Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 187052, 29 pages
doi:10.1155/2011/187052

Research Article

A New General Iterative Method for Solution of a New General System of Variational Inclusions for Nonexpansive Semigroups in Banach Spaces

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Received 4 April 2011; Accepted 8 May 2011

Academic Editor: Yansheng Liu

We introduce a new general system of variational inclusions in Banach spaces and propose a new iterative scheme for finding common element of the set of solutions of the variational inclusion with set-valued maximal monotone mapping and Lipschitzian relaxed cocoercive mapping and the set of fixed point of nonexpansive semigroups in a uniformly convex and 2-uniformly smooth Banach space. Furthermore, strong convergence theorems are established under some certain control conditions. As applications, finding a common solution for a system of variational inequality problems and minimization problems is given.

1. Introduction

In the theory of variational inequalities and variational inclusions, the development of an efficient and implementable iterative algorithm is interesting and important. The important generalization of variational inequalities called *variational inclusions*, have been extensively studied and generalized in different directions to study a wide class of problems arising in optimization, nonlinear programming, finance, economics, and applied sciences.

Variational inequalities are being used as a mathematical programming tool in modeling a wide class of problems arising in several branches of pure and applied mathematics. Several numerical techniques for solving variational inequalities and the related optimization problem have been considered by many authors.

Throughout this paper, we denoted by $\mathbb{N}$ and $\mathbb{R}^+$ the set of all positive integers and all positive real numbers, respectively. Let X be a real Banach space and X^* be its dual space. Let

$U = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. X is said to be *uniformly convex* if for each $\epsilon \in (0,2]$, there exists a constant $\delta > 0$ such that for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \text{ implies } \left\| \frac{x+y}{2} \right\| \leq 1 - \delta. \tag{1.1}$$

The norm on X is said to be *Gâteaux differentiable* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.2}$$

exists for each $x, y \in U$, and in this case X is *smooth*. Moreover, we say that the norm X is said to have a *uniformly Gâteaux differentiable* if the above limit is attained uniformly for all $x, y \in U$ and in this case X is said to be *uniformly smooth*. We define a function $\rho : [0,\infty) \to [0,\infty)$, called the *modulus of smoothness* of X, as follows:

$$\rho(\tau) = \sup\left\{ \frac{1}{2}\left(\|x + y\| + \|x - y\|\right) - 1 : x, y \in X,\ \|x\| = 1,\ \|y\| = \tau \right\}. \tag{1.3}$$

It is know that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number $1 < q \leq 2$. A Banach space X is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. From [1], we know the following property.

Let q be a real number with $1 < q \leq 2$ and let X be a Banach space. Then, X is q-uniformly smooth if and only if there exists a constant $K \geq 1$ such that

$$\|x + y\|^q + \|x - y\|^q \leq 2\left(\|x\|^q + \|Ky\|^q\right), \quad \forall x, y \in X. \tag{1.4}$$

The best constant K in the above inequality is called the q-uniformly smoothness constant of X (see [1] for more details).

Let X be a real Banach space and X^* the dual space of X. Let $\langle \cdot, \cdot \rangle$ denote the pairing between X and X^*. For $q > 1$, the generalized duality mapping

$$J_q(x) = \left\{ f \in X^* : \langle x, f \rangle = \|x\|^q,\ \|f\| = \|x\|^{q-1} \right\}, \quad \forall x \in X. \tag{1.5}$$

In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and usually, we write $J_2 = J$. If X is a Hilbert space, then $J = I$ is the identity. Further, we have the following properties of the generalized duality mapping J_q:

(1) $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \in X$ with $x \neq 0$,

(2) $J_q(tx) = t^{q-1} J_q(x)$ for all $x \in X$ and $t \in [0,\infty)$,

(3) $J_q(-x) = -J_q(x)$ for all $x \in X$.

It is know that if X is smooth, then J is single-valued, which is denoted by j.

Definition 1.1. Let C be a nonempty closed convex subset of X. A mapping $T : C \to X$ is said to be

(i) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C, \tag{1.6}$$

(ii) *Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C, \tag{1.7}$$

(iii) *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.8}$$

Remark 1.2. We denote $F(T)$ as the set of fixed points of T. We know that $F(T)$ is nonempty if C is bounded; for more detail see [2].

Definition 1.3. A one-parameter family $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ from C of X into itself is said to be a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$,

(ii) $T(s + t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}^+$,

(iii) for each $x \in C$ the mapping $t \mapsto T(t)x$ is continuous,

(iv) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \in \mathbb{R}^+$.

Remark 1.4. We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is $F(\mathcal{S}) := \bigcap_{t\in\mathbb{R}^+} F(T(t)) = \{x \in C : T(t)x = x\}$. We know that $F(\mathcal{S})$ is nonempty if C is bounded, see [3].

Let C be a nonempty closed convex subset of a smooth Banach space X. Recall the following definitions of a nonlinear mapping $B : C \to X$, the following are mentioned.

Definition 1.5. Given a mapping $B : C \to X$,

(i) B is said to be *accretive*

$$\langle Bx - By, J(x - y)\rangle \geq 0, \quad \forall x, y \in C, \tag{1.9}$$

(ii) B is said to be *α-strongly accretive* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, J(x - y)\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C, \tag{1.10}$$

(iii) B is said to be *α-inverse-strongly accretive* or *α-cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, J(x-y) \rangle \geq \alpha \|Bx - By\|^2, \quad \forall x, y \in C, \tag{1.11}$$

(iv) B is said to be *α-relaxed cocoercive* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, J(x-y) \rangle \geq -\alpha \|Bx - By\|^2, \quad \forall x, y \in C, \tag{1.12}$$

(v) B is said to be *(α, β)-relaxed cocoercive* if there exist positive constants $\alpha > 0$ and $\beta > 0$ such that

$$\langle Bx - By, J(x-y) \rangle \geq -\alpha \|Bx - By\|^2 + \beta \|x - y\|^2, \quad \forall x, y \in C. \tag{1.13}$$

Remark 1.6. (1) Every α-strongly accretive mapping is an accretive mapping.

(2) Every α-strongly accretive mapping is a (β, α)-relaxed cocoercive mapping for any positive constant β but the converse is not true in general. Then, the class of relaxed cocoercive operators is more general than the class of strongly accretive operators.

(3) Evidently, the definition of the inverse-strongly accretive operator is based on that of the inverse-strongly monotone operator in real Hilbert spaces (see, e.g., [4]).

(4) The notion of the cocoercivity is applied in several directions, especially for solving variational inequality problems using the auxiliary problem principle and projection methods [5]. Several classes of relaxed cocoercive variational inequalities have been studied in [6, 7].

The resolvent operator technique for solving variational inequalities and variational inclusions is interesting and important. The resolvent equation technique is used to develop powerful and efficient numerical techniques for solving various classes of variational inequalities, inclusions, and related optimization problems.

Definition 1.7. Let $M : X \to 2^X$ be a multivalued maximal accretive mapping. The single-valued mapping $J_{(M,\rho)} : X \to X$, defined by

$$J_{(M,\rho)}(u) = (I + \rho M)^{-1}(u), \quad \forall u \in X, \tag{1.14}$$

is called *resolvent operator* associated with M, where ρ is any positive number and I is the identity mapping.

In 2010, Qin et al. [8] introduced a *system of quasivariational inclusions* as follows. Find $(x^*, y^*) \in X \times X$ such that

$$\begin{aligned} &0 \in x^* - y^* + \rho_1 (B_1 y^* + M_1 x^*), \\ &0 \in y^* - x^* + \rho_2 (B_2 x^* + M_2 y^*), \end{aligned} \tag{1.15}$$

where $B_i : X \to X$ and $M_i : X \to 2^X$ are nonlinear mappings for all $i = 1, 2$. As special cases of problem (1.15), we have the following.

(1) If $B_1 = B_2 = B$ and $M_1 = M_2 = M$, then problem (1.15) is reduced to the following. Find $(x^*, y^*) \in X \times X$ such that

$$\begin{aligned} 0 &\in x^* - y^* + \rho_1(By^* + Mx^*), \\ 0 &\in y^* - x^* + \rho_2(Bx^* + My^*). \end{aligned} \tag{1.16}$$

(2) Further, if $x^* = y^*$ in problem (1.16), then problem (1.16) is reduced to the following. Find $x^* \in X$ such that

$$0 \in Bx^* + Mx^*. \tag{1.17}$$

The problem (1.17) is called *variational inclusion problem* denoted by $V(X, B, M)$.

Here we have examples of the variational inclusion (1.17). If $M = \partial\delta_C$, where C is a nonempty closed convex subset of X, and $\delta_C : X \to [0, \infty)$

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases} \tag{1.18}$$

then the variational inclusion problem (1.17) is equivalent (see [9]) to finding $u \in C$ such that

$$\langle Bu, v - u \rangle \geq 0, \quad \forall x \in C. \tag{1.19}$$

This problem is called *Hartman-Stampacchia variational inequality problem* denoted by $\mathrm{VI}(C, B)$.

Let D be a subset of C, and let P be a mapping of C into D. Then, P is said to be *sunny* if

$$P(Px + t(x - Px)) = Px, \tag{1.20}$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. A mapping P of C into itself is called a *retraction* if $P^2 = P$. If a mapping P of C into itself is a retraction, then $Pz = z$ for all $z \in R(P)$, where $R(P)$ is the range of P. A subset D of C is called a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction from C onto D.

In 2006, Aoyama et al. [10] considered the following problem: find $u \in C$ such that

$$\langle Au, J(v - u) \rangle \geq 0, \quad \forall v \in C. \tag{1.21}$$

They proved that the variational inequality (1.21) is equivalent to a fixed point problem. The element $u \in C$ is a solution of the variational inequality (1.21) if and only if $u \in C$ satisfies the following equation:

$$u = P_C(u - \lambda Au), \tag{1.22}$$

where $\lambda > 0$ is a constant and P_C is a sunny nonexpansive retraction from X onto C.

The following results describe a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 1.8 (see [11]). *Let X be a smooth Banach space and C a nonempty subset of X. Let $P : X \to C$ be a retraction and J the normalized duality mapping on X. Then the following are equivalent:*

(1) *P is sunny and nonexpansive,*

(2) $\langle x - Px, J(y - Px)\rangle \leq 0$, *for all $x \in X$, $y \in C$.*

Proposition 1.9 (see [12]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and T a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

For the class of nonexpansive mappings, one classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [13, 14]. More precisely, take $t \in (0,1)$, and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)T_t x, \quad \forall x \in C, \tag{1.23}$$

where $u \in C$ is a fixed point. Banachs contraction mapping principle guarantees that T_t has a unique fixed point $x_t \in C$, that is,

$$x_t = tu + (1-t)T_t x_t. \tag{1.24}$$

It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Ceng et al. [15] proved that, if X is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [14] extended Browders result to the setting of Banach spaces and proved that, if X is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T, and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$.

Reich [14] showed that, if X is uniformly smooth and D is the fixed point set of a nonexpansive mapping from C into itself, then there is a unique sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Proposition 1.10 (see [14]). *Let X be a uniformly smooth Banach space and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. For each fixed $u \in C$ and every $t \in (0,1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $P : C \to D$ by $Pu = s - \lim_{t\to 0} x_t$. Then P is the unique sunny nonexpansive retract from C onto D; that is, P satisfies the property,*

$$\langle x - Px, J(y - Px)\rangle \leq 0, \quad \forall x \in C, \ y \in D. \tag{1.25}$$

Many authors have studied the problems of finding a common element of the set of fixed points of a nonexpansive mapping and one of the sets of solutions to the variational inclusion and variational inequalities (1.15)–(1.17) and (1.21) by using different iterative methods (see, e.g., [10, 16–27]).

Recently, Qin et al. [8] considered the problem of finding the solutions of a general system of variational inclusion (1.15) with α-inverse strongly accretive mappings. To be more precise, they obtained the following results.

Lemma 1.11. *For any* $(x^*, y^*) \in X \times X$, *where* $y^* = J_{(M_2,\rho_2)}(x^* - \rho_2 B_2 x^*)$, (x^*, y^*) *is a solution of the problem* (1.15) *if and only if* x^* *is a fixed point of the mapping* Q *defined by*

$$Qx := J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(x - \rho_2 B_2 x) - \rho_1 B_1 J_{(M_2,\rho_2)}(x - \rho_2 B_2 x)\right]. \tag{1.26}$$

Theorem QCCK 1.12 (see [8]). *Let* X *be a uniformly convex and 2-uniformly smooth Banach space with the smoothness constant* K. *Let* $M_i : X \to 2^X$ *be a maximal monotone mapping and* $B_i : X \to X$ *be a* ρ_i*-inverse-strongly accretive mapping, respectively, for all* $i = 1, 2$. *Let* $T : X \to X$ *be a* ϵ*-strict pseudocontraction such that* $F(T) \neq \emptyset$. *Define a mapping* S *by* $Sx = (1 - \epsilon/K^2)x + (\epsilon/K^2)Tx$, *for all* $x \in X$. *Assume that* $\Omega = F(T) \cap F(Q)$, *where* Q *is defined as in Lemma 1.11. Let* $x_1 = u \in C$ *and let* $\{x_n\}$ *be a sequence defined by*

$$\begin{aligned} z_n &= J_{(M_2,\rho_2)}(x_n - \rho_2 B_2 x_n), \\ y_n &= J_{(M_1,\rho_1)}(z_n - \rho_1 B_1 z_n), \\ x_{n+1} &= \alpha_n u + \beta_n x_n + (1 - \beta_n - \alpha_n)\left[\mu S x_n + (1 - \mu) y_n\right], \quad \forall n \geq 1, \end{aligned} \tag{1.27}$$

where $\mu \in (0, 1)$, $\rho_1 \in (0, \gamma_1/K^2]$, $\rho_2 \in (0, \gamma_2/K^2]$, *and* $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* $(0, 1)$. *If the control consequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following restrictions:*

(C1) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

then $\{x_n\}$ *converges strongly to* $x^* = P_\Omega u$, *where* P_Ω *is the sunny nonexpansive retraction from* X *onto* Ω *and* (x^*, y^*) *is a solution of the problem* (1.15), *where* $y^* = J_{(M_2,\rho_2)}(x^* - \rho_2 B_2 x^*)$.

Iterative methods for nonexpansive mappings have recently been applied to solve minimization problems; see, for example, [28–32]. Let H be a real Hilbert space, whose inner product and norm are denoted by $\|\cdot,\cdot\|$ and $\langle\cdot\rangle$, respectively. Let A be a strongly positive bounded linear operator on H: that is, there is a constant $\gamma > 0$ with property

$$\langle Ax, x \rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.28}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H

$$\min_{x\in F} \frac{1}{2}\langle Ax, x \rangle - \langle x, u \rangle, \tag{1.29}$$

where F is the fixed point set of a nonexpansive mapping T on H and u is a given point in H.

In [33], Moudafi introduced the viscosity approximation method and proved that if H is a real Hilbert space, the sequence $\{x_n\}$ generated by the following algorithm:

$$\begin{gathered} x_0 = u \in H \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T x_n, \quad \forall n \geq 0, \end{gathered} \tag{1.30}$$

where $f : C \to C$ is a contraction mapping with a constant $\alpha \in (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfies certain conditions, converges strongly to a fixed point of T in C which is the unique solution of the following variational inequality:

$$\left\langle (f - I)x^*, x - x^* \right\rangle \leq 0, \quad \forall x \in F(T). \tag{1.31}$$

In 2006, Marino and Xu [34] introduced the following general iterative method:

$$\begin{gathered} x_0 = u \in H \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad \forall n \geq 0, \end{gathered} \tag{1.32}$$

where A is a strongly positive bounded linear operator on a Hilbert space H. They proved that, if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.32) converges strongly to the unique solution of the variational inequality:

$$\left\langle (\gamma f - A)x^*, x - x^* \right\rangle \leq 0, \quad \forall x \in F(T), \tag{1.33}$$

which is the optimality condition for the minimization problem:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.34}$$

where C is the fixed point set of a nonexpansive mapping T and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$).

In a smooth Banach space, we always assume that A is *strongly positive* (see [35]), that is, a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, J(x) \rangle \geq \overline{\gamma} \|x\|^2, \quad \|aI - bA\| = \sup_{\|x\| \leq 1} |\langle (aI - bA)x, J(x) \rangle|, \quad a \in [0,1], \; b \in [-1,1], \tag{1.35}$$

where I is the identity mapping and J is the normalized duality mapping.

Recently, Sunthrayuth and Kumam [36] introduced the following iterative method for nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ in Banach spaces,

$$\begin{gathered} x_0 = u \in C \text{ chosen arbitrarily}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n) I - \alpha_n A\right) \frac{1}{t_n} \int_0^{t_n} T(s) x_n ds, \quad \forall n \geq 0. \end{gathered} \tag{1.36}$$

They proved strong convergence theorem of the iterative scheme $\{x_n\}$ defined by (1.36) converges strongly to the common fixed point of $x^* \in F(\mathcal{S})$ solving the variational inequality

$$\langle (\gamma f - A)x^*, J(x - x^*) \rangle \leq 0, \quad \forall x \in F(\mathcal{S}). \tag{1.37}$$

In 2010, Kamraksa and Wangkeeree [37] introduced a general iterative approximation method for finding common elements of the set of solutions to a general system of variational inclusions with Lipschitzian and relaxed cocoercive mappings and the set common fixed points of a countable family of strict pseudocontractions. They proved the strong convergence theorems of such iterative scheme for finding a common element of such two sets which is a unique solution of some variational inequality and is also the optimality condition for some minimization problems in a strictly convex and 2-uniformly smooth Banach space.

In this paper, we are motivated and inspire by idea of Qin et al. [8] and Sunthrayuth and Kumam [36].

First, we introduce a new general system of variational inclusions in Banach spaces as follows.

Let X be Banach spaces. We consider a system of quasivariational inclusions as follows. Finding $(x^*, y^*, z^*) \in X \times X \times X$ such that

$$\begin{aligned} 0 &\in x^* - y^* + \rho_1(B_1 y^* + M_1 x^*), \\ 0 &\in y^* - z^* + \rho_2(B_2 z^* + M_2 y^*), \\ 0 &\in z^* - x^* + \rho_3(B_3 x^* + M_3 z^*), \end{aligned} \tag{1.38}$$

which is called a *new general system of variational inclusions in Banach spaces*, $B_i : X \to X$ and $M_i : X \to 2^X$ are nonlinear mappings for all $i = 1, 2, 3$. As special cases of problem (1.38), we have the following.

(1) If $B_1 = B_2 = B_3 = B$ and $M_1 = M_2 = M_3 = M$, then problem (1.38) is reduced to the following. Finding $(x^*, y^*, z^*) \in X \times X \times X$ such that

$$\begin{aligned} 0 &\in x^* - y^* + \rho_1(B y^* + M x^*), \\ 0 &\in y^* - z^* + \rho_2(B z^* + M y^*), \\ 0 &\in z^* - x^* + \rho_3(B x^* + M z^*). \end{aligned} \tag{1.39}$$

(2) Further, if $B_3 = M_3 = 0$, $z^* = x^*$, then problem (1.38) is reduced to problem (1.15).

Second, we study a general iterative approximation method (3.1) below, for finding common elements of the set of solutions of a new general system of variational inclusions (1.38) with set-valued maximal monotone mapping and Lipschitzian relaxed cocoercive mappings and the set common fixed points of nonexpansive semigroup in the framework of Banach spaces. Moreover, we prove the strong convergence of the proposed iterative method under some certain control conditions. The results presented in this paper extend and improve the results of Qin et al. [8] and Sunthrayuth and Kumam [36], and many authors.

2. Preliminaries

This section collects some results that will be used in the proofs of our main results.

Lemma 2.1 (see [38]). *The resolvent operator $J_{(M,\rho)}$ associated with M is single valued and nonexpansive for all $\rho > 0$.*

Lemma 2.2 (see [39]). *Let X be a real 2-uniformly smooth Banach space with the best smoothness constant K. Then, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in X. \tag{2.1}$$

Lemma 2.3 (see [40]). *In a real Banach space X, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad \forall x, y \in X. \tag{2.2}$$

Now, we present the concept of a uniformly asymptotically regular semigroup (see [41–43]).

Definition 2.4. Let C be a nonempty closed convex subset of a Banach space X, $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a continuous operator semigroup on C. Then $\mathcal{S}$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) on C if for all $h \geq 0$ and any bounded subset B of C such that

$$\lim_{t\to\infty} \sup_{x\in B} \|T(h)T(t)x - T(t)x\| = 0. \tag{2.3}$$

Lemma 2.5 (see [44]). *Let C be a nonempty closed convex subset of a uniformly Banach space X, B be a bounded closed convex subset of C. If we denote $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ a nonexpansive semigroup on C such that $F(\mathcal{S}) := \bigcap_{t\in\mathbb{R}^+} F(T(t)) \neq \emptyset$. For all $h \geq 0$, the set $\sigma_t(x) = (1/t)\int_0^t T(s)x\,ds$, then*

$$\lim_{t\to\infty} \sup_{x\in B} \|\sigma_t(x) - T(h)\sigma_t(x)\| = 0. \tag{2.4}$$

Remark 2.6. It is easy to check that the set $\{\sigma_t : t \in \mathbb{R}^+\}$ defined by Lemma 2.5 is a u.a.r. nonexpansive semigroup on C (see [45] for more detail).

Lemma 2.7 (see [46]). *Let C be a nonempty closed convex subset of X and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a u.a.r. nonexpansive semigroup on C such $F(\mathcal{S}) := \bigcap_{t\in\mathbb{R}^+} F(T(t)) \neq \emptyset$ and at least there exists a $T(t)$ which is demicompact. Then, for each $x \in C$, there exists a sequence $\{T(t_n) : t_n \in \mathbb{R}^+, n \in \mathbb{N}\} \subset T(t)$, such that $\{T(t_n)x\}$ converges strongly to some point in $F(\mathcal{S})$, where $\lim_{n\to\infty} t_n = \infty$.*

Remark 2.8. By Lemma 2.7, we can see that, for each $x \in C$, there is a corresponding unique point $y \in F(\mathcal{S})$, thus we can define a mapping T such that $Tx = y$ and it is easy to see that $F(T) = F(\mathcal{S})$.

Remark 2.9. From the definition of T, we can see that T is a nonexpansive mapping. Actually, by Lemma 2.7, let $x_1 \in C$; then there exists a sequence $\{T(t_n)\}$ of $\{T(h)\}$ such that

$Tx_1 = \lim_{n\to\infty} T(t_n)x_1 = y_1 \in F(\mathcal{S})$. Further, for any other point $x_2 \in C$, by the definition of a u.a.r., we can get a subsequence $\{T(t_{n_k})\}$ of $\{T(t_n)\}$ such that $T(t_{n_k}) \to y_2 \in F(\mathcal{S})$, then

$$\|Tx_1 - Tx_2\| = \lim_{k\to\infty} \|T(t_{n_k})x_1 - T(t_{n_k})x_2\| \le \|x_1 - x_2\|. \tag{2.5}$$

Lemma 2.10 (see [35]). *Let C be a nonempty closed convex subset of a reflexive, smooth Banach space X which admits a weakly sequentially continuous duality mapping J from X into X^*, T be a nonexpansive mapping such that $F(T) \neq \emptyset$, $f : C \to C$ be a contraction mapping with a coefficient $\alpha \in (0,1)$ and A be a strongly positive bounded linear operator with a coefficient $\overline{\gamma} > 0$. Let $t \in (0,1)$ such that $t \le \|A\|^{-1}$ and $0 < \gamma < \overline{\gamma}/\alpha$ which satisfies $t \to 0$. Then the sequence $\{x_t\}$ defined by*

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t, \tag{2.6}$$

converges strongly to the common fixed point x^* as $t \to 0$, where x^* is a unique solution in $F(\mathcal{S})$ of the variational inequality

$$\langle (\gamma f - A)x^*, J(z - x^*) \rangle \le 0, \quad \forall z \in F(\mathcal{S}). \tag{2.7}$$

Lemma 2.11 (see [47]). *Let C be a closed convex subset of a strictly convex Banach space X. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\mu_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \mu_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=1}^{\infty} \mu_n T_n x$ for all $x \in C$ is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ holds.*

Lemma 2.12 (see [48]). *Let $\{x_n\}$ and $\{v_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)v_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|v_n - x_n\| = 0$.*

Lemma 2.13 (see [35]). *Assume that A is a strongly positive linear bounded operator on a smooth Banach space X with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

If a Banach space X admits a sequentially continuous duality mapping J from weak topology to weak star topology, then by Lemma 1 of [49], we have that duality mapping J is a single value. In this case, the duality mapping J is said to be a weakly sequentially continuous duality mapping, that is, for each $\{x_n\} \subset X$ with $x_n \rightharpoonup x$, we have $J(x_n) \rightharpoonup^* J(x)$ (see [49–51] for more details).

A Banach space X is said to be satisfying Opial's condition if for any sequence $x_n \rightharpoonup x$ for all $x \in X$ implies

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|, \quad \forall y \in X, \text{ with } x \neq y. \tag{2.8}$$

By Theorem 1 in [49], it is well known that if X admits a weakly sequentially continuous duality mapping, then X satisfies Opial's condition, and X is smooth.

Lemma 2.14 (see [50], Demiclosed principle). *Let C be a nonempty closed convex subset of a reflexive Banach space X which satisfies Opial's condition and suppose $T : C \to X$ is nonexpansive. Then the mapping $I - T$ is demiclosed at zero, that is, $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ implies $x = Tx$.*

Lemma 2.15 (see [30]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\tau_n)a_n + \sigma_n, \tag{2.9}$$

where $\{\tau_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \tau_n = \infty$,

(ii) $\limsup_{n\to\infty}(\sigma_n/\tau_n) \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Let X be a real 2-uniformly smooth Banach space X with the smoothness constant K. Let $B : X \to X$ be an L_B-Lipschitzian and relaxed (c,d)-cocoercive mapping, we defined a function $p : (0,+\infty) \to (-\infty,+\infty)$ by

$$p(\rho) := 1 + 2\rho c L_B^2 - 2\rho d + 2\rho^2 K^2 L_B^2, \quad \forall \rho \in (0,+\infty). \tag{2.10}$$

Consequence, we put

$$\theta_\rho = \begin{cases} \sqrt{p(\rho)}, & \text{if } p(\rho) > 0, \\ \dfrac{1}{1+K}, & \text{if } p(\rho) \leq 0. \end{cases} \tag{2.11}$$

In order to prove our main result, the following lemmas are needed.

Lemma 2.16. *Let X be a real 2-uniformly smooth Banach space X with the smoothness constant K. Let $B : X \to X$ be an L_B-Lipschitzian and relaxed (c,d)-cocoercive mapping. Then*

$$\|(I-\rho B)x - (I-\rho B)y\|^2 \leq \theta_\rho^2 \|x-y\|^2. \tag{2.12}$$

In particular, if $0 < \rho \leq (d - cL_B^2)/K^2 L_B^2$, then $I - \rho B$ is nonexpansive.

Proof. For all $x, y \in X$, from Lemma 2.2 and by the cocoercivity of the mapping B, we have

$$\begin{aligned}
\|(I-\rho B)x - (I-\rho B)y\|^2 &= \|(x-y) - (\rho Bx - \rho By)\|^2 \\
&\leq \|x-y\|^2 - 2\rho\langle Bx - By, J(x-y)\rangle + 2\rho^2 K^2 \|Bx - By\|^2 \\
&\leq \|x-y\|^2 - 2\rho\left[-c\|Bx-By\|^2 + d\|x-y\|^2\right] + 2\rho^2 K^2\|Bx-By\|^2 \\
&= \|x-y\|^2 - 2\rho d\|x-y\|^2 + 2\rho c\|Bx-By\|^2 + 2\rho^2K^2\|Bx-By\|^2 \\
&\leq \left(1 + 2\rho c L_B^2 - 2\rho d + 2\rho^2 K^2 L_B^2\right)\|x-y\|^2 \\
&= \theta_\rho^2 \|x-y\|^2.
\end{aligned} \tag{2.13}$$

It follows that

$$\left\|(I-\rho B)x-(I-\rho B)y\right\| \le \theta_\rho \left\|x-y\right\|. \tag{2.14}$$

It is clear that, if $0<\rho \le (d-cL_B^2)/K^2L_B^2$. Thus, we have $I-\rho B$ is nonexpansive. □

Lemma 2.17. *Let X be a 2-uniformly smooth Banach space X. Let $M_i : X \to 2^X$ be the a maximal monotone mapping and $B_i : X \to X$ be an L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping for $i=1,2,3$. Let $Q : X \to X$ be a mapping defined by*

$$\begin{aligned} Qx := J_{(M_1,\rho_1)}\big(&J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(x-\rho_3B_3x)-\rho_2B_2J_{(M_3,\rho_3)}(x-\rho_3B_3x) \\ &-\rho_1B_1J_{(M_2,\rho_2)}J_{(M_3,\rho_3)}(x-\rho_3B_3x)-\rho_2B_2J_{(M_3,\rho_3)}(x-\rho_3B_3x)\big)\big), \quad \forall x\in X. \end{aligned} \tag{2.15}$$

If $0<\rho_i\le (d_i-c_iL_i^2)/K^2L_i^2$ and $\theta_\rho=\max\{\theta_{\rho_i}\}$ for all $i=1,2,3$, then the mapping Q is nonexpansive.

Proof. For all $x, y\in X$, we have

$$\begin{aligned} \|Qx-Qy\| \le \big\|&J_{(M_1,\rho_1)}\big[J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(x-\rho_3B_3x)-\rho_2B_2J_{(M_3,\rho_3)}(x-\rho_3B_3x)\big) \\ &\qquad -\rho_1B_1J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(x-\rho_3B_3x)-\rho_2B_2J_{(M_3,\rho_3)}(x-\rho_3B_3x)\big)\big] \\ &-J_{(M_1,\rho_1)}\big[J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(y-\rho_3B_3y)-\rho_2B_2J_{(M_3,\rho_3)}(y-\rho_3B_3y)\big) \\ &\qquad -\rho_1B_1J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(y-\rho_3B_3y)-\rho_2B_2J_{(M_3,\rho_3)}(y-\rho_3B_3y)\big)\big]\big\| \\ \le \big\|&J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(x-\rho_3B_3x)-\rho_2B_2J_{(M_3,\rho_3)}(x-\rho_3B_3x)\big) \\ &-\rho_1B_1J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(x-\rho_3B_3x)-\rho_2B_2J_{(M_3,\rho_3)}(x-\rho_3B_3x)\big) \\ &-J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(y-\rho_3B_3y)-\rho_2B_2J_{(M_3,\rho_3)}(y-\rho_3B_3y)\big) \\ &-\rho_1B_1J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(y-\rho_3B_3y)-\rho_2B_2J_{(M_3,\rho_3)}(y-\rho_3B_3y)\big)\big\| \\ = \big\|&J_{(M_2,\rho_2)}(I-\rho_2B_2)J_{(M_3,\rho_3)}(I-\rho_3B_3)(I-\rho_1B_1)x \\ &-J_{(M_2,\rho_2)}(I-\rho_2B_2)J_{(M_3,\rho_3)}(I-\rho_3B_3)(I-\rho_1B_1)y\big\| \\ \le \big\|&(I-\rho_2B_2)J_{(M_3,\rho_3)}(I-\rho_3B_3)(I-\rho_1B_1)x \\ &-(I-\rho_2B_2)J_{(M_3,\rho_3)}(I-\rho_3B_3)(I-\rho_1B_1)y\big\|. \end{aligned} \tag{2.16}$$

From Lemma 2.16 and by the nonexpansiveness of $J_{(M_i,\rho_i)}$ for all $i=1,2,3$, we have $(I-\rho_2B_2)J_{(M_3,\rho_3)}(I-\rho_3B_3)(I-\rho_1B_1)$ is a nonexpansive mapping, which implies that the mapping Q is nonexpansive. □

Lemma 2.18. *For all $(x^*, y^*, z^*) \in X\times X\times X$, where $y^* = J_{(M_2,\rho_2)}(z^*-\rho_2 B_2 z^*)$ and $z^* = J_{(M_3,\rho_3)}(x^*-\rho_3 B_3 x^*)$, (x^*, y^*, z^*) is a solution of the problem* (1.38) *if and only if x^* is a fixed point of the mapping Q defined as in Lemma 2.17.*

Proof. Let $(x^*, y^*, z^*) \in X \times X \times X$ be a solution of the problem (1.38). Then, we have

$$\begin{aligned} y^* - \rho_1 B_1 y^* &\in (I + \rho_1 M_1)x^*, \\ z^* - \rho_2 B_2 z^* &\in (I + \rho_2 M_2)y^*, \\ x^* - \rho_3 B_3 x^* &\in (I + \rho_3 M_3)z^*, \end{aligned} \tag{2.17}$$

which implies that

$$\begin{aligned} x^* &= J_{(M_1,\rho_1)}(y^* - \rho_1 B_1 y^*), \\ y^* &= J_{(M_2,\rho_2)}(z^* - \rho_2 B_2 z^*), \\ z^* &= J_{(M_3,\rho_3)}(x^* - \rho_3 B_3 x^*). \end{aligned} \tag{2.18}$$

We can deduce that (2.18) is equivalent to

$$\begin{aligned} x^* = J_{(M_1,\rho_1)}\big[&J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(x^* - \rho_3 B_3 x^*) - \rho_2 B_2 J_{(M_3,\rho_3)}(x^* - \rho_3 B_3 x^*)\big) \\ &-\rho_1 B_1 J_{(M_2,\rho_2)} J_{(M_3,\rho_3)}(x^* - \rho_3 B_3 x^*) - \rho_2 B_2 J_{(M_3,\rho_3)}(x^* - \rho_3 B_3 x^*)\big)\big]. \end{aligned} \tag{2.19}$$

This completes the proof. □

3. Main Results

In this section, we prove that the iterative scheme (3.1) below converges strongly to common element of the set of solutions of the variational inclusion with set-valued maximal monotone mapping and Lipschitzian relaxed cocoercive mapping and the set of fixed point of a family of nonexpansive semigroup in a uniformly convex and 2-uniformly smooth Banach space under some certain control conditions.

Now, we prove the strong convergence theorem of the sequence (3.1) for solving the problem (1.38).

Theorem 3.1. *Let X be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly continuous duality mapping and has the smoothness constant K. Let $M_i : X \to 2^X$ be a maximal monotone mapping and $B_i : X \to X$ be a L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping with $\rho_i \in (0, (d_i - c_i L_i^2)/K^2 L_i^2]$ and $\theta_\rho = \max\{\theta_{\rho_i}\}$ for all $i = 1, 2, 3$. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup from X into itself and at least there exists a $T(t)$ which is demicompact. Assume that $\Omega := F(\mathcal{S}) \cap F(Q) \neq \emptyset$, where Q is defined as in Lemma 2.17. Let $f : X \to X$ be a contraction mapping with a coefficient $\alpha \in (0, 1)$ and A be a strongly positive linear bounded* *self adjoint operator with a coefficient* $\overline{\gamma} \in (0,1)$ *such that* $\|A\| \leq 1$ *and* $0 < \gamma < \overline{\gamma}/\alpha$. *Let* $\{x_n\}$ *be a sequence defined by*

$$\begin{gathered}
x_1 \in X \text{ chosen arbitrarily}, \\
z_n = J_{(M_3,\rho_3)}(x_n - \rho_3 B_3 x_n), \\
y_n = J_{(M_2,\rho_2)}(z_n - \rho_2 B_2 z_n), \\
v_n = J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n), \\
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\left[\mu \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + (1-\mu)v_n\right], \quad \forall n \geq 1,
\end{gathered} \tag{3.1}$$

where $\mu \in (0,1)$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *are the sequences in* $(0,1)$ *which satisfies* $\alpha_n + \beta_n \leq 1$ *and* $\{t_n\}_{n=1}^{\infty}$ *is a positive real divergent sequence satisfy the following restrictions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}|t_{n+1} - t_n|/t_{n+1} = 0$.

Then the sequence $\{x_n\}$ *defined by* (3.1) *converges strongly to* x^*, *which* x^* *solves the variational inequality*

$$\langle (\gamma f - A)x^*, J(z - x^*)\rangle \leq 0, \quad \forall z \in \Omega, \tag{3.2}$$

and (x^*, y^*, z^*) *is a solution of general system of variational inequality problem* (1.38), *where* $y^* = J_{(M_2,\rho_2)}(z^* - \rho_2 B_2 z^*)$ *and* $z^* = J_{(M_3,\rho_3)}(x^* - \rho_3 B_3 x^*)$.

Proof. First, we show $\{x_n\}$ is bounded. By the condition (C1), we may assume, with no loss of generality, that $\alpha_n \leq (1-\beta_n)\|A\|^{-1}$. Since A is a linear bounded operator on X, by (1.35), we have $\|A\| = \sup\{|\langle Au, J(u)\rangle| : u \in X, \|u\| = 1\}$. Observe that

$$\begin{aligned}
\langle ((1-\beta_n)I - \alpha_n A)u, J(u)\rangle &= 1 - \beta_n - \alpha_n \langle Au, J(u)\rangle \\
&\geq 1 - \beta_n - \alpha_n \|A\| \\
&\geq 0,
\end{aligned} \tag{3.3}$$

that is, $(1-\beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}
\|(1-\beta_n)I - \alpha_n A\| &= \sup\{\langle ((1-\beta_n)I - \alpha_n A)u, J(u)\rangle : u \in X, \ \|u\| = 1\} \\
&= \sup\{1 - \beta_n - \alpha_n\langle Au, J(u)\rangle : u \in X, \ \|u\| = 1\} \\
&\leq 1 - \beta_n - \alpha_n\overline{\gamma}.
\end{aligned} \tag{3.4}$$

Taking $\overline{x} \in \Omega$, it follows from Lemma 2.17 that

$$\begin{aligned}\overline{x} = J_{(M_1,\rho_1)}\big[&J_{(M_2,\rho_2)}\big(J_{(M_3,\rho_3)}(\overline{x} - \rho_3 B_3 \overline{x}) - \rho_2 B_2 J_{(M_3,\rho_3)}(\overline{x} - \rho_3 B_3 \overline{x})\\ &-\rho_1 B_1 J_{(M_2,\rho_2)} J_{(M_3,\rho_3)}(\overline{x} - \rho_3 B_3 \overline{x}) - \rho_2 B_2 J_{(M_3,\rho_3)}(\overline{x} - \rho_3 B_3 \overline{x})\big)\big].\end{aligned} \tag{3.5}$$

Putting $\overline{y} = J_{(M_2,\rho_2)}(\overline{z} - \rho_2 B_2 \overline{z})$ and $\overline{z} = J_{(M_1,\rho_1)}(\overline{x} - \rho_3 B_3 \overline{x})$. Then $\overline{x} = J_{(M_1,\rho_1)}(\overline{y} - \rho_1 B_1 \overline{y})$. It follows from Lemmas 2.1 and 2.16 that

$$\begin{aligned}\|v_n - \overline{x}\| &= \big\|J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n) - J_{(M_1,\rho_1)}(\overline{y} - \rho_1 B_1 \overline{y})\big\|\\ &\le \big\|(y_n - \rho_1 B_1 y_n) - (\overline{y} - \rho_1 B_1 \overline{y})\big\|\\ &\le \|y_n - \overline{y}\|\\ &= \big\|J_{(M_2,\rho_2)}(z_n - \rho_2 B_2 z_n) - J_{(M_2,\rho_2)}(\overline{z} - \rho_2 B_2 \overline{z})\big\|\\ &\le \big\|(z_n - \rho_2 B_2 z_n) - (\overline{z} - \rho_2 B_2 \overline{z})\big\|\\ &\le \|z_n - \overline{z}\|\\ &= \big\|J_{(M_3,\rho_3)}(x_n - \rho_3 B_3 x_n) - J_{(M_3,\rho_3)}(\overline{x} - \rho_3 B_3 \overline{x})\big\|\\ &\le \big\|(x_n - \rho_3 B_3 x_n) - (\overline{x} - \rho_3 B_3 \overline{x})\big\|\\ &\le \|x_n - \overline{x}\|\end{aligned} \tag{3.6}$$

and setting $\epsilon_n = \mu(1/t_n)\int_0^{t_n} T(s)x_n ds + (1-\mu)v_n$. From (3.6), we obtain

$$\begin{aligned}\|\epsilon_n - \overline{x}\| &= \left\|\mu\left(\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - \overline{x}\right) + (1-\mu)(v_n - \overline{x})\right\|\\ &\le \mu\left\|\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds - \overline{x}\right\| + (1-\mu)\|v_n - \overline{x}\|\\ &\le \mu\|x_n - \overline{x}\| + (1-\mu)\|x_n - \overline{x}\|\\ &= \|x_n - \overline{x}\|.\end{aligned} \tag{3.7}$$

It follows from (3.7) that

$$\begin{aligned}\|x_{n+1} - \overline{x}\| &= \big\|\alpha_n(\gamma f(x_n) - A\overline{x}) + \beta_n(x_n - \overline{x}) + ((1-\beta_n)I - \alpha_n A)(\epsilon_n - \overline{x})\big\|\\ &\le \alpha_n\big\|\gamma f(x_n) - A\overline{x}\big\| + \beta_n\|x_n - \overline{x}\| + (1-\beta_n - \alpha_n\overline{\gamma})\|\epsilon_n - \overline{x}\|\\ &\le \alpha_n\gamma\big\|f(x_n) - f(\overline{x})\big\| + \alpha_n\big\|\gamma f(\overline{x}) - A\overline{x}\big\| + \beta_n\|x_n - \overline{x}\| + (1-\beta_n - \alpha_n\overline{\gamma})\|x_n - \overline{x}\|\\ &\le (1-\alpha_n\overline{\gamma})\|x_n - \overline{x}\| + \alpha_n\gamma\alpha\|x_n - \overline{x}\| + \alpha_n\big\|\gamma f(\overline{x}) - A\overline{x}\big\|\\ &= (1-\alpha_n(\overline{\gamma} - \gamma\alpha))\|x_n - \overline{x}\| + \alpha_n\big\|\gamma f(\overline{x}) - A\overline{x}\big\|\\ &\le \max\left\{\|x_n - \overline{x}\|, \frac{\|\gamma f(\overline{x}) - A\overline{x}\|}{\overline{\gamma} - \gamma\alpha}\right\}.\end{aligned} \tag{3.8}$$

By induction, we have

$$\|x_n - \overline{x}\| \leq \max\left\{ \|x_1 - \overline{x}\|, \frac{\|\gamma f(\overline{x}) - A\overline{x}\|}{\overline{\gamma} - \gamma\alpha} \right\}, \quad \forall n \geq 1. \tag{3.9}$$

Hence, $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{y_n\}$, and $\{v_n\}$. On the other hand, by nonexpansiveness of $J_{(M_i,\rho_i)}$ and $I - \rho_i B_i$ for all $i = 1, 2, 3$, we have

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \left\| J_{(M_1,\rho_1)}(y_{n+1} - \rho_1 B_1 y_{n+1}) - J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n) \right\| \\
&\leq \left\| (y_{n+1} - \rho_1 B_1 y_{n+1}) - (y_n - \rho_1 B_1 y_n) \right\| \\
&\leq \left\| y_{n+1} - y_n \right\| \\
&= \left\| J_{(M_2,\rho_2)}(z_{n+1} - \rho_2 B_2 z_{n+1}) - J_{(M_2,\rho_2)}(z_n - \rho_2 B_2 z_n) \right\| \\
&\leq \|z_{n+1} - z_n\| \\
&= \left\| J_{(M_3,\rho_3)}(x_{n+1} - \rho_3 B_3 x_{n+1}) - J_{(M_3,\rho_3)}(x_n - \rho_3 B_3 x_n) \right\| \\
&\leq \|x_{n+1} - x_n\|.
\end{aligned} \tag{3.10}$$

Now, we estimate

$$\begin{aligned}
&\left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} T(s) x_{n+1} ds - \frac{1}{t_n} \int_0^{t_n} T(s) x_n ds \right\| \\
&\quad = \left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} [T(s)x_{n+1} - T(s)x_n] ds + \frac{1}{t_{n+1}} \int_0^{t_{n+1}} T(s) x_n ds - \frac{1}{t_n} \int_0^{t_n} T(s) x_n ds \right\| \\
&\quad = \left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} [T(s)x_{n+1} - T(s)x_n] ds \right. \\
&\qquad \left. + \frac{1}{t_{n+1}} \int_0^{t_n} T(s) x_n ds + \frac{1}{t_{n+1}} \int_{t_n}^{t_{n+1}} T(s) x_n ds - \frac{1}{t_n} \int_0^{t_n} T(s) x_n ds \right\| \\
&\quad = \left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} [T(s)x_{n+1} - T(s)x_n] ds + \left(\frac{1}{t_{n+1}} - \frac{1}{t_n} \right) \int_0^{t_n} T(s) x_n ds + \frac{1}{t_{n+1}} \int_{t_n}^{t_{n+1}} T(s) x_n ds \right\|,
\end{aligned} \tag{3.11}$$

for $\overline{x} \in \Omega$, it follows that

$$\begin{aligned}
&\left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} T(s)x_{n+1}ds - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds \right\| \\
&\quad = \left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} [T(s)x_{n+1} - T(s)x_n] ds \right. \\
&\qquad \left. + \left(\frac{1}{t_{n+1}} - \frac{1}{t_n} \right) \int_0^{t_n} [T(s)x_n - \overline{x}] ds + \frac{1}{t_{n+1}} \int_{t_n}^{t_{n+1}} [T(s)x_n - \overline{x}] ds \right\| \\
&\quad \leq \frac{1}{t_{n+1}} \int_0^{t_{n+1}} \|T(s)x_{n+1} - T(s)x_n\| ds + \left| \frac{1}{t_{n+1}} - \frac{1}{t_n} \right| \int_0^{t_n} \|T(s)x_n - \overline{x}\| ds \\
&\qquad + \frac{1}{t_{n+1}} \int_{t_n}^{t_{n+1}} \|T(s)x_n - \overline{x}\| ds \\
&\quad \leq \|x_{n+1} - x_n\| + \left[\left| \frac{1}{t_{n+1}} - \frac{1}{t_n} \right| t_n + \frac{1}{t_{n+1}} |t_{n+1} - t_n| \right] \|x_n - \overline{x}\| \\
&\quad \leq \|x_{n+1} - x_n\| + \left(\frac{2|t_{n+1} - t_n|}{t_{n+1}} \right) \|x_n - \overline{x}\|.
\end{aligned} \tag{3.12}$$

It follows from (3.10) and (3.12) that

$$\begin{aligned}
\|\epsilon_{n+1} - \epsilon_n\| &= \left\| \left(\mu \frac{1}{t_{n+1}} \int_0^{t_{n+1}} T(s)x_{n+1}ds + (1-\mu)v_{n+1} \right) - \left(\mu \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds + (1-\mu)v_n \right) \right\| \\
&\leq \mu \left\| \frac{1}{t_{n+1}} \int_0^{t_{n+1}} T(s)x_{n+1}ds - \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds \right\| + (1-\mu)\|v_{n+1} - v_n\| \\
&\leq \|x_{n+1} - x_n\| + \left(\frac{2|t_{n+1} - t_n|}{t_{n+1}} \right) \|x_n - \overline{x}\| + (1-\mu)\|x_{n+1} - x_n\| \\
&\leq \|x_{n+1} - x_n\| + \left(\frac{2|t_{n+1} - t_n|}{t_{n+1}} \right) M_1,
\end{aligned} \tag{3.13}$$

where $M_1 > 0$ is an appropriate constant such that $M_1 = \sup_{n\geq 1}\{\|x_n - \overline{x}\|\}$.

Setting $l_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n)$, for all $n \in \mathbb{N}$. Then $x_{n+1} = \beta_n x_n + (1-\beta_n)l_n$, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
l_{n+1} - l_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \alpha_{n+1}A)\epsilon_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n \gamma f(x_n) + ((1-\beta_n)I - \alpha_n A)\epsilon_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1}) - A\epsilon_{n+1}) + \frac{\alpha_n}{1-\beta_n}(A\epsilon_n - \gamma f(x_n)) + \epsilon_{n+1} - \epsilon_n
\end{aligned} \tag{3.14}$$

and, hence,

$$\|l_{n+1} - l_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left\|\gamma f(x_{n+1}) - A\epsilon_{n+1}\right\| + \frac{\alpha_n}{1-\beta_n} \left\|A\epsilon_n - \gamma f(x_n)\right\| + \|\epsilon_{n+1} - \epsilon_n\|, \tag{3.15}$$

which, combined with (3.13) yields that

$$\begin{aligned}\|l_{n+1} - l_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left\|\gamma f(x_{n+1}) - A\epsilon_{n+1}\right\| + \frac{\alpha_n}{1-\beta_n} \left\|A\epsilon_n - \gamma f(x_n)\right\| \\ &+ \|x_{n+1} - x_n\| + \left(\frac{2|t_{n+1} - t_n|}{t_{n+1}}\right) M_1.\end{aligned} \tag{3.16}$$

It follows that

$$\begin{aligned}\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left\|\gamma f(x_{n+1}) - A\epsilon_{n+1}\right\| \\ &+ \frac{\alpha_n}{1-\beta_n} \left\|A\epsilon_n - \gamma f(x_n)\right\| + \left(\frac{2|t_{n+1} - t_n|}{t_{n+1}}\right) M_1.\end{aligned} \tag{3.17}$$

By conditions (C1)–(C3), we have

$$\limsup_{n\to\infty} \left(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|\right) \leq 0. \tag{3.18}$$

Then, from Lemma 2.12, we obtain

$$\lim_{n\to\infty} \|l_n - x_n\| = 0, \tag{3.19}$$

observing that

$$x_{n+1} - x_n = (1-\beta_n)(l_n - x_n), \tag{3.20}$$

and, hence,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.21}$$

Since $x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\epsilon_n$, then we have

$$x_{n+1} - x_n = \alpha_n\left(\gamma f(x_n) - Ax_n\right) + \left((1-\beta_n)I - \alpha_n A\right)(\epsilon_n - x_n). \tag{3.22}$$

It follows that

$$\left(1 - \beta_n - \alpha_n \overline{\gamma}\right)\|\epsilon_n - x_n\| \leq \|x_n - x_{n+1}\| + \alpha_n \left\|\gamma f(x_n) - Ax_n\right\|. \tag{3.23}$$

From (3.21) and by the conditions (C1), (C2), we obtain that

$$\lim_{n\to\infty}\|\epsilon_n - x_n\| = 0. \tag{3.24}$$

Define the mapping G_n by

$$G_n x = \mu \frac{1}{t_n}\int_0^{t_n} T(s)x\,ds + (1-\mu)Qx, \quad \forall x \in X, \tag{3.25}$$

where Q is defined as in Lemma 2.17. By Lemma 2.11, we see that G_n is nonexpansive such that

$$F(G_n) = F(\mathcal{S}) \cap F(Q). \tag{3.26}$$

From (3.24), it follows that

$$\lim_{n\to\infty}\|G_n x_n - x_n\| = 0. \tag{3.27}$$

Since $\{T(t) : t \in \mathbb{R}^+\}$ is a u.a.r. nonexpansive semigroup by Remark 2.6, and at least there exists a $T(t)$ which is demicompact, then by Lemma 2.7, we can define the mapping $T : X \to X$ such that $Ty = \lim_{n\to\infty}(1/t_n)\int_0^{t_n} T(s)y\,ds$ for all $y \in X$. Furthermore, let $G : X \to X$ be the mapping define by $Gy = \lim_{n\to\infty} G_n y$, for all $y \in X$. It follows from the nonexpansivity of G and

$$Gy := \mu Ty + (1-\mu)Qy, \quad \forall y \in X, \tag{3.28}$$

such that

$$F(G) = F(T) \cap F(Q) = F(\mathcal{S}) \cap F(Q) = F(G_n). \tag{3.29}$$

On the other hand, we observe that for any bounded subset C of X containing $\{x_n\}$,

$$\begin{aligned}\|x_n - Gx_n\| &\le \|x_n - G_n x_n\| + \|G_n x_n - Gx_n\| \\ &\le \|x_n - G_n x_n\| + \sup_{\omega\in C}\|G_n\omega - G\omega\|.\end{aligned} \tag{3.30}$$

By Lemma 2.7 and (3.27), we obtain that

$$\lim_{n\to\infty}\|x_n - Gx_n\| = 0. \tag{3.31}$$

Since X is a uniformly convex, hence it is reflexive and $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $z \in X$ as $j \to \infty$. Again, since

Banach space X has a weakly sequentially continuous duality mapping satisfying Opial's condition, then by (3.32), we have by Lemma 2.14, that $z \in F(G) = \Omega$.

Next, we show that $\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle \leq 0$, where $x^* = \lim_{t\to 0} x_t$ with x_t be the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)Gx$. Since G is a nonexpansive mapping, it follows from Lemma 2.10 that $x^* \in F(G) = \Omega$, which solves the variational inequality

$$\langle (\gamma f - A)x^*, J(z - x^*)\rangle \leq 0, \quad \forall z \in F(G) \tag{3.32}$$

and (x^*, y^*, z^*) is a solution of general system of variational inequality problem (1.38) such that $y^* = J_{(M_2,\rho_2)}(z^* - \rho_2 B_2 z^*)$ and $z^* = J_{(M_3,\rho_3)}(x^* - \rho_3 B_3 x^*)$.

Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle = \lim_{j\to\infty}\left\langle \gamma f(x^*) - Ax^*, J\left(x_{n_j} - x^*\right)\right\rangle. \tag{3.33}$$

Since the duality mapping J is single-valued and weakly sequentially continuous from X to X^*, we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(x^*) - Ax^*, J(x_n - x^*)\rangle &= \lim_{j\to\infty}\left\langle \gamma f(x^*) - Ax^*, J\left(x_{n_j} - x^*\right)\right\rangle \\ &= \langle \gamma f(x^*) - Ax^*, J(z - x^*)\rangle \leq 0,\end{aligned} \tag{3.34}$$

as required.

Finally, we show that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Now, from Lemma 2.3, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(\epsilon_n - x^*)\right\|^2 \\ &\leq \left\|((1-\beta_n)I - \alpha_n A)(\epsilon_n - x^*) + \beta_n(x_n - x^*)\right\|^2 \\ &\quad + 2\alpha_n\langle \gamma f(x_n) - Ax^*, J(x_{n+1} - x^*)\rangle \\ &\leq \left((1 - \beta_n - \alpha_n\overline{\gamma})\|\epsilon_n - x^*\| + \beta_n\|x_n - x^*\|\right)^2 + 2\alpha_n\langle \gamma f(x_n) - Ax^*, J(x_{n+1} - x^*)\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\gamma\langle f(x_n) - f(x^*), J(x_{n+1} - x^*)\rangle \\ &\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + 2\alpha_n\gamma\alpha\|x_n - x^*\|\|x_{n+1} - x^*\| \\ &\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle \\ &\leq (1 - \alpha_n\overline{\gamma})^2\|x_n - x^*\|^2 + \alpha_n\gamma\alpha\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\ &\quad + 2\alpha_n\langle \gamma f(x^*) - Ax^*, J(x_{n+1} - x^*)\rangle,\end{aligned} \tag{3.35}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{(1-\alpha_n\overline{\gamma})^2+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\big\langle \gamma f(x^*)-Ax^*, J(x_{n+1}-x^*)\big\rangle \\
&= \frac{1-2\alpha_n\overline{\gamma}+\alpha_n\gamma\alpha}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2+\frac{\alpha_n^2\overline{\gamma}^2}{1-\alpha_n\gamma\alpha}\|x_n-x^*\|^2 \\
&\quad+\frac{2\alpha_n}{1-\alpha_n\gamma\alpha}\big\langle \gamma f(x^*)-Ax^*, J(x_{n+1}-x^*)\big\rangle \\
&\le \left[1-\frac{2\alpha_n(\overline{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\right]\|x_n-x^*\|^2 \\
&\quad+\frac{2\alpha_n(\overline{\gamma}-\gamma\alpha)}{1-\alpha_n\gamma\alpha}\left[\frac{1}{\overline{\gamma}-\gamma\alpha}\big\langle \gamma f(x^*)-Ax^*, J(x_{n+1}-x^*)\big\rangle+\frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma}-\gamma\alpha)}M_2\right],
\end{aligned}\tag{3.36}$$

where $M_2 > 0$ is an appropriate constant such that $M_2 = \sup_{n\ge1}\{\|x_n - x^*\|\}$. Put $\tau_n = 2\alpha_n(\overline{\gamma} - \gamma\alpha)/(1-\alpha_n\gamma\alpha)$ and $\delta_n = (1/(\overline{\gamma}-\gamma\alpha))\langle \gamma f(x^*) - Ax^*, J(x_{n+1}-x^*)\rangle + (\alpha_n\overline{\gamma}^2/2(\overline{\gamma}-\gamma\alpha))M_2$. Then (3.32) reduces to formula

$$\|x_{n+1}-x^*\|^2 \le (1-\tau_n)\|x_n-x^*\|^2+\tau_n\delta_n. \tag{3.37}$$

By the condition (C1) and (3.31), it easy to see that

$$\lim_{n\to\infty}\tau_n = 0, \qquad \sum_{n=1}^{\infty}\tau_n=\infty, \qquad \limsup_{n\to\infty}\delta_n \le 0. \tag{3.38}$$

Applying Lemma 2.15 to (3.37), we obtain $x_n \to x^*$ as $n \to \infty$, that is $\lim_{n\to\infty}\|x_n - x^*\| = 0$. This completes the proof. □

Remark 3.2. Theorem 3.1 mainly improves Theorem 2.1 of Qin et al. [8], in the following respects:

(1) from a system variational inclusion to a general system of variational inclusions,

(2) from the class of inverse-strongly accretive mappings to the class of Lipchitzian and relaxed cocoercive mappings.

Taking $A \equiv I$ and $\gamma = 1$ in Theorem 3.1, we can obtain the following result.

Corollary 3.3. *Let X be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly continuous duality mapping and has the smoothness constant K. Let $M_i : X \to 2^X$ be a maximal monotone mapping and $B_i : X \to X$ be a L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping with $\rho_i \in (0, (d_i - c_iL_i^2)/K^2L_i^2]$ and $\theta_\rho = \max\{\theta_{\rho_i}\}$ for all $i = 1, 2, 3$. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$*

be a nonexpansive semigroup from X into itself and at least there exists a $T(t)$ which is demicompact. Assume that $\Omega := F(\mathcal{S}) \cap F(Q) \neq \emptyset$, where Q is defined as in Lemma 2.17. Let $f : X \to X$ be a contraction mapping with a coefficient $\alpha \in (0,1)$. Let $\{x_n\}$ be a sequence defined by

$$\begin{gathered} x_1 \in X \text{ chosen arbitrarily}, \\ z_n = J_{(M_3,\rho_3)}(x_n - \rho_3 B_3 x_n), \\ y_n = J_{(M_2,\rho_2)}(z_n - \rho_2 B_2 z_n), \\ v_n = J_{(M_1,\rho_1)}(y_n - \rho_1 B_1 y_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\beta_n-\alpha_n)\left[\mu \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + (1-\mu)v_n\right], \quad \forall n \geq 1, \end{gathered} \tag{3.39}$$

where $\mu \in (0,1)$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are the sequences in $(0,1)$ which satisfies $\alpha_n + \beta_n \leq 1$ and $\{t_n\}_{n=1}^{\infty}$ is a positive real divergent sequence satisfy the following restrictions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}(|t_{n+1} - t_n|/t_{n+1}) = 0$.

Then the sequence $\{x_n\}$ defined by (3.39) *converges strongly to x^*, which x^* solves the variational inequality*

$$\langle (f-I)x^*, J(z-x^*)\rangle \leq 0, \quad \forall z \in \Omega, \tag{3.40}$$

and (x^*, y^*, z^*) is a solution of general system of variational inequality problem (1.38), *where $y^* = J_{(M_2,\rho_2)}(z^* - \rho_2 B_2 z^*)$ and $z^* = J_{(M_3,\rho_3)}(x^* - \rho_3 B_3 x^*)$.*

Taking $f(x) = u$ in Corollary 3.3, we can obtain the following result.

Corollary 3.4. *Let X be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly continuous duality mapping and has the smoothness constant K. Let $M_i : X \to 2^X$ be a maximal monotone mapping and $B_i : X \to X$ be a L_i-Lipschitzian and relaxed (c_i, d_i)-cocoercive mapping with $\rho_i \in (0, (d_i - c_i L_i^2)/K^2 L_i^2]$ and $\theta_\rho = \max\{\theta_{\rho_i}\}$ for all $i = 1,2,3$. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup from X into itself and at least there exists a $T(t)$ which is demicompact. Assume that $\Omega := F(\mathcal{S}) \cap F(Q) \neq \emptyset$, where Q is defined as in Lemma 2.17. Let $\{x_n\}$ be a sequence defined by*

$$\begin{gathered} x_1 \in X \text{ chosen arbitrarily}, \\ z_n = J_{(M,\rho_3)}(x_n - \rho_3 B_3 x_n), \\ y_n = J_{(M,\rho_2)}(z_n - \rho_2 B_2 z_n), \\ v_n = J_{(M,\rho_1)}(y_n - \rho_1 B_1 y_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + (1-\beta_n-\alpha_n)\left[\mu \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + (1-\mu)v_n\right], \quad \forall n \geq 1, \end{gathered} \tag{3.41}$$

where $\mu \in (0,1)$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are the sequences in $(0,1)$ which satisfies $\alpha_n + \beta_n \le 1$ and $\{t_n\}_{n=1}^{\infty}$ is a positive real divergent sequence satisfy the following restrictions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}(|t_{n+1} - t_n|/t_{n+1}) = 0$.

Then the sequence $\{x_n\}$ defined by (3.41) *converges strongly to x^*, which x^* solves the variational inequality*

$$\langle u - x^*, J(z - x^*)\rangle \le 0, \quad \forall z \in \Omega, \tag{3.42}$$

and (x^*, y^*, z^*) is a solution of general system of variational inequality problem (1.38)*, where $y^* = J_{(M_2,\rho_2)}(z^* - \rho_2 B_2 z^*)$ and $z^* = J_{(M_3,\rho_3)}(x^* - \rho_3 B_3 x^*)$.*

4. Some Applications

As some applications of Theorem 3.1, we obtain the following results.

Lemma 4.1. *For all $(x^*, y^*, z^*) \in X \times X \times X$, where $y^* = J_{(M,\rho_2)}(z^* - \rho_2 B z^*)$ and $z^* = J_{(M,\rho_3)}(x^* - \rho_3 B x^*)$, (x^*, y^*, z^*) is a solution of the problem* (1.39) *if and only if x^* is a fixed point of the mapping Q' defined by*

$$\begin{aligned} Q'x := J_{(M,\rho_1)}\big(&J_{(M,\rho_2)}\big(J_{(M,\rho_3)}(x - \rho_3 Bx) - \rho_2 B J_{(M,\rho_3)}(x - \rho_3 Bx) \\ &-\rho_1 B J_{(M,\rho_2)} J_{(M,\rho_3)}(x - \rho_3 Bx) - \rho_2 B J_{(M,\rho_3)}(x - \rho_3 Bx)\big)\big), \quad \forall x \in X. \end{aligned} \tag{4.1}$$

Corollary 4.2. *Let X be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly continuous duality mapping and has the smoothness constant K. Let $M : X \to 2^X$ be a maximal monotone mapping and $B : X \to X$ be a L-Lipschitzian and relaxed (c,d)-cocoercive mapping with $\rho_i \in (0, (d - cL^2)/K^2L^2]$ and $\theta_\rho = \max\{\theta_{\rho_i}\}$ for all $i = 1,2,3$. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a nonexpansive semigroup from X into itself and at least there exists a $T(t)$ which is demicompact. Assume that $\Omega := F(\mathcal{S}) \cap F(Q') \ne \emptyset$, where Q' is defined as in Lemma 4.1. Let $f : X \to X$ be a contraction mapping with a coefficient $\alpha \in (0,1)$ and A be a strongly positive linear bounded self adjoint operator with a coefficient $\overline{\gamma} \in (0,1)$ such that $\|A\| \le 1$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned} &x_1 \in X \text{ chosen arbitrarily}, \\ &z_n = J_{(M,\rho_3)}(x_n - \rho_3 B x_n), \\ &y_n = J_{(M,\rho_2)}(z_n - \rho_2 B z_n), \\ &v_n = J_{(M,\rho_1)}(y_n - \rho_1 B y_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + &((1-\beta_n)I - \alpha_n A)\left[\mu \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + (1-\mu)v_n\right], \quad \forall n \ge 1, \end{aligned} \tag{4.2}$$

where $\mu \in (0,1)$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *are the sequences in* $(0,1)$ *which satisfies* $\alpha_n + \beta_n \le 1$ *and* $\{t_n\}_{n=1}^{\infty}$ *is a positive real divergent sequence satisfy the following restrictions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}(|t_{n+1} - t_n|/t_{n+1}) = 0$.

Then the sequence $\{x_n\}$ *defined by* (4.2) *converges strongly to* x^**, which* x^* *solves the variational inequality*

$$\left\langle (\gamma f - A)x^*, J(z - x^*)\right\rangle \le 0, \quad \forall z \in \Omega, \tag{4.3}$$

and (x^*, y^*, z^*) *is a solution of system of variational inequality problem* (1.39)*, where* $y^* = J_{(M,\rho_2)}(z^* - \rho_2 B z^*)$ *and* $z^* = J_{(M,\rho_3)}(x^* - \rho_3 B x^*)$.

Lemma 4.3 (see [8]). *For all* $(x^*, y^*) \in X \times X$*, where* $y^* = J_{(M_2,\rho_2)}(x^* - \rho_2 B_2 x^*)$*,* (x^*, y^*) *is a solution of the problem* (1.15) *if and only if* x^* *is a fixed point of the mapping* Q' *defined by*

$$Q'x := J_{(M_1,\rho_1)}\left[J_{(M_2,\rho_2)}(x - \rho_2 B_2 x) - \rho_1 B_1 J_{(M_2,\rho_2)}(x - \rho_2 B_2 x)\right]. \tag{4.4}$$

Corollary 4.4. *Let* X *be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly continuous duality mapping and has the smoothness constant* K*. Let* $M_i : X \to 2^X$ *be a maximal monotone mapping and* $B_i : X \to X$ *be a* L_i*-Lipschitzian and relaxed* (c_i, d_i)*-cocoercive mapping with* $\rho_i \in (0, (d_i - c_i L_i^2)/K^2 L_i^2]$ *and* $\theta_\rho = \max\{\theta_{\rho_i}\}$ *for all* $i = 1, 2$*. Let* $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ *be a nonexpansive semigroup from* X *into itself and at least there exists a* $T(t)$ *which is demicompact. Assume that* $\Omega := F(\mathcal{S}) \cap F(Q') \neq \emptyset$*, where* Q' *is defined as in Lemma 4.3. Let* $f : X \to X$ *be a contraction mapping with a coefficient* $\alpha \in (0,1)$ *and* A *be a strongly positive linear bounded self adjoint operator with a coefficient* $\overline{\gamma} \in (0,1)$ *such that* $\|A\| \le 1$ *and* $0 < \gamma < \overline{\gamma}/\alpha$*. Let* $\{x_n\}$ *be a sequence defined by*

$$\begin{gathered} x_1 \in X \text{ chosen arbitrarily}, \\ y_n = J_{(M,\rho_2)}(x_n - \rho_2 B_2 x_n), \\ v_n = J_{(M,\rho_1)}(y_n - \rho_1 B_1 y_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\left[\mu \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + (1-\mu)v_n\right], \quad \forall n \ge 1, \end{gathered} \tag{4.5}$$

where $\mu \in (0,1)$, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *are the sequences in* $(0,1)$ *which satisfies* $\alpha_n + \beta_n \le 1$ *and* $\{t_n\}_{n=1}^{\infty}$ *is a positive real divergent sequence satisfy the following restrictions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}(|t_{n+1} - t_n|/t_{n+1}) = 0$.

Then the sequence $\{x_n\}$ *defined by* (4.5) *converges strongly to* x^*, *which* x^* *solves the variational inequality*

$$\langle (\gamma f - A)x^*, J(z - x^*)\rangle \leq 0, \quad \forall z \in \Omega, \tag{4.6}$$

and (x^*, y^*) *is a solution of system of variational inequality problem* (1.15), *where* $y^* = J_{(M_2,\rho_2)}(x^* - \rho_2 B_2 x^*)$.

If we set $B_1 = B_2 = B$ and $M_1 = M_2 = M$, we obtain the following result.

Lemma 4.5 (see [8]). *For all* $(x^*, y^*) \in X \times X$, *where* $y^* = J_{(M,\rho_2)}(x^* - \rho_2 Bx^*)$, (x^*, y^*) *is a solution of the problem* (1.16) *if and only if* x^* *is a fixed point of the mapping* Q'' *defined by*

$$Q''x := J_{(M,\rho_1)}\left[J_{(M,\rho_2)}(x - \rho_2 Bx) - \rho_1 B J_{(M,\rho_2)}(x - \rho_2 Bx)\right]. \tag{4.7}$$

Corollary 4.6. *Let* X *be a uniformly convex and 2-uniformly smooth Banach space which admits a weakly continuous duality mapping and has the smoothness constant* K. *Let* $M : X \to 2^X$ *be a maximal monotone mapping and* $B : X \to X$ *be a* L*-Lipschitzian and relaxed* (c, d)*-cocoercive mapping with* $\rho_i \in (0, (d - cL^2)/K^2L^2]$ *and* $\theta_\rho = \max\{\theta_{\rho_i}\}$ *for all* $i = 1, 2$. *Let* $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ *be a nonexpansive semigroup from* X *into itself and at least there exists a* $T(t)$ *which is demicompact. Assume that* $\Omega := F(\mathcal{S}) \cap F(Q') \neq \emptyset$, *where* Q'' *is defined as in Lemma 4.5. Let* $f : X \to X$ *be a contraction mapping with a coefficient* $\alpha \in (0, 1)$ *and* A *be a strongly positive linear bounded self adjoint operator with a coefficient* $\overline{\gamma} \in (0, 1)$ *such that* $\|A\| \leq 1$ *and* $0 < \gamma < \overline{\gamma}/\alpha$. *Let* $\{x_n\}$ *be a sequence defined by*

$$\begin{aligned}
&x_1 \in X \text{ chosen arbitrarily},\\
&y_n = J_{(M,\rho_2)}(x_n - \rho_2 Bx_n),\\
&v_n = J_{(M,\rho_1)}(y_n - \rho_1 By_n),\\
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)&\left[\mu \frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + (1 - \mu)v_n\right], \quad \forall n \geq 1,
\end{aligned} \tag{4.8}$$

where $\mu \in (0, 1)$, $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ *are the sequences in* $(0, 1)$ *which satisfies* $\alpha_n + \beta_n \leq 1$ *and* $\{t_n\}_{n=1}^\infty$ *is a positive real divergent sequence satisfy the following restrictions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(C3) $\lim_{n\to\infty}(|t_{n+1} - t_n|/t_{n+1}) = 0$.

Then the sequence $\{x_n\}$ *defined by* (4.8) *converges strongly to* x^*, *which* x^* *solves the variational inequality*

$$\langle (\gamma f - A)x^*, J(z - x^*)\rangle \leq 0, \quad \forall z \in \Omega, \tag{4.9}$$

and (x^*, y^*) *is a solution of system of variational inequality problem* (1.16), *where* $y^* = J_{(M,\rho_2)}(x^* - \rho_2 Bx^*)$.

Acknowledgments

The authors wish to express gratitude to the referees for their careful reading of the manuscript and helpful suggestions. This research was partially supported by the Centre of Excellence in Mathematics under the Commission on Higher Education, Thailand, and P. Sunthrayuth was partially supported by the Centre of Excellence in Mathematics for Ph.D. Program at KMUTT. Moreover, we also would like to thank the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission for financial support (under the CSEC project no. 54000267).

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