Infinite Dimensional Quantum Information Geometry

Matheus R. Grasselli[1]

King's College London
Strand,
London, WC2R 2LS

Abstract. We present the construction of an infinite dimensional Banach manifold of quantum mechanical states on a Hilbert space $\mathcal{H}$ using different types of small perturbations of a given Hamiltonian H_0. We provide the manifold with a flat connection, called the exponential connection, and comment on the possibility of introducing the dual mixture connection

INTRODUCTION

In finite dimensional quantum information geometry, the set upon which the geometric structures are defined is simply the set of all (invertible) density matrices on a finite dimensional Hilbert space [3]. Already in the definition of the underlying set in infinite dimensions, we need to be slightly more careful and take a more restrictive set than just that of all (invertible) density operators. The reason for this is that, if we modify a given state in the set with a small perturbation, we want the perturbed state to have the same properties as the original one.

Let $\mathcal{C}_p, 0 < p < 1$, denote the set of compact operators $A : \mathcal{H} \mapsto \mathcal{H}$ such that $|A|^p \in \mathcal{C}_1$, where $\mathcal{C}_1$ is the set of trace-class operators on $\mathcal{H}$. Define $\mathcal{C}_{<1} := \bigcup_{0<p<1} \mathcal{C}_p$. We take the underlying set of the quantum information manifold to be $\mathcal{M} = \mathcal{C}_{<1} \cap \Sigma$ where $\Sigma \subseteq \mathcal{C}_1$ denotes the set of density operators. This guarantees that, if $\rho_0 \in \mathcal{M}$, there exists a $\beta_0 < 1$ such that $\rho_0^{\beta_0}$ is a density as well as ρ_0 itself. This set has an affine structure induced from the linear structure of each $\mathcal{C}_p$ in the following way: let $\rho_1 \in \mathcal{C}_{p_1} \cap \Sigma$ and $\rho_2 \in \mathcal{C}_{p_2} \cap \Sigma$; take $p = \max\{p_1, p_2\}$, then $\rho_1, \rho_2 \in \mathcal{C}_p \cap \Sigma$, since $p \leq q$ implies $\mathcal{C}_p \subseteq \mathcal{C}_q$ [4]; define "$\lambda\rho_1 + (1-\lambda)\rho_2, 0 \leq \lambda \leq 1$" as the usual sum of operators in $\mathcal{C}_p$. This is called the (-1)-affine structure. However, this is not the affine structure we use to define a flat connection on the manifold. Instead, we equip $\mathcal{M}$ with an exponential affine structure and then define the natural connection associated with it.

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To each $\rho_0 \in \mathcal{C}_{\beta_0} \cap \Sigma$, $\beta_0 < 1$, let $H_0 = -\log \rho_0 + cI \geq I$ be a self-adjoint operator with domain $\mathcal{D}(H_0)$ such that $\rho_0 = Z_0^{-1} e^{-H_0} = e^{-(H_0+\Psi_0)}$.

The idea is to perturb the Hamiltonian H_0, obtaining a new Hamiltonian H_X and then construct a neighbourhood of the point ρ_0 consisting of the perturbed states ρ_X. The perturbations considered are of three different types, and that is the content of the next section.

PERTURBATIONS

The most general class of perturbations we use are the form bounded perturbations. Given a positive self-adjoint operator H with associated form q_H and form domain $Q(H)$, we say that a symmetric quadratic form X (or the symmetric sesquiform obtained from it by polarization) is q_H-*bounded* if

i. $Q(H) \subset Q(X)$ and

ii. there exist positive constants a and b such that $|X(\psi, \psi)| \leq a q_H(\psi, \psi) + b(\psi, \psi)$, for all $\psi \in Q(H)$.

Although form bounded perturbations are of much interest in the study of Schrödinger operators for a variety of quantum systems, they provide very little regularity for the quantum information manifold. In [7], Streater was able to show that the manifold constructed using then has Lipschitz structure. However, more regularity is needed if we want to define a metric on it by, say, the second derivative of the free energy. This led to the idea of looking at the more restrictive case of operator bounded perturbations. Given operators H and X defined on dense domains $\mathcal{D}(H)$ and $\mathcal{D}(X)$ in a Hilbert space $\mathcal{H}$, we say that X is H-*bounded* if

i. $\mathcal{D}(H) \subset \mathcal{D}(X)$ and

ii. there exist positive constants a and b such that $\|X\psi\| \leq a\|H\psi\| + b\|\psi\|$, for all $\psi \in \mathcal{D}(H)$.

For both form bounded and operator bounded perturbations, the infimum of such a is called the relative bound of X (with respect to H or with respect to q_H, accordingly). If $a < 1$, the perturbation is said to be small.

Operator bounded perturbations are also used in the study of Schrödinger operators for quite a broad range of quantum systems [6], with the additional property of providing enough regularity for the manifold $\mathcal{M}$ to have an analytic free energy, for instance.

The following lemma tells us how to characterise operator bounded and form bounded perturbations in terms of certain norms. Before we state it, we need to say what we mean by a form multiplied from both sides by operators. Suppose that X is a quadratic form with domain $Q(X)$ and A, B are operators on $\mathcal{H}$ such that A^* and B are densely defined. Suppose further that $A^* : \mathcal{D}(A^*) \to Q(X)$ and $B : \mathcal{D}(B) \to Q(X)$. Then the expression AXB means the form defined by

$$\phi, \psi \mapsto X(A^*\phi, B\psi), \qquad \phi \in \mathcal{D}(A^*), \quad \psi \in \mathcal{D}(B).$$

Consider now the case where $H_0 \geq I$ is a self-adjoint operator with domain $\mathcal{D}(H_0)$, quadratic form q_0 and form domain $Q_0 = \mathcal{D}(H_0^{1/2})$, and let $R_0 = H_0^{-1}$ be its resolvent at the origin.

Lemma 1 *A symmetric operator $X : \mathcal{D}(H_0) \to \mathcal{H}$ is H_0-bounded if and only if $\|XR_0\| < \infty$. Analogously, a symmetric quadratic form X defined on Q_0 is q_0-bounded if and only if $R_0^{1/2}XR_0^{1/2}$ is a bounded symmetric form defined everywhere. Moreover, if $\left\|R_0^{1/2}XR_0^{1/2}\right\| < \infty$ then the relative bound a of X with respect to q_0 satisfies $a \leq \left\|R_0^{1/2}XR_0^{1/2}\right\|$.*

The set $\mathcal{T}_\omega(0)$ of all H_0-bounded symmetric operators X is a Banach space with norm $\|X\|_\omega(0) := \|XR_0\|$, since the map $A \mapsto AH_0$ from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{T}_\omega(0)$ is an isometry. The set $\mathcal{T}_0(0)$ of all q_0-bounded symmetric forms X is also a Banach space with norm $\|X\|_0(0) := \left\|R_0^{1/2}XR_0^{1/2}\right\|$, since the map $A \mapsto H_0^{1/2}AH_0^{1/2}$ from the set of all bounded self-adjoint operators on $\mathcal{H}$ onto $\mathcal{T}_0(0)$ is again an isometry.

Motivated by Banach space interpolation theory, let us consider, for $\varepsilon \in (0, 1/2)$, the set $\mathcal{T}_\varepsilon(0)$ of all symmetric forms X with $\mathcal{D}(H_0^{\frac{1}{2}-\varepsilon}) \subset Q(X)$ and such that $\|X\|_\varepsilon(0) := \left\|R_0^{\frac{1}{2}+\varepsilon}XR_0^{\frac{1}{2}-\varepsilon}\right\|$ is finite. Then the map $A \mapsto H_0^{\frac{1}{2}-\varepsilon}AH_0^{\frac{1}{2}+\varepsilon}$ is an isometry from the set of all bounded self-adjoint operators on $\mathcal{H}$ onto $\mathcal{T}_\varepsilon(0)$. Hence $\mathcal{T}_\varepsilon(0)$ is a Banach space with the ε-norm $\|\cdot\|_\varepsilon(0)$. Such an X will be called a ε-bounded perturbation and it is shown to interpolate between the extreme cases of form bounded perturbations, for which $\varepsilon = 0$, and operator bounded perturbations for which $\varepsilon = 1/2$.

Lemma 2 *For fixed symmetric X, $\|X\|_\varepsilon$ is a monotonically increasing function of $\varepsilon \in [0, 1/2]$.*

In the next section, we carry out the programme of using these perturbations to obtain the hoods in the manifold. In what follows, we write $\mathcal{T}_{(\cdot)}(0)$ to indicate that we can use any of the three Banach spaces $\mathcal{T}_\omega(0)$, $\mathcal{T}_0(0)$ or $\mathcal{T}_\varepsilon(0)$.

CONSTRUCTION OF THE MANIFOLD

The two technical tools used in the construction of our manifold are the following.

Theorem 3 (KLMN) *Let H_0 be a positive self-adjoint operator with quadratic form q_0 and form domain Q_0; let X be a q_0-small symmetric quadratic form. Then there exists a unique self-adjoint operator H_X with form domain Q_0 such that*

$$\langle H_X^{1/2}\phi, H_X^{1/2}\psi\rangle = q_0(\phi, \psi) + X(\phi, \psi), \qquad \phi, \psi \in Q_0.$$

Moreover, H_X is bounded below by $-b$.

Lemma 4 (Streater 98) *Let X be a q_0-small with bound $a < 1 - \beta_0$. Denote by H_X the unique operator given by the KLMN theorem. Then $\exp(-\beta H_X)$ is of trace class for all $\beta > \beta_X = \beta_0/(1-a)$.*

The construction of the neighbourhood of ρ_0 goes as follows. In $\mathcal{T}_{(\cdot)}(0)$, take X such that $\|X\|_{(\cdot}(0) < 1 - \beta_0$. Since $\|X\|_0(0) \leq \|X\|_{(\cdot)}(0) < 1 - \beta_0$, X is also q_0-bounded with bound a_0 less than $1 - \beta_0$. The *KLMN* theorem then tells us that there exists a unique semi-bounded self-adjoint operator H_X with form $q_X = q_0 + X$ and form domain $Q_X = Q_0$. We write $H_X = H_0 + X$ for this operator and consider the state $\rho_X = Z_X^{-1} e^{-(H_0+X)} = Z_X^{-1} e^{-(H_0+X+\Psi_X)}$

Then, from lemma 4, $\rho_X \in \mathcal{C}_{\beta_X} \cap \Sigma$, where $\beta_X = \frac{\beta_0}{1-a_0} < 1$. If we add to H_X a multiple of the identity, we can still have the same state ρ_X, by simply adjusting the partition function Z_X; so we can always assume that, for the perturbed state, we have $H_X \geq I$. We take as a hood $\mathcal{M}_0$ of ρ_0 the set of all such states, that is, $\mathcal{M}_0 = \{\rho_X : \|X\|_{(\cdot)}(0) < 1 - \beta_0\}$.

To give a topology to $\mathcal{M}_0$, we first introduce in $\mathcal{T}_{(\cdot)}(0)$ the equivalence relation $X \sim Y$ iff $X - Y = \alpha I$ for some $\alpha \in \mathbf{R}$, precisely because $\rho_X = \rho_{X+\alpha I}$, as remarked above. We then identify ρ_X in $\mathcal{M}_0$ with the line $\{Y \in \mathcal{T}_{(\cdot)}(0) : Y = X + \alpha I, \alpha \in \mathbf{R}\}$ in $\mathcal{T}_{(\cdot)}(0)/\sim$. This is a bijection from $\mathcal{M}_0$ onto the subset of $\mathcal{T}_{(\cdot)}(0)/\sim$ defined by $\left\{\{X + \alpha I\}_{\alpha \in \mathbf{R}} : \|X\|_{(\cdot)}(0) < 1 - \beta_0\right\}$. The topology in $\mathcal{M}_0$ is then given by transfer of structure. Now that $\mathcal{M}_0$ is a (Hausdorff) topological space, we want to parametrise it by an open set in a Banach space. As in the classical case [5], we choose the Banach subspace of centred variables in $\mathcal{T}_{(\cdot)}(0)$; in our terms, perturbations with zero mean (the 'scores'). The only problem is that, when X is not an operator, it is not immediately clear what its mean in the state ρ_0 should be, let alone the question of whether or not it is finite. To deal with this, define the regularised mean of $X \in \mathcal{T}_{(\cdot)}(0)$ in the state ρ_0 as $\rho_0 \cdot X := \mathrm{Tr}(\rho_0^\lambda X \rho_0^{1-\lambda})$, for $0 < \lambda < 1$.

Since $\rho_0 \in \mathcal{C}_{\beta_0} \cap \Sigma$ and X is q_0-bounded, lemma 5 of [7] ensures that $\rho_0 \cdot X$ is finite and independent of λ. It was a shown there that $\rho_0 \cdot X$ is a continuous map from $\mathcal{T}_0(0)$ to $\mathbf{R}$, because its bound contained a factor $\|X\|_0(0)$. Exactly the same proof shows that $\rho_0 \cdot X$ is a continuous map from $\mathcal{T}_{(\cdot)}(0)$ to $\mathbf{R}$. Thus the set $\widehat{\mathcal{T}}_{(\cdot)}(0) := \left\{X \in \mathcal{T}_{(\cdot)}(0) : \rho_0 \cdot X = 0\right\}$ is a closed subspace of $\mathcal{T}_{(\cdot)}(0)$ and so is a Banach space with the norm $\|\cdot\|_\varepsilon$ restricted to it. We notice that for the case of operator bounded perturbations, the regularised mean of X coincides with the usual mean $\mathrm{Tr}(\rho_0 X)$.

To each $\rho_X \in \mathcal{M}_0$, consider the point in the line $\{X + \alpha I\}_{\alpha \in \mathbf{R}}$ with $\alpha = -\rho_0 \cdot X$. Write $\widehat{X} = X - \rho_0 \cdot X$ for this point. The map $\rho_X \mapsto \widehat{X}$ is a homeomorphism between $\mathcal{M}_0$ and the open subset of $\widehat{\mathcal{T}}_{(\cdot)}(0)$ defined by $\left\{\widehat{X} : \widehat{X} = X - \rho_0 \cdot X, \|X\|_{(\cdot)} < 1 - \beta_0\right\}$. The map $\rho_X \mapsto \widehat{X}$ is then a global chart for the Banach manifold $\mathcal{M}_0$ modeled by $\widehat{\mathcal{T}}_{(\cdot)}(0)$. The tangent space at ρ_0 is given by $\widehat{\mathcal{T}}_{(\cdot)}(0)$, with the curve $\left\{\rho(\lambda) = Z_{\lambda X}^{-1} e^{-(H_0+\lambda X)}, \lambda \in [-\delta, \delta]\right\}$ having tangent vector $\widehat{X} = X - \rho_0 \cdot X$.

We extend our manifold by adding new patches compatible with $\mathcal{M}_0$. The idea is to construct a chart around each perturbed state ρ_X as we did around ρ_0. Let $\rho_X \in \mathcal{M}_0$ with Hamiltonian $H_X \geq I$ and consider the Banach space $\mathcal{T}_{(\cdot)}(X)$ of all symmetric forms Y such that the norm $\|Y\|_{(\cdot)}(X)$ is finite, where the expression for this norm is the same as $\|Y\|_{(\cdot)}(0)$ but with all the Hamiltonians replaced by H_X. Then we repeat exactly the same process, namely, take sufficiently small Y (with $\|Y\|_{(\cdot)}(X) < 1-\beta_X$), obtain from the *KLMN* theorem the Hamiltonian $H_{X=Y}$ with form $q_{X+Y} = q_X + Y = q_0 + X + Y$ and form domain $Q_{X+Y} = Q_X = Q_0$ and take the hood of X to be the set $\mathcal{M}_X$ of all states of the form $\rho_{X+Y} = Z_{X+Y}^{-1} e^{-H_{X+Y}} = Z_{X+Y}^{-1} e^{-(H_0+X+Y)}$. The topology and coordinates for $\mathcal{M}_X$ are then introduced in a completely similar fashion.

We then turn to the union of $\mathcal{M}_0$ and $\mathcal{M}_X$. We need to show that our two previous charts are compatible in the overlapping region $\mathcal{M}_0 \cap \mathcal{M}_X$. For the case of form bounded and operator bounded perturbations, the equivalence of the norms is achieved by a straightforward series of operator identities [7,8]. For the case of ε-bounded perturbations, the argument is more subtle and involves careful consideration of the domains of the operators. Nevertheless, the result still follows [2].

Theorem 5 *$\|\cdot\|_\varepsilon(X)$ and $\|\cdot\|_\varepsilon(0)$ are equivalent norms.*

We can repeat the construction again, starting from any point in $\mathcal{M}_0 \bigcup \mathcal{M}_X$. We then obtain the following definition.

Definition 6 *The information manifold $\mathcal{M}(H_0)$ defined by H_0 consists of all states obtainable in a finite numbers of steps, by extending $\mathcal{M}_0$ as explained above.*

AFFINE GEOMETRY IN $\mathcal{M}(H_0)$

The set $A = \left\{\widehat{X} \in \widehat{\mathcal{T}}_{(\cdot)}(0) : \widehat{X} = X - \rho_0 \cdot X, \|X\|_{(\cdot)}(0) < 1 - \beta_0\right\}$ is a convex subset of the Banach space $\widehat{\mathcal{T}}_{(\cdot)}(0)$ and so has an affine structure coming from its linear structure. We provide $\mathcal{M}_0$ with an affine structure induced from A using the patch $\widehat{X} \mapsto \rho_X$ and call this the canonical or (+1)-affine structure. The (+1)-convex mixture of ρ_X and ρ_Y in $\mathcal{M}_0$ is then $\rho_{\lambda X+(1-\lambda)Y}$, $(0 \leq \lambda \leq 1)$, which differs from the previously defined (-1)-convex mixture $\lambda\rho_X + (1-\lambda)\rho_Y$.

Given two points ρ_X and ρ_Y in $\mathcal{M}_0$ and their tangent spaces $\widehat{\mathcal{T}}_\varepsilon(X)$ and $\widehat{\mathcal{T}}_\varepsilon(Y)$, we define the (+1)-parallel transport U_L of $(Z - \rho_X \cdot Z) \in \widehat{\mathcal{T}}_\varepsilon(X)$ along any continuous path L connecting ρ_X and ρ_Y in the manifold to be the point $(Z - \rho_Y \cdot Z) \in \widehat{\mathcal{T}}_\varepsilon(Y)$. Clearly U_L is independent of L by construction, thus the (+1)-affine connection is flat. We see that the (+1)-parallel transport just moves the representative point in the line $\{Z + \alpha I\}_{\alpha \in \mathbf{R}}$ from one hyperplane to another.

DISCUSSION

The manifold $\mathcal{M}(H_0)$ constructed here does not cover the whole set $\mathcal{M}$ at once. It could not possibly do so, since our small perturbations do not change the domain of the original Hamiltonian H_0, and certainly $\mathcal{M}$ contains states defined by Hamiltonians with plenty of different domains. Also, although we can reach far removed points with a finite number of small perturbations, we can not move in arbitrary directions. For instance, we cannot reach $X = -H_0$ as the result of our perturbations, since the identity is not an operator of trace class. To cover $\mathcal{M}$ entirely, we have to start at several different points. The whole manifold thus obtained consists of several disconnected parts, pointing towards positive directions with respect to the given Hamiltonians.

Finally, it is clear that $\lambda\rho_X + (1-\lambda)\rho_Y$, for, say, $\rho_X, \rho_Y \in \mathcal{M}_0$, defines a new state in the underlying set $\mathcal{M}$. Nonetheless, we were not yet able to prove that it belongs to the neighbourhood of the original states. This is the main obstacle to define a mixture connection in our manifold and thence develop a quantum version of Amari's duality theory [1]. One possibility is to change the definition of the neighbourhoods altogether and use, for instance, the condition that states in the same neighbourhood all have finite relative entropy with respect to each other, besides having finite von Neumann entropy.

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