamma_{n-1}|/(\alpha_n\gamma_n)^2) = 0$.

Xu [10] proved that the sequence $\{x_n\}$ converges strongly to a minimizer of (1.1).

Motivated by Xu's work, in the present paper, we further investigate the gradient projection method (1.6). Under some different control conditions, we prove that this gradient projection algorithm strongly converges to the minimum norm solution of the minimization problem (1.1).

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2.1}$$

We will use $\mathrm{Fix}(T)$ to denote the set of fixed points of T, that is, $\mathrm{Fix}(T) = \{x \in C : x = Tx\}$. A mapping $T : C \to C$ is said to be ν-inverse strongly monotone (ν-ism), if there exists a constant $\nu > 0$ such that

$$\langle x - y, Tx - Ty \rangle \geq \nu \|Tx - Ty\|^2, \quad \forall x, y \in C. \tag{2.2}$$

Recall that the (nearest point or metric) projection from H onto C, denoted P_C, assigns, to each $x \in H$, the unique point $P_C(x) \in C$ with the property

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}. \tag{2.3}$$

It is well known that the metric projection P_C of H onto C has the following basic properties:

(a) $\|P_C(x) - P_C(y)\| \leq \|x - y\|$ for all $x, y \in H$;

(b) $\langle x - y, P_C(x) - P_C(y) \rangle \geq \|P_C(x) - P_C(y)\|^2$ for every $x, y \in H$;

(c) $\langle x - P_C(x), y - P_C(x) \rangle \leq 0$ for all $x \in H$, $y \in C$.

Next we adopt the following notation:

(i) $x_n \to x$ means that x_n converges strongly to x;

(ii) $x_n \rightharpoonup x$ means that x_n converges weakly to x;

(iii) $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ is the weak ω-limit set of the sequence $\{x_n\}$.

Lemma 2.1 (see [11]). *Given $T : H \to H$ and letting $V = I - T$ be the complement of T, given also $S : H \to H$,*

(a) *T is nonexpansive if and only if V is $(1/2)$-ism;*

(b) *if S is ν-ism, then, for $\gamma > 0$, γS is (ν/γ)-ism;*

(c) *S is averaged if and only if the complement $I - S$ is ν-ism for some $\nu > 1/2$.*

Lemma 2.2 (see [12], (demiclosedness principle)). *Let C be a closed and convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then*

$$(I - T)x = y. \tag{2.4}$$

In particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.

Lemma 2.3 (see [13]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with*

$$0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1. \tag{2.5}$$

Suppose that

$$x_{n+1} = (1 - \beta_n) y_n + \beta_n x_n \tag{2.6}$$

for all $n \geq 0$ and

$$\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\right)\leq 0. \tag{2.7}$$

Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.4 (see [14]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1}\leq (1-\gamma_n)a_n+\delta_n, \tag{2.8}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(2) $\limsup_{n\to\infty}(\delta_n/\gamma_n)\leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then, $\lim_{n\to\infty}a_n = 0$.

3. Main Result

In this section, we will state and prove our main result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to R$ be a real-valued Fréchet differentiable convex function. Assume $\Omega \neq \emptyset$. Assume that the gradient ∇f is L-Lipschitzian. Let $\{x_n\}$ be a sequence generated by the following hybrid gradient projection algorithm:*

$$x_{n+1} = P_C\left(I-\gamma_n(\nabla f+\alpha_n I)\right)x_n, \quad n\geq 0, \tag{3.1}$$

where the sequences $\{\theta_n\}\subset(0,1)$ and $\{\gamma_n\}\subset(0,2/(L+2\alpha_n))$ satisfy the following conditions:

(1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(2) $0<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}\gamma_n<2/L$ *and* $\lim_{n\to\infty}(\gamma_{n+1}-\gamma_n) = 0$.

Then, the sequence $\{x_n\}$ generated by (3.1) *converges to a minimizer $\hat{x}$ of* (1.1).

Proof. Note that the Lipschitz condition implies that the gradient ∇f is $(1/L)$-ism [10]. Then, we have

$$\begin{aligned}
&\left\|P_C(I-\gamma(\nabla f+\alpha I))x-P_C(I-\gamma(\nabla f+\alpha I))y\right\|^2\\
&\quad\leq\left\|(I-\gamma(\nabla f+\alpha I))x-(I-\gamma(\nabla f+\alpha I))y\right\|^2\\
&\quad=\left\|(1-\alpha\gamma)(x-y)-\gamma(\nabla f(x)-\nabla f(y))\right\|^2\\
&\quad=(1-\alpha\gamma)^2\|x-y\|^2-2(1-\alpha\gamma)\gamma\langle x-y,\nabla f(x)-\nabla f(y)\rangle+\gamma^2\|\nabla f(x)-\nabla f(y)\|^2\\
&\quad\leq(1-\alpha\gamma)^2\|x-y\|^2-2(1-\alpha\gamma)\gamma\frac{1}{L}\|\nabla f(x)-\nabla f(y)\|^2+\gamma^2\|\nabla f(x)-\nabla f(y)\|^2.
\end{aligned}\tag{3.2}$$

If $\gamma \in [0, 2/(L+2\alpha)]$, then $2(1-\alpha\gamma)\gamma(1/L) \geq \gamma^2$. It follows that

$$\left\| P_C(I-\gamma(\nabla f+\alpha I))x - P_C(I-\gamma(\nabla f+\alpha I))y \right\|^2 \leq (1-\alpha\gamma)^2 \left\| x-y \right\|^2. \tag{3.3}$$

Thus,

$$\left\| P_C(I-\gamma(\nabla f+\alpha I))x - P_C(I-\gamma(\nabla f+\alpha I))y \right\| \leq (1-\alpha\gamma) \left\| x-y \right\|, \tag{3.4}$$

for all $x, y \in C$.

Take any $x^* \in S$. Since $x^* \in C$ solves the minimization problem (1.1) if and only if x^* solves the fixed-point equation $x^* = P_C(I-\gamma\nabla f)x^*$ for any fixed positive number γ, so we have $x^* = P_C(I-\gamma_n\nabla f)x^*$ for all $n \geq 0$. From (3.1) and (3.4), we get

$$\begin{aligned} \|x_{n+1}-x^*\| &= \left\| P_C(I-\gamma_n(\nabla f+\alpha_n I))x_n - P_C(I-\gamma_n\nabla f)x^* \right\| \\ &= \left\| P_C(I-\gamma_n(\nabla f+\alpha_n I))x_n - P_C(I-\gamma_n(\nabla f+\alpha_n I))x^* \right\| \\ &\quad + \left\| P_C(I-\gamma_n(\nabla f+\alpha_n I))x^* - P_C(I-\gamma_n\nabla f)x^* \right\| \\ &\leq (1-\alpha_n\gamma_n)\|x_n-x^*\| + \alpha_n\gamma_n\|x^*\| \\ &\leq \max\{\|x_n-x^*\|, \|x^*\|\}. \end{aligned} \tag{3.5}$$

Thus, we deduce by induction that

$$\|x_n-x^*\| \leq \max\{\|x_0-x^*\|, \|x^*\|\}. \tag{3.6}$$

This indicates that the sequence $\{x_n\}$ is bounded.

Since the gradient ∇f is $(1/L)$-ism, $\gamma\nabla f$ is $(1/\gamma L)$-ism. So by Lemma 2.1, $I-\gamma_n\nabla f$ is $(\gamma_n L/2)$-averaged; that is, $I-\gamma_n\nabla f = (1-(\gamma_n L/2))I + (\gamma_n L/2)T$ for some nonexpansive mapping T. Since P_C is $(1/2)$-averaged, $P_C = (I+S)/2$ for some nonexpansive mapping S. Then, we can rewrite x_{n+1} as

$$\begin{aligned} x_{n+1} &= \frac{1}{2}(I-\gamma_n(\nabla f+\alpha_n I))x_n + \frac{1}{2}S(I-\gamma_n(\nabla f+\alpha_n I))x_n \\ &= \frac{1}{2}(x_n-\gamma_n\nabla f(x_n)) - \frac{1}{2}\gamma_n\alpha_n x_n + \frac{1}{2}S(I-\gamma_n(\nabla f+\alpha_n I))x_n \\ &= \frac{2-\gamma_n L}{4}x_n + \frac{\gamma_n L}{4}Tx_n - \frac{1}{2}\gamma_n\alpha_n x_n + \frac{1}{2}S(I-\gamma_n(\nabla f+\alpha_n I))x_n \\ &= \frac{2-\gamma_n L}{4}x_n + \frac{2+\gamma_n L}{4}y_n, \end{aligned} \tag{3.7}$$

where

$$y_n = \frac{4}{2+\gamma_n L}\left(\frac{\gamma_n L}{4}Tx_n - \frac{1}{2}\gamma_n\alpha_n x_n + \frac{1}{2}S(I-\gamma_n(\nabla f+\alpha_n I))x_n\right). \tag{3.8}$$

It follows that

$$
\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\quad = \left\| \frac{4}{2+\gamma_{n+1}L}\left(\frac{\gamma_{n+1}L}{4}Tx_{n+1} - \frac{1}{2}\gamma_{n+1}\alpha_{n+1}x_{n+1} + \frac{1}{2}S(I-\gamma_{n+1}(\nabla f + \alpha_{n+1}I))x_{n+1}\right)\right. \\
&\qquad \left. - \frac{4}{2+\gamma_n L}\left(\frac{\gamma_n L}{4}Tx_n - \frac{1}{2}\gamma_n\alpha_n x_n + \frac{1}{2}S(I-\gamma_n(\nabla f + \alpha_n I))x_n\right)\right\| \\
&\quad \le \frac{4}{2+\gamma_{n+1}L}\left\|\left(\frac{\gamma_{n+1}L}{4}Tx_{n+1} - \frac{1}{2}\gamma_{n+1}\alpha_{n+1}x_{n+1} + \frac{1}{2}S(I-\gamma_{n+1}(\nabla f + \alpha_{n+1}I))x_{n+1}\right)\right. \\
&\qquad\qquad \left. -\left(\frac{\gamma_n L}{4}Tx_n - \frac{1}{2}\gamma_n\alpha_n x_n + \frac{1}{2}S(I-\gamma_n(\nabla f + \alpha_n I))x_n\right)\right\| \\
&\qquad + \left|\frac{4}{2+\gamma_{n+1}L} - \frac{4}{2+\gamma_n L}\right|\left\|\frac{\gamma_n L}{4}Tx_n - \frac{1}{2}\gamma_n\alpha_n x_n + \frac{1}{2}S(I-\gamma_n(\nabla f + \alpha_n I))x_n\right\| \\
&\quad \le \frac{4}{2+\gamma_{n+1}L}\left(\left\|\frac{\gamma_{n+1}L}{4}Tx_{n+1} - \frac{\gamma_n L}{4}Tx_n\right\| + \frac{1}{2}\gamma_{n+1}\alpha_{n+1}\|x_{n+1}\| + \frac{1}{2}\gamma_n\alpha_n\|x_n\|\right) \\
&\qquad + \frac{2}{2+\gamma_{n+1}L}\|(I-\gamma_{n+1}(\nabla f + \alpha_{n+1}I))x_{n+1} - (I-\gamma_n(\nabla f + \alpha_n I))x_n\| \\
&\qquad + \left|\frac{4}{2+\gamma_{n+1}L} - \frac{4}{2+\gamma_n L}\right|\left\|\frac{\gamma_n L}{4}Tx_n - \frac{1}{2}\gamma_n\alpha_n x_n + \frac{1}{2}S(I-\gamma_n(\nabla f + \alpha_n I))x_n\right\|.
\end{aligned}
\tag{3.9}
$$

Now we choose a constant M such that

$$
\sup_n\left\{\|x_n\|, L\|Tx_n\|, \|\nabla f(x_n)\|, \left\|\frac{\gamma_n L}{4}Tx_n - \frac{1}{2}\gamma_n\alpha_n x_n + \frac{1}{2}S(I-\gamma_n(\nabla f + \alpha_n I))x_n\right\|\right\} \le M.
\tag{3.10}
$$

We have the following estimates:

$$
\begin{aligned}
\left\|\frac{\gamma_{n+1}L}{4}Tx_{n+1} - \frac{\gamma_n L}{4}Tx_n\right\| &= \left\|\frac{\gamma_{n+1}L}{4}(Tx_{n+1} - Tx_n) + \left(\frac{\gamma_{n+1}L}{4} - \frac{\gamma_n L}{4}\right)Tx_n\right\| \\
&\le \frac{\gamma_{n+1}L}{4}\|Tx_{n+1} - Tx_n\| + |\gamma_{n+1} - \gamma_n|\frac{\|Tx_n\|}{4} \\
&\le \frac{\gamma_{n+1}L}{4}\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|M, \\
\|(I-\gamma_{n+1}(\nabla f + \alpha_{n+1}I))x_{n+1} &- (I-\gamma_n(\nabla f + \alpha_n I))x_n\| \\
&\le \|(I-\gamma_{n+1}\nabla f)x_{n+1} - (I-\gamma_{n+1}\nabla f)x_n\| + |\gamma_{n+1} - \gamma_n|\|\nabla f(x_n)\| \\
&\quad + \gamma_{n+1}\alpha_{n+1}\|x_{n+1}\| + \gamma_n\alpha_n\|x_n\| \\
&\le \|x_{n+1} - x_n\| + (|\gamma_{n+1} - \gamma_n| + \gamma_{n+1}\alpha_{n+1} + \gamma_n\alpha_n)M.
\end{aligned}
\tag{3.11}
$$

Thus, we deduce

$$\begin{aligned}\|y_{n+1}-y_n\| &\le \frac{4}{2+\gamma_{n+1}L}\left(\frac{\gamma_{n+1}L}{4}\|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|M+(\gamma_{n+1}\alpha_{n+1}+\gamma_n\alpha_n)M\right)\\ &\quad+\frac{2}{2+\gamma_{n+1}L}\left(\|x_{n+1}-x_n\|+\left(|\gamma_{n+1}-\gamma_n|+\gamma_{n+1}\alpha_{n+1}+\gamma_n\alpha_n\right)M\right)\\ &\quad+\left|\frac{4}{2+\gamma_{n+1}L}-\frac{4}{2+\gamma_nL}\right|M\\ &\le \|x_{n+1}-x_n\|+\frac{6}{2+\gamma_{n+1}L}\left(|\gamma_{n+1}-\gamma_n|+\gamma_{n+1}\alpha_{n+1}+\gamma_n\alpha_n\right)M\\ &\quad+\frac{4L}{(2+\gamma_{n+1}L)(2+\gamma_nL)}|\gamma_{n+1}-\gamma_n|M.\end{aligned}\tag{3.12}$$

Note that $\alpha_n \to 0$ and $\gamma_{n+1}-\gamma_n \to 0$. Hence, by Lemma 2.3, we get

$$\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\right)\le 0.\tag{3.13}$$

It follows that

$$\lim_{n\to\infty}\|y_n-x_n\|=0.\tag{3.14}$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}\frac{2+\gamma_nL}{4}\|y_n-x_n\|=0.\tag{3.15}$$

Now we show that the weak limit set $\omega_w(x_n)\subset\Omega$. Choose any $\tilde{x}\in\omega_w(x_n)$. Since $\{x_n\}$ is bounded, there must exist a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\rightharpoonup\tilde{x}$. At the same time, the real number sequence $\{\gamma_{n_j}\}$ is bounded. Thus, there exists a subsequence $\{\gamma_{n_{j_i}}\}$ of $\{\gamma_{n_j}\}$ which converges to γ. Without loss of generality, we may assume that $\gamma_{n_j}\to\gamma$. Note that $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<2/L$. So, $\gamma\in(0,2/L)$; that is, $\gamma_{n_j}\to\gamma\in(0,2/L)$ as $j\to\infty$. Next, we only need to show that $\tilde{x}\in\Omega$. First, from (3.15) we have that $\|x_{n_j+1}-x_{n_j}\|\to 0$. Then, we have

$$\begin{aligned}\left\|x_{n_j}-P_C(I-\gamma\nabla f)x_{n_j}\right\| &\le \left\|x_{n_j}-x_{n_j+1}\right\|+\left\|x_{n_j+1}-P_C\left(I-\gamma_{n_j}\nabla f\right)x_{n_j}\right\|\\ &\quad+\left\|P_C\left(I-\gamma_{n_j}\nabla f\right)x_{n_j}-P_C(I-\gamma\nabla f)x_{n_j}\right\|\\ &=\left\|P_C\left(I-\gamma_{n_j}\left(\nabla f+\alpha_{n_j}I\right)\right)x_{n_j}-P_C\left(I-\gamma_{n_j}\nabla f\right)x_{n_j}\right\|\\ &\quad+\left\|P_C\left(I-\gamma_{n_j}\nabla f\right)x_{n_j}-P_C(I-\gamma\nabla f)x_{n_j}\right\|+\left\|x_{n_j}-x_{n_j+1}\right\|\\ &\le \alpha_{n_j}\gamma_{n_j}\left\|x_{n_j}\right\|+\left|\gamma_{n_j}-\gamma\right|\left\|\nabla f\left(x_{n_j}\right)\right\|+\left\|x_{n_j}-x_{n_j+1}\right\|\\ &\longrightarrow 0.\end{aligned}\tag{3.16}$$

Since $\gamma \in (0, 2/L)$, $P_C(I - \gamma\nabla f)$ is nonexpansive. It then follows from Lemma 2.2 (demiclosedness principle) that $\tilde{x} \in \mathrm{Fix}(P_C(I - \gamma\nabla f))$. Hence, $\tilde{x} \in \Omega$ because of $\Omega = \mathrm{Fix}(P_C(I - \gamma\nabla f))$. So, $\omega_w(x_n) \subset \Omega$.

Finally, we prove that $x_n \to \hat{x}$, where $\hat{x}$ is the minimum norm solution of (1.1). First, we show that $\limsup_{n\to\infty}\langle \hat{x}, x_n - \hat{x}\rangle \geq 0$. Observe that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying

$$\limsup_{n\to\infty}\langle \hat{x}, x_n - \hat{x}\rangle = \lim_{j\to\infty}\left\langle \hat{x}, x_{n_j} - \hat{x}\right\rangle. \tag{3.17}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_i}}\}$ of $\{x_{n_j}\}$ such that $x_{n_{j_i}} \rightharpoonup \tilde{x}$. Without loss of generality, we assume that $x_{n_j} \rightharpoonup \tilde{x}$. Then, we obtain

$$\limsup_{n\to\infty}\langle \hat{x}, x_n - \hat{x}\rangle = \lim_{j\to\infty}\left\langle \hat{x}, x_{n_j} - \hat{x}\right\rangle = \langle \hat{x}, \tilde{x} - \hat{x}\rangle \geq 0. \tag{3.18}$$

Since $\gamma_n < 2/(L + 2\alpha_n)$, $\gamma_n/(1 - \alpha_n\gamma_n) < 2/L$. So, $I - (\gamma_n/(1 - \alpha_n\gamma_n))\nabla f$ is nonexpansive. By using the property (b) of P_C, we have

$$\begin{aligned}
\|x_{n+1} - \hat{x}\|^2 &= \left\|P_C(I - \gamma_n(\nabla f + \alpha_n I))x_n - P_C(\hat{x} - \gamma_n\nabla f(\hat{x}))\right\|^2 \\
&\leq \langle (I - \gamma_n(\nabla f + \alpha_n I))x_n - (\hat{x} - \gamma_n\nabla f(\hat{x})), x_{n+1} - \hat{x}\rangle \\
&= (1 - \alpha_n\gamma_n)\left\langle \left(I - \frac{\gamma_n}{1 - \alpha_n\gamma_n}\nabla f\right)x_n - \left(I - \frac{\gamma_n}{1 - \alpha_n\gamma_n}\nabla f\right)\hat{x}, x_{n+1} - \hat{x}\right\rangle \\
&\quad - \alpha_n\gamma_n\langle \hat{x}, x_{n+1} - \hat{x}\rangle \\
&\leq (1 - \alpha_n\gamma_n)\left\|\left(I - \frac{\gamma_n}{1 - \alpha_n\gamma_n}\nabla f\right)x_n - \left(I - \frac{\gamma_n}{1 - \alpha_n\gamma_n}\nabla f\right)\hat{x}\right\|\|x_{n+1} - \hat{x}\| \\
&\quad - \alpha_n\gamma_n\langle \hat{x}, x_{n+1} - \hat{x}\rangle \\
&\leq (1 - \alpha_n\gamma_n)\|x_n - \hat{x}\|\|x_{n+1} - \hat{x}\| - \alpha_n\gamma_n\langle \hat{x}, x_{n+1} - \hat{x}\rangle \\
&\leq \frac{1 - \alpha_n\gamma_n}{2}\|x_n - \hat{x}\|^2 + \frac{1}{2}\|x_{n+1} - \hat{x}\|^2 - \alpha_n\gamma_n\langle \hat{x}, x_{n+1} - \hat{x}\rangle.
\end{aligned} \tag{3.19}$$

It follows that

$$\|x_{n+1} - \hat{x}\|^2 \leq (1 - \alpha_n\gamma_n)\|x_n - \hat{x}\|^2 + \alpha_n\gamma_n\langle -\hat{x}, x_{n+1} - \hat{x}\rangle. \tag{3.20}$$

From Lemma 2.4, (3.18) and (3.20), we deduce that $x_n \to \hat{x}$. This completes the proof. □

Remark 3.2. We obtain the strong convergence of the regularized gradient projection method (3.1) under some different control conditions.

Remark 3.3. From the proof of result, we observe that our algorithm (3.1) converges to a special solution $\hat{x}$ of the minimization (1.1). As a matter of fact, this special solution $\hat{x}$ is the minimum-norm solution of the minimization (1.1). Finding the minimum-norm solution of practical problem is an interesting work due to its applications. A typical example is the least-squares solution to the constrained linear inverse problem; see, for example, [15]. For some related works on the minimum-norm solution and the minimization problems, please see [16–22].

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