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*Research Article*

# Strong Convergence of Non-Implicit Iteration Process with Errors in Banach Spaces

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The purpose of this paper is to study the strong convergence of a non-implicit iteration process with errors for asymptotically $I$-nonexpansive mappings in the intermediate sense in the framework of Banach spaces. The results presented in this paper extend and improve the corresponding results recently announced.

## 1. Introduction and Preliminaries

Let $K$ be a nonempty, closed, and convex subset of a real Banach space $X$ and let $T : K \to K$ be a mapping. In this paper, we use $F(T)$ to stand for the set of fixed points of $T$, that is $F(T) = \{x \in K : Tx = x\}$.

Recall that $T$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in K. \tag{1.1}$$

$T$ is said to be asymptotically nonexpansive if there exists a sequence $\{h_n\}$ with $h_n \subset [1, +\infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\|T^n x - T^n y\| \leq h_n \|x - y\|, \quad \forall x, y \in K,\ n \geq 1. \tag{1.2}$$

$T$ is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in k} \left(\|T^n x - T^n y\| - \|x - y\|\right) \le 0. \tag{1.3}$$

Observe that if we define $a_n = \sup_{x,y\in k}(\|T^n x - T^n y\| - \|x - y\|)$, $\sigma_n = \max\{0, a_n\}$, then $\sigma_n \to 0$ as $n \to \infty$ and (1.3) reduces to

$$\|T^n x - T^n y\| \le \|x - y\| + \sigma_n, \quad \forall x, y \in K,\ n \ge 1. \tag{1.4}$$

It is easy to see that every nonexpansive mapping is asymptotically nonexpansive. And every asymptotically nonexpansive mapping is asymptotically nonexpansive in the intermediated sense. In [1], Goebel and Kirk proved that, if $K$ is a nonempty closed convex bounded subset of a real uniformly convex Banach space $X$, and $T$ is an asymptotically nonexpansive self-mapping on $K$, then $T$ has a fixed point in $K$. The class of mappings which are asymptotically nonexpansive in the intermediat sense was investigated by Bruck et al. [2] and Kirk [3]. Since then, many authors have investigated the fixed point problem of these mappings based on implicit iterative methods or non-implicit iterative methods; see, for example, [4–21].

Let $I : K \to K$ be a mapping. Recall that $T$ is said to be asymptotically $I$-nonexpansive if there exists a sequence $\{h_n\}$ with $\{h_n\} \subset [1, +\infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\|T^n x - T^n y\| \le h_n \|I^n x - I^n y\|, \quad \forall x, y \in K,\ n \ge 1. \tag{1.5}$$

Recently, weak and strong convergence theorems for fixed points of $I$-nonexpansive mappings, and asymptotically $I$-nonexpansive mappings have been established by many scholar, see, for example, [22–25].

In this paper, we consider a new mapping based on asymptotically nonexpansive mappings in the intermediate sense and asymptotically $I$-nonexpansive mappings.

Let $T : K \to K$, $I : K \to K$ be two mappings. $T$ is said to be asymptotically $I$-nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in k} \left(\|T^n x - T^n y\| - \|I^n x - I^n y\|\right) \le 0. \tag{1.6}$$

Observe that if we define $a_n = \sup_{x,y\in k}(\|T^n x - T^n y\| - \|I^n x - I^n y\|)$, $\sigma_n = \max\{0, a_n\}$, then $\sigma_n \to 0$ as $n \to \infty$ and (1.6) reduces to

$$\|T^n x - T^n y\| \le \|I^n x - I^n y\| + \sigma_n, \quad \forall x, y \in K,\ n \ge 1. \tag{1.7}$$

Note that if $I = \mathrm{Id}$, where Id is the identity mapping, then (1.7) reduces to (1.4).

In this paper, we investigate asymptotically $I$-nonexpansive mappings in the intermediate sense based on a non-implicit iterative algorithm. Strong convergence of the implicit iterative algorithm is obtained in the framework of Banach spaces.

In order to prove our main results, we need the following lemmas.

**Lemma 1.1** (see [21]). *let $X$ be a uniformly convex Banach space. Let $b$ and $c$ be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that*

$$\limsup_{n\to\infty}\|x_n\| \le d, \qquad \limsup_{n\to\infty}\|y_n\| \le d,$$
$$\lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = d \tag{1.8}$$

*hold for some $d \ge 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

**Lemma 1.2** (see [26]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \le (1+b_n)a_n + c_n, \quad \forall n \ge n_0, \tag{1.9}$$

*where $n_0$ is some nonnegative integer, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.*

## 2. Main Results

**Lemma 2.1.** *Let $X$ be a real Banach space and $K$ a nonempty closed and convex subset of $X$. Let $T : K \to K$ be a asymptotically $I$-nonexpansive in the intermediate sense and $I : K \to K$ a asymptotically nonexpansive in the intermediate sense. Assume that $F := F(T) \cap F(I) \neq \emptyset$. Let $\sigma_n = \max\{0, \sup_{x,y\in k}(\|T^n x - T^n y\| - \|I^n x - I^n y\|)\}$ and $\overline{\sigma}_n = \max\{0, \sup_{x,y\in k}(\|T^n x - T^n y\| - \|x - y\|)\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\widehat{\gamma}_n\}$ be six real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{aligned} x_1 &\in C, \\ y_n &= \widehat{\alpha}_n x_n + \widehat{\beta}_n I^n x_n + \widehat{\gamma}_n v_n, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad n \ge 1, \end{aligned} \tag{2.1}$$

*where $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in $K$. Assume that the following restrictions are satisfied:*

(a) $\alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1$;

(b) $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \overline{\sigma}_n < \infty$;

(c) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \widehat{\gamma}_n < \infty$.

*Then $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$.*

*Proof.* Letting $p \in F$, we see that

$$\begin{aligned}
\|y_n - p\| &= \left\|\left(1-\widehat{\beta}_n-\widehat{\gamma}_n\right)x_n + \widehat{\beta}_n I^n x_n + \widehat{\gamma}_n v_n - p\right\| \\
&\le \left(1-\widehat{\beta}_n-\widehat{\gamma}_n\right)\|x_n - p\| + \widehat{\beta}_n\|I^n x_n - p\| + \widehat{\gamma}_n\|v_n - p\| \\
&\le \left(1-\widehat{\beta}_n-\widehat{\gamma}_n\right)\|x_n - p\| + \widehat{\beta}_n(\|x_n - p\| + \overline{\sigma}_n) + \widehat{\gamma}_n\|v_n - p\| \\
&= (1-\widehat{\gamma}_n)\|x_n - p\| + \widehat{\gamma}_n\|v_n - p\| + \widehat{\beta}_n\overline{\sigma}_n \\
&\le \|x_n - p\| + \widehat{\gamma}_n\|v_n - p\| + \widehat{\beta}_n\overline{\sigma}_n,
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\beta_n-\gamma_n)x_n + \beta_n T^n y_n + \gamma_n u_n - p\| \\
&\le (1-\beta_n-\gamma_n)\|x_n - p\| + \beta_n\|T^n y_n - p\| + \gamma_n\|u_n - p\| \\
&= (1-\beta_n-\gamma_n)\|x_n - p\| + \beta_n\|T^n y_n - T^n p\| + \gamma_n\|u_n - p\| \\
&\le (1-\beta_n-\gamma_n)\|x_n - p\| + \beta_n(\|I^n y_n - I^n p\| + \sigma_n) + \gamma_n\|u_n - p\| \\
&= (1-\beta_n-\gamma_n)\|x_n - p\| + \beta_n\|I^n y_n - I^n p\| + \beta_n\sigma_n + \gamma_n\|u_n - p\| \\
&\le (1-\beta_n-\gamma_n)\|x_n - p\| + \beta_n(\|y_n - p\| + \overline{\sigma}_n) + \beta_n\sigma_n + \gamma_n\|u_n - p\| \\
&= (1-\beta_n-\gamma_n)\|x_n - p\| + \beta_n\|y_n - p\| + \gamma_n\|u_n - p\| + \beta_n(\sigma_n + \overline{\sigma}_n) \\
&\le (1-\beta_n)\|x_n - p\| + \beta_n\|y_n - p\| + \gamma_n\|u_n - p\| + \beta_n(\sigma_n + \overline{\sigma}_n).
\end{aligned} \tag{2.3}$$

Substituting (2.2) into (2.3),we obtain that

$$\begin{aligned}
\|x_{n+1} - p\| &\le (1-\beta_n)\|x_n - p\| + \beta_n\left(\|x_n - p\| + \widehat{\gamma}_n\|v_n - p\| + \widehat{\beta}_n\overline{\sigma}_n\right) \\
&\quad + \gamma_n\|u_n - p\| + \beta_n(\sigma_n + \overline{\sigma}_n) \\
&= \|x_n - p\| + \left[\beta_n\widehat{\gamma}_n\|v_n - p\| + \gamma_n\|u_n - p\| + \beta_n\sigma_n + \beta_n\overline{\sigma}_n\left(1+\widehat{\beta}_n\right)\right].
\end{aligned} \tag{2.4}$$

Let $a_n = \|x_n - p\|$, $b_n = 0$, and

$$c_n = \beta_n\widehat{\gamma}_n\|v_n - p\| + \gamma_n\|u_n - p\| + \beta_n\sigma_n + \beta_n\overline{\sigma}_n\left(1+\widehat{\beta}_n\right). \tag{2.5}$$

It follows from (2.4) that

$$a_{n+1} \le a_n + c_n. \tag{2.6}$$

In view of the restrictions (b) and (c), we see that $\sum_{n=1}^{\infty} c_n < \infty$. We can easily conclude the desired conclusion with the aid of Lemma 1.2. This completes the proof of Lemma 2.1. □

**Theorem 2.2.** *Let $X$ be a real Banach space and $K$ a nonempty closed and convex subset of $X$. Let $T : K \to K$ be a asymptotically $I$-nonexpansive in the intermediate sense and $I : K \to K$ a asymptotically nonexpansive in the intermediate sense. Assume that $F := F(T) \cap F(I) \neq \emptyset$. Let $\sigma_n = \max\{0, \sup_{x,y\in k}(\|T^n x - T^n y\| - \|I^n x - I^n y\|)\}$ and $\overline{\sigma}_n = \max\{0, \sup_{x,y\in k}(\|T^n x - T^n y\| - \|x - y\|)\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\widehat{\alpha}_n\}, \{\widehat{\beta}_n\}, \{\widehat{\gamma}_n\}$ be six real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{gathered} x_1 \in C, \\ y_n = \widehat{\alpha}_n x_n + \widehat{\beta}_n I^n x_n + \widehat{\gamma}_n v_n, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad n \geq 1, \end{gathered} \tag{2.7}$$

*where $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in $K$. Assume that the following restrictions are satisfied:*

(a) $\alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1$;

(b) $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \overline{\sigma}_n < \infty$;

(c) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \widehat{\gamma}_n < \infty$.

*If both $T$ and $I$ are continuous, then the sequence $\{x_n\}$ strongly converges to a common fixed point of $T$ and $I$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{2.8}$$

*Proof.* The necessity is obvious. Next, we prove the sufficiency part of the theorem. Note that continuity of $T$ and $I$ implies that the set $F(T)$ and $F(I)$ are closed. It follows from (2.6) that

$$\|x_{n+1} - p\| \leq \|x_n - p\| + c_n. \tag{2.9}$$

This implies in turn that

$$d(x_{n+1}, F) \leq d(x_n, F) + c_n. \tag{2.10}$$

Now applying Lemma 1.2 to (2.10), we obtain the existence of the limit $\lim_{n\to\infty} d(x_n, F)$. By condition (2.8), we have

$$\lim_{n\to\infty} d(x_n, F) = \liminf_{n\to\infty} d(x_n, F) = 0. \tag{2.11}$$

Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence in $K$. For any positive integers $n$, $m$, from (2.9) it follows that

$$\begin{aligned}\|x_{n+m}-p\| &\le \|x_{n+m-1}-p\|+c_{n+m-1}\\ &\le (\|x_{n+m-2}-p\|+c_{n+m-2})+c_{n+m-1}\\ &\le \cdots\\ &\le \|x_n-p\|+\sum_{i=n}^{n+m-1}c_i\\ &\le \|x_n-p\|+\sum_{i=n}^{\infty}c_i.\end{aligned} \tag{2.12}$$

Since $\lim_{n\to\infty}d(x_n,F)=0$, and $\sum_{n=1}^{\infty}c_n<\infty$, for any given $\epsilon>0$, there exists a positive integer $n_0$ such that

$$d(x_n,F)<\frac{\epsilon}{8},\quad \sum_{i=n}^{\infty}c_i<\frac{\epsilon}{2},\quad \forall n\ge n_0. \tag{2.13}$$

Therefore there exists $p_1\in F$ such that $d(x_n,p_1)<(\epsilon/4)$, $\forall n\ge n_0$. Consequently, for any $n\ge n_0$ and for all $m\ge 1$, we have

$$\begin{aligned}\|x_{n+m}-x_n\| &\le \|x_{n+m}-p_1\|+\|x_n-p_1\|\\ &\le 2\|x_n-p_1\|+\sum_{i=n}^{\infty}c_i\\ &\le \frac{\epsilon}{4}\cdot 2+\frac{\epsilon}{2}=\epsilon.\end{aligned} \tag{2.14}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $K$. Let $x_n\to x^*\in K$. Since $F$ is closed, this implies that $x^*\in F$. This shows that $\{x_n\}$ strongly converges to a common fixed of $T$ and $I$. This completes the proof of Theorem 2.2. □

**Lemma 2.3.** *Let $X$ be a real Banach space and $K$ a nonempty closed and convex subset of $X$. Let $T:K\to K$ be a asymptotically $I$-nonexpansive in the intermediate sense and $I:K\to K$ a asymptotically nonexpansive in the intermediate sense. Assume that $F:=F(T)\cap F(I)\neq\emptyset$. Let $\sigma_n=\max\{0,\sup_{x,y\in k}(\|T^nx-T^ny\|-\|I^nx-I^ny\|)\}$ and $\overline{\sigma}_n=\max\{0,\sup_{x,y\in k}(\|T^nx-T^ny\|-\|x-y\|)\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\widehat{\gamma}_n\}$ be six real number sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{aligned}x_1&\in C,\\ y_n&=\widehat{\alpha}_nx_n+\widehat{\beta}_nI^nx_n+\widehat{\gamma}_nv_n,\\ x_{n+1}&=\alpha_nx_n+\beta_nT^ny_n+\gamma_nu_n,\quad n\ge 1,\end{aligned} \tag{2.15}$$

*where $\{u_n\}$ and $\{v_n\}$ be two bounded sequences in $K$. Assume that the following restrictions are satisfied:*

(a) $\alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1, \forall n \geq 1$;

(b) $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \overline{\sigma}_n < \infty$;

(c) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that* $\tau_1 \leq \beta_n, \widehat{\beta}_n \leq \tau_2, \forall n \geq 1$;

(d) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \widehat{\gamma}_n < \infty$.

*Then*

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0, \tag{2.16}$$

$$\lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \tag{2.17}$$

*Proof.* According to Lemma 2.1, for any $p \in F$, we have $\lim_{n\to\infty}\|x_n - p\|$ exists. Without loss of generality, we may assume that

$$\lim_{n\to\infty} \|x_n - p\| = d, \tag{2.18}$$

where $d > 0$ is some constant. It follows that

$$\lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \left\| (1-\beta_n)\left[x_n - p + \gamma_n(u_n - x_n)\right] + \beta_n\left[T^n y_n - p + \gamma_n(u_n - x_n)\right] \right\| = d. \tag{2.19}$$

Notice that

$$\|x_n - p + \gamma_n(u_n - x_n)\| \leq \|x_n - p\| + \gamma_n\|u_n - x_n\|. \tag{2.20}$$

It follows from the restriction (d) and (2.18) that

$$\limsup_{n\to\infty} \|x_n - p + \gamma_n(u_n - x_n)\| \leq d. \tag{2.21}$$

Notice that

$$\begin{aligned}\limsup_{n\to\infty}\left\|T^n y_n - p + \gamma_n(u_n - x_n)\right\| &\le \limsup_{n\to\infty}\left\|T^n y_n - p\right\| + \limsup_{n\to\infty}\gamma_n\|u_n - x_n\| \\ &= \limsup_{n\to\infty}\left\|T^n y_n - p\right\| \\ &\le \limsup_{n\to\infty}\left(\left\|I^n y_n - I^n p\right\| + \sigma_n\right) \\ &\le \limsup_{n\to\infty}\left(\left\|y_n - p\right\| + \overline{\sigma}_n + \sigma_n\right) \\ &= \limsup_{n\to\infty}\left\|y_n - p\right\| \\ &\le \limsup_{n\to\infty}\Big(\left\|x_n - p\right\| + \widehat{\gamma}_n\left\|v_n - p\right\| + \widehat{\beta}_n\overline{\sigma}_n\Big) \\ &= d.\end{aligned} \tag{2.22}$$

In view of (2.19), (2.21) and (2.22), we obtain from Lemma 1.1 that

$$\lim_{n\to\infty}\left\|x_n - T^n y_n\right\| = 0. \tag{2.23}$$

Notice that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \left\|\beta_n\left(T^n y_n - x_n\right) + \gamma_n(u_n - x_n)\right\| \\ &\le \beta_n\left\|T^n y_n - x_n\right\| + \gamma_n\|u_n - x_n\|.\end{aligned} \tag{2.24}$$

It follows from (2.23) and the restriction (d) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.25}$$

Notice that

$$\left\|I^n x_n - p\right\| \le \left\|x_n - p\right\| + \overline{\sigma}_n. \tag{2.26}$$

It follows that

$$\limsup_{n\to\infty}\left\|I^n x_n - p\right\| \le d. \tag{2.27}$$

On the other hand, we have

$$\lim_{n\to\infty}\left\|y_n - p\right\| = \lim_{n\to\infty}\left\|\left(1 - \widehat{\beta}_n\right)\left[x_n - p + \widehat{\gamma}_n(v_n - x_n)\right] + \widehat{\beta}_n\left[I^n x_n - p + \widehat{\gamma}_n(v_n - x_n)\right]\right\| = d. \tag{2.28}$$

Notice that

$$\left\|x_n - p + \widehat{\gamma}_n(v_n - x_n)\right\| \leq \left\|x_n - p\right\| + \widehat{\gamma}_n\|v_n - x_n\|. \tag{2.29}$$

It follows that

$$\limsup_{n\to\infty}\left\|x_n - p + \widehat{\gamma}_n(v_n - x_n)\right\| \leq d. \tag{2.30}$$

Notice that

$$\left\|I^n x_n - p + \widehat{\gamma}_n(v_n - x_n)\right\| \leq \left\|I^n x_n - p\right\| + \widehat{\gamma}_n\|v_n - x_n\|. \tag{2.31}$$

It follows from (2.27) that

$$\limsup_{n\to\infty}\left\|I^n x_n - p + \widehat{\gamma}_n(v_n - x_n)\right\| \leq d. \tag{2.32}$$

In view of (2.28), (2.30), and (2.32), we obtain from Lemma 1.1 that

$$\lim_{n\to\infty}\|x_n - I^n x_n\| = 0. \tag{2.33}$$

On the other hand, we have

$$\begin{aligned}
\|x_n - T^n x_n\| &\leq \left\|x_n - T^n y_n\right\| + \left\|T^n y_n - T^n x_n\right\| \\
&\leq \left\|x_n - T^n y_n\right\| + \left\|y_n - x_n\right\| + \sigma_n + \overline{\sigma}_n \\
&\leq \left\|x_n - T^n y_n\right\| + \widehat{\beta}_n\|I^n x_n - x_n\| + \widehat{\gamma}_n\|v_n - x_n\| + \sigma_n + \overline{\sigma}_n.
\end{aligned} \tag{2.34}$$

In view of (2.23) and (2.33), we have $\lim_{n\to\infty}\|x_n - T^n x_n\| = 0$. This completes the proof of Lemma 2.3. □

**Theorem 2.4.** *Let $X$ be a real Banach space and $K$ a nonempty closed and convex subset of $X$. Let $T : K \to K$ be a asymptotically $I$-nonexpansive in the intermediate sense and $I : K \to K$ a asymptotically nonexpansive in the intermediate sense. Assume that $F := F(T) \cap F(I) \neq \emptyset$. Let $\sigma_n = \max\{0, \sup_{x,y\in k}(\|T^n x - T^n y\| - \|I^n x - I^n y\|)\}$ and $\overline{\sigma}_n = \max\{0, \sup_{x,y\in k}(\|T^n x - T^n y\| - \|x - y\|)\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\widehat{\gamma}_n\}$ be six real number sequences in $(0,1)$. Assume that both $T$ and $I$ are Lipschitz continuous. Let $\{x_n\}$ are a sequence generated in the following iterative process:*

$$\begin{aligned}
&x_1 \in C, \\
&y_n = \widehat{\alpha}_n x_n + \widehat{\beta}_n I^n x_n + \widehat{\gamma}_n v_n, \\
&x_{n+1} = \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \quad n \geq 1,
\end{aligned} \tag{2.35}$$

*where* $\{u_n\}$ *and* $\{v_n\}$ *be two bounded sequences in* $K$*. Assume that the following restrictions are satisfied:*

(a) $\alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1$, $\forall n \geq 1$;

(b) $\sum_{n=1}^{\infty} \sigma_n < \infty$, $\sum_{n=1}^{\infty} \overline{\sigma}_n < \infty$;

(c) *there exist constants* $\tau_1, \tau_2 \in (0,1)$ *such that* $\tau_1 \leq \beta_n$, $\widehat{\beta}_n \leq \tau_2$, $\forall n \geq 1$;

(d) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \widehat{\gamma}_n < \infty$.

*If at least one of the mappings* $T$ *and* $I$ *is compact, then the sequence convergence strongly to a common fixed point of* $T$ *and* $I$*.*

*Proof.* Without loss of generality, we may assume that $T$ is compact; this means that there exists a subsequence $\{T^{n_k}x_{n_k}\}$ of $\{T^n x_n\}$ such that $\{T^{n_k}x_{n_k}\}$ converges strongly to $x^* \in K$, then (2.16) implies that $\{x_{n_k}\}$ converges strongly to $x^*$. Since $T$ is continuous, then $\{T^{n_k+1}x_{n_k}\}$ converges strongly to $Tx^*$. On the other hand, according to (2.17) and the continuity of $I$, we obtain that $\{I^{n_k}x_{n_k}\}$, $\{I^{n_k+1}x_{n_k}\}$ converge strongly to $x^*$, $Ix^*$, respectively. Since $\lim_{k\to\infty}\|x_{n_k+1} - x_{n_k}\| = 0$, then

$$\begin{aligned} \left\|I^{n_k+1}x_{n_k+1} - I^{n_k+1}x_{n_k}\right\| &\leq \|x_{n_k+1} - x_{n_k}\| + \overline{\sigma}_{n_k} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty, \\ \left\|T^{n_k+1}x_{n_k+1} - T^{n_k+1}x_{n_k}\right\| &\leq \left\|I^{n_k+1}x_{n_k+1} - I^{n_k+1}x_{n_k}\right\| + \sigma_{n_k} \\ &\leq \|x_{n_k+1} - x_{n_k}\| + \overline{\sigma}_{n_k} + \sigma_{n_k} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{2.36}$$

Observe that

$$\begin{aligned} \|x^* - Tx^*\| &\leq \|x^* - x_{n_k+1}\| + \left\|x_{n_k+1} - T^{n_k+1}x_{n_k+1}\right\| \\ &\quad + \left\|T^{n_k+1}x_{n_k+1} - T^{n_k+1}x_{n_k}\right\| + \left\|T^{n_k+1}x_{n_k} - Tx^*\right\|, \\ \|x^* - Ix^*\| &\leq \|x^* - x_{n_k+1}\| + \left\|x_{n_k+1} - I^{n_k+1}x_{n_k+1}\right\| \\ &\quad + \left\|I^{n_k+1}x_{n_k+1} - I^{n_k+1}x_{n_k}\right\| + \left\|I^{n_k+1}x_{n_k} - Ix^*\right\|. \end{aligned} \tag{2.37}$$

Taking limit as $k \to \infty$ in the above inequality, we find $x^* = Tx^*$, $x^* = Ix^*$, which means $x^* \in F$. However, due to Lemma 2.1, the limit $\lim_{n\to\infty}\|x_n - x^*\|$ exists, therefore

$$\lim_{n\to\infty}\|x_n - x^*\| = \lim_{k\to\infty}\|x_{n_k} - x^*\| = 0, \tag{2.38}$$

which means that $\{x_n\}$ converges strongly to $x^* \in F$. This completes the proof of Theorem 2.4. □

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## References


[1] K. Goebel and W. A. Kirk, "A fixed point theorem for asymptotically nonexpansive mapping," *Proceedings of the American Mathematical Society*, vol. 35, pp. 171–174, 1972.

[2] B. Bruck, T. Kuczumow, and S. Reich, "Convergence of iterates of asymptotically nonexpansive mappings in Banach space with the uniform Opial property," *Colloquium Mathematicum*, vol. 65, pp. 169–179, 1993.

[3] W. A. Kirk, "Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type," *Israel Journal of Mathematics*, vol. 17, no. 4, pp. 339–346, 1974.

[4] X. Qin, S. S. Chang, and Y. J. Cho, "Iterative methods for generalized equilibrium problems and fixed point problems with applications," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 4, pp. 2963–2972, 2010.

[5] S. Y. Cho and S. M. Kang, "Approximation of fixed points of pseudocontraction semigroups based on a viscosity iterative process," *Applied Mathematics Letters*, vol. 24, no. 2, pp. 224–228, 2011.

[6] S. Y. Cho and S. M. Kang, "Approximation of common solutions of variational inequalities via strict pseudocontractions," *Acta Mathematica Scientia*, vol. 32, no. 4, pp. 1607–1618, 2012.

[7] J. K. Kim, Y. M. Nam, and J. Y. Sim, "Convergence theorems of implicit iterative sequences for a finite family of asymptotically quasi-nonexpansive type mappings," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 71, no. 12, pp. e2839–e2848, 2009.

[8] S. M. Kang, S. Y. Cho, and Z. Liu, "Convergence of iterative sequences for generalized equilibrium problems involving inverse-strongly monotone mappings," *Journal of Inequalities and Applications*, vol. 2010, Article ID 827082, 16 pages, 2010.

[9] J. K. Kim, S. Y. Cho, and X. Qin, "Hybrid projection algorithms for generalized equilibrium problems and strictly pseudocontractive mappings," *Journal of Inequalities and Applications*, vol. 2010, Article ID 312602, 18 pages, 2010.

[10] S. Y. Cho, S. M. Kang, and X. Qin, "Hybrid projection algorithms for treating common fixed points of a family of demicontinuous pseudocontractions," *Applied Mathematics Letters*, vol. 25, no. 5, pp. 854–857, 2012.

[11] Y. Su and S. Li, "Composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 2, pp. 882–891, 2006.

[12] X. Qin, S. Y. Cho, and S. M. Kang, "Iterative algorithms for variational inequality and equilibrium problems with applications," *Journal of Global Optimization*, vol. 48, no. 3, pp. 423–445, 2010.

[13] S. Yang and W. Li, "Iterative solutions of a system of equilibrium problems in Hilbert spaces," *Advances in Fixed Point Theory*, vol. 1, no. 1, pp. 15–26, 2011.

[14] X. Qin, S. Y. Cho, and S. M. Kang, "Strong convergence of shrinking projection methods for quasi-$\phi$-nonexpansive mappings and equilibrium problems," *Journal of Computational and Applied Mathematics*, vol. 234, no. 3, pp. 750–760, 2010.

[15] J. Ye and J. Huang, "Strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces," *Journal of Mathematical and Computational Science*, vol. 1, no. 1, pp. 1–18, 2011.

[16] S. Husain and S. Gupta, "A resolvent operator technique for solving generalized system of nonlinear relaxed cocoercive mixed variational inequalities," *Advances in Fixed Point Theory*, vol. 2, no. 1, pp. 18–28, 2012.

[17] X. Qin, Y. J. Cho, and S. M. Kang, "Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 20–30, 2009.

[18] X. Qin, S. Y. Cho, and S. M. Kang, "On hybrid projection methods for asymptotically quasi-$\phi$-nonexpansive mappings," *Applied Mathematics and Computation*, vol. 215, no. 11, pp. 3874–3883, 2010.

[19] F. Gu, "Some convergence theorems of non-implicit iteration process with errors for a finite families of $I$-asymptotically nonexpansive mappings," *Applied Mathematics and Computation*, vol. 216, no. 1, pp. 161–172, 2010.

[20] S. Lv and C. Wu, "Convergence of iterative algorithms for a generalized variational inequality and a nonexpansive mapping," *Engineering Mathematics Letters*, vol. 1, no. 1, pp. 44–57, 2012.

[21] J. Schu, "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.

[22] F. Mukhamedov and M. Saburov, "Strong convergence of an explicit iteration process for a totally asymptotically $I$-nonexpansive mapping in Banach spaces," *Applied Mathematics Letters*, vol. 23, no. 12, pp. 1473–1478, 2010.

[23] S. Temir, "On the convergence theorems of implicit iteration process for a finite family of $I$-asymptotically nonexpansive mappings," *Journal of Computational and Applied Mathematics*, vol. 225, no. 2, pp. 398–405, 2009.

[24] F. Mukhamedov and M. Saburov, "Weak and strong convergence of an implicit iteration process for an asymptotically quasi-$I$-nonexpansive mapping in Banach space," *Fixed Point Theory and Applications*, vol. 2010, Article ID 719631, 13 pages, 2010.

[25] N. Shahzad, "Generalized $I$-nonexpansive maps and best approximations in Banach spaces," *Demonstratio Mathematica*, vol. 37, pp. 597–600, 2004.

[26] K. K. Tan and H. K. Xu, "Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process," *Journal of Mathematical Analysis and Applications*, vol. 178, no. 2, pp. 301–308, 1993.