General Solution of the Rayleigh Equation for the Description of Bubble Oscillations Near a Wall

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Abstract. We consider a generalization of the Rayleigh equation for the description of the dynamics of a spherical gas bubble oscillating near an elastic or rigid wall. We show that in the non–dissipative case, i.e. neglecting the liquid viscosity and compressibility, it is possible to construct the general analytical solution of this equation. The corresponding general solution is expressed via the Weierstrass elliptic function. We analyze the dependence of this solution properties on the physical parameters.

1 Introduction

Understanding the dynamics of the spherical gas bubbles in a liquid is of high importance for various applications such as biomedical and industrial ones [1]. There are several models for the description of gas bubbles oscillations, which mostly generalize the Rayleigh equation (see, e.g. [1, 2]). Although, usually, these models are studied numerically, it has been recently shown [3–5] that it is possible to study such models analytically. For example, in [3, 4] the general solutions of the Rayleigh equation and its N-dimensional generalization for both an empty and a gas–filled bubble were found. In this work we consider another generalization of the Rayleigh equation, which describes oscillations of a spherical gas bubble near an elastic or rigid wall [6, 7]. This model has not been studied analytically yet and, therefore, the aim of this work is to find its general solution and investigate the influence of the physical parameters on the properties of this solution.

2 Main results

We consider the following equation

$$R_{tt} + \frac{3}{2} \cdot \frac{1+4AR}{1+3AR} \cdot \frac{R_t^2}{R} - \frac{\widetilde{P_0} R_0^{3\gamma} R^{-3\gamma-1}}{\rho_1(1+3AR)} + \frac{2\sigma}{\rho_1 R^2 (1+3AR)} + \frac{P}{\rho_1 R(1+3AR)} = 0, \quad (1)$$

which describes the motion of a spherical gas bubble near an elastic or rigid wall [6, 7], if one neglects the liquid viscosity and compressibility. Here t is the time, R is the radius of the bubble, γ is the polytropic exponent, ρ_1 is the liquid density, σ is the surface tension, $\widetilde{P_0} = P_0 + 2\sigma/R_0$, P_0 is the

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ambient pressure of the gas inside the bubble, R_0 is the ambient radius of the bubble and P is the far–field pressure, which we assume to be constant. The parameter A is $1/(6d)$ for the rigid wall or

$$A = \frac{(\rho_1 - \beta)(\beta - \rho_3)}{6h(\rho_1 + \beta)(\rho_3 + \beta)} - \frac{\rho_1 - \beta}{6d(\rho_1 + \beta)} - \frac{\beta - \rho_3}{(\beta + \rho_3)6(d + h)} \tag{2}$$

for the elastic wall. Here ρ_2 is the wall density, ρ_3 is the density of the material surrounding the wall, $\beta = \rho_2\nu/(1-\nu)$ is a characteristic of the wall, ν is the Poisson ratio, h is the thickness of the wall given in mm and d denotes the distance between the wall and the center of the bubble.

For finding the general solution of (1) we use the approach that was proposed in [3, 4]. First, we introduce the dimensionless variables $u = R/R_0$, $\tau = t\omega_0$, where

$$\omega_0^2 = \frac{3\gamma\tilde{P}_0 - 2\sigma/R_0}{\rho_1 R_0^2(1 + 3AR_0)}\,.$$

Then, equation (1) can be rewritten as

$$u_{\tau\tau} + \frac{3}{2}\cdot\frac{1 + 4AR_0u}{1 + 3AR_0u}\cdot\frac{u_\tau^2}{u} - \frac{\alpha u^{-3\gamma-1}}{1 + 3AR_0u} + \frac{\delta}{u^2(1 + 3AR_0u)} + \frac{\epsilon}{u(1 + 3AR_0u)} = 0\,, \tag{3}$$

where

$$\alpha = \frac{\widetilde{P_0}}{\rho_1 R_0^2\omega_0^2}, \qquad \delta = \frac{2\sigma}{\rho_1 R_0^3\omega_0^2}, \qquad \epsilon = \frac{P}{\rho_1 R_0^2\omega_0^2}\,.$$

Note that equation (3) admits the following first integral for $\gamma \neq 1$

$$u^3(1 + 3AR_0u)u_\tau^2 + \frac{2}{3}\epsilon u^3 + \delta u^2 + \frac{2\alpha}{3(\gamma - 1)}u^{3(1-\gamma)} = C_1, \tag{4}$$

where C_1 is an integration constant. This constant can be considered as the total energy of the bubble and, therefore, is positive. Below we consider the case $\gamma = 4/3$, which corresponds to the behaviour of diatomic gases between isothermal and adiabatic [3]. Note that the case of $\gamma = 1$, which correspond to isothermal behaviour, is not considered due to appearance of a logarithmic term in the first integral of (3).

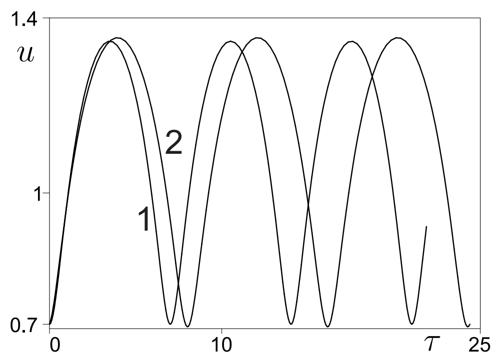

Figure 1. Solution (10) for the case of the elastic (curve 1) and rigid (curve 2) walls corresponding to the initial conditions $u(0) = 0.7$, $u_t(0) = 0$

Then, at $\gamma = 4/3$, with the help of the transformations

$$v = \frac{1}{u}\,, \qquad d\zeta = \frac{1}{u^2\sqrt{1 + 3AR_0u}}\,d\tau \tag{5}$$

from (3) we get

$$v_{\zeta\zeta} - \frac{3}{2}\frac{v_\zeta^2}{v} + \alpha v^3 - \delta - \frac{\epsilon}{v} = 0. \tag{6}$$

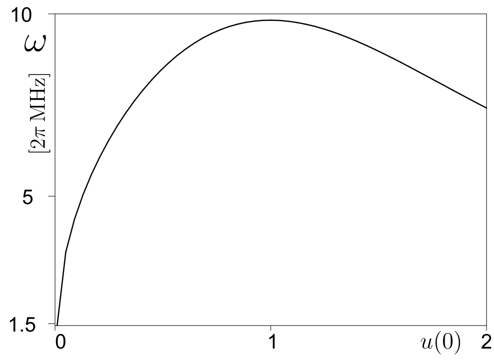

Figure 2. Dependence of the angular frequency of solution (10) on the initial strain

The general solution of (6) is obtained in terms of the Weierstrass elliptic function

$$v = \frac{8\mu^2\wp\{\zeta - \zeta_0, g_2, g_3\} + 8C_1\mu - \delta\mu^3 - 32\alpha}{4\mu^3\wp\{\zeta - \zeta_0, g_2, g_3\} - 2C_1\mu^2 + 16\alpha\mu}\,, \tag{7}$$

where ζ_0 is an arbitrary constant,

$$\begin{aligned} g_2 &= \frac{\delta\,\mu^4 C_1 + 64\,C_1\,\mu\,\alpha - 16\,\alpha\,\delta\,\mu^3 - 256\,\alpha^2}{4\mu^4} = \frac{\delta C_1}{4} + \frac{4\alpha\epsilon}{3}, \\ g_3 &= \frac{\left(4\,C_1 - \delta\,\mu^2\right)\left(C_1^2\mu - 4\alpha\,C_1 - \alpha\,\delta\,\mu^2\right)}{8\mu^4} = \frac{\epsilon C_1^2}{24} + \frac{\delta^2\alpha}{8}, \end{aligned} \tag{8}$$

and the parameter μ is a real solution of the following equation

$$\epsilon\mu^4 - 3(4C_1\mu - 16\alpha - \delta\mu^3) = 0. \tag{9}$$

Finally, inverting transformations (5) we can find the general solution of (3) as follows

$$u = \frac{4\mu^3\wp\{\zeta - \zeta_0, g_2, g_3\} - 2C_1\mu^2 + 16\alpha\mu}{8\mu^2\wp\{\zeta - \zeta_0, g_2, g_3\} + 8C_1\mu - \delta\mu^3 - 32\alpha}, \qquad \tau = \int\limits_0^\zeta u^2\sqrt{1 + 3AR_0u}\,d\xi. \tag{10}$$

Formula (10) gives us parametric representation of the general solution of (3), where ζ can be considered as a parameter.

It can be shown that, for the physically relevant values of the parameters, solution (10) is periodic with a real period. One can investigate the dependence of the period or angular frequency of this solution on the parameters. For all analytical calculations we use the same values of the physical parameters as in [6]. For the results presented in Figs. 1 and 2 we also assume that $d = R_0$, i.e. the bubble touches the wall. Notice that the typical range of bubbles oscillations frequency is $\nu = \omega/2\pi = 10^6 \div 10^7$ Hz.

Plots of solution (10) for both rigid and elastic wall cases are shown in Fig. 1. One can see that both the period and magnitude of the oscillations in the case of an elastic wall are slightly smaller

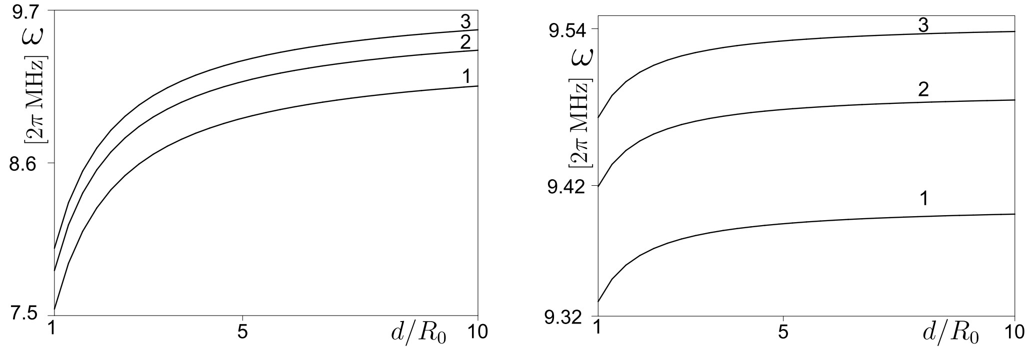

Figure 3. Dependence of angular frequency on the distance from the elastic wall for different values of the initial strain for the rigid wall curve 1: u(0)=0.7; curve 2: u(0)=0.8; curve 3: u(0)=0.9 (left figure) and for the elastic wall curve 1: u(0)=0.7; curve 2: u(0)=0.73; curve 3: u(0)=0.75 (right figure).

than those in the case of a rigid wall. In Fig. 2 the dependence of the dimensional angular frequency of solution (10) is plotted in terms of the initial strain, i.e. on the ratio $R(0)/R_0 = u(0)$. This figure shows us that the bubble oscillations are essentially nonlinear, since their frequency strongly depends on the initial values. In Fig. 3 the dependence of the angular frequency of the bubble oscillations is plotted in term of the distance from the bubble to the elastic or the rigid wall for different values of the initial strain. One can see that in the both cases the frequency of oscillations decreases when the bubble comes closer to the wall and also decreases with the initial strain. However, in the case of the elastic wall the oscillation frequency is smaller than that in the case of the rigid wall.

3 Conclusion

In this work we have considered a model for the description of the dynamics of a spherical gas bubble near an elastic or a rigid wall. We have shown that it is possible to construct the general analytical solution of this model in the non–dissipative case. We have found this solution in explicit form in the adiabatic case with the help of nonlocal transformations. Some physical properties of the obtained solution have been discussed.

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