Updated Tables of Parameters of (T, M, S)-Nets

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Abstract: **We present an updated survey of the known constructions and bounds for (t, m, s)-nets as well as tables of upper and lower bounds on their parameters for various bases.** © 1999 John Wiley & Sons, Inc. J Combin Designs 7: 381–393, 1999

1. INTRODUCTION

The main purpose of this article is to update the tables and the list of constructions for (t, m, s)-nets and (t, s)-sequences given in Mullen, Mahalanabis, and Niederreiter [35]. To this end, we begin by giving definitions for the fundamental concepts of (t, m, s)-nets and (t, s)-sequences in base b as first discussed by Niederreiter in [38].

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Let $s \geq 1$ be a fixed integer. For an integer $b \geq 2$, an *elementary interval in base b* is an interval of the form

$$E = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1) b^{-d_i})$$

with integers $d_i \geq 0$ and integers $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$. For integers $0 \leq t \leq m$, a (t, m, s)*-net in base* b is a point set of b^m points in $[0, 1)^s$ such that every elementary interval E in base b of volume b^{t-m} contains exactly b^t points of the point set. For an integer $t \geq 0$, a sequence $x_1, x_2, \cdots$ of points in $[0, 1)^s$ is called a (t, s)*-sequence in base* b if for all integers $k \geq 0$ and $m > t$, the point set $\{x_n : kb^m < n \leq (k + 1)b^m\}$ is a (t, m, s)-net in base b. We refer to [38] for a systematic development of the theory and various constructions of such nets and sequences and to [38, 39] for applications of nets and sequences to various areas of numerical analysis.

There has been considerable progress both in finding constructions as well as in obtaining bounds on the parameters of (t, m, s)-nets, and so the tables presented here should be of value to researchers and to those who use nets. We also note that in recent years, considerable emphasis has been placed by a number of researchers on the subclass of (t, m, s)-nets known as *digital* (t, m, s)*-nets in base* b; see for example [11, 19, 22, 51, 53].

2. NEW (t, m, s)-NET CONSTRUCTIONS

In this section we briefly describe the new constructions for (t, m, s)-nets which have arisen since the publication of [35]. We continue with the numbering which in [35] ended with Construction 16.

Construction 17. In [31] the authors show that given a linear $[n, l, d]$ code over the finite field F_q (q a prime power) with $d \geq 3$, one can construct a digital $(n - l - d + 1, n - l, s)$ -net in base q where $s = \lfloor \frac{n-1}{h} \rfloor$ if $d = 2h + 2$; and $s = \lfloor \frac{n}{h} \rfloor$ if $d = 2h + 1$. The existing tables of codes [6] were then electronically scanned and the results are tabulated for bases $q = 2, 3$ in [31].

Construction 18. This is, in fact, a whole family of constructions due to Niederreiter and Xing that are based on algebraic curves over finite fields or, equivalently, on algebraic function fields over finite fields. These constructions yield (t, s)-sequences in base b, and nets are obtained by applying the standard principle that the existence of a (t, s)-sequence in base b implies the existence of a $(t, m, s + 1)$-net in base b for all integers $m \geq t$. We refer to Section 3 for further information on the construction of (t, s)-sequences by these methods.

Construction 19. A short article by Clayman and Mullen [8] gives a computer implementation of the Gilbert–Varshamov bound from coding theory which provides a sufficient condition for the existence of a linear code with a given set of parameters. Schmid [51, p. 33] provides an analogous sufficient condition for the existence of a digital net. Most of the improvements arising from [8] are for base 5.

Construction 20. Schmid and Wolf in [53] provide a series of bounds for both digital (t, m, s)-nets and digital (t, s)-sequences. In addition they obtain a few improvements in bases 2 and 3 to the tables in [35].

Construction 21. In [52] Schmid provides a new method for the construction of digital nets in base 2 which he calls the *shift-net* method.

Construction 22. The main focus of Lawrence [27] is a combinatorial equivalence between the existence of a (t, m, s)-net in base b and the existence of what he calls a *generalized orthogonal array* (GOA), which may be thought of as a three-dimensional orthogonal array in which only certain collections of rows are required to have the orthogonal array property. In addition to this important equivalence, he also provides a new method for constructing GOAs and hence nets which uses orthogonal arrays (see Section 4 for the definition of an orthogonal array). In [28], Lawrence exploits the links between orthogonal arrays, finite projective geometry and both linear and nonlinear error-correcting codes to construct GOAs via the aforementioned construction method from [27]. With this approach he is able to obtain a number of improvements in net parameters.

Construction 23. In [1] Adams and Shader use arrays, called M *arrays*, of elements of the finite field F_q in which certain collections of rows are linearly independent, along with linear codes to give a new construction of generalized orthogonal arrays. The authors then apply the combinatorial equivalence of Lawrence and obtain new classes of (t, m, s)-nets in base q, where q is a prime power.

Construction 24. Edel in [10] uses *ordered orthogonal arrays* (which are analogous to generalized orthogonal arrays) and linear codes over finite fields to construct several classes of nets in base q where q is any prime power. A number of specific examples are constructed in bases $q = 2, 3, 4$, with some being obtained by machine calculation.

Construction 25. In [11] Edel and Bierbrauer provide three families of digital nets in base $b = 2$ and one family in base $b = 3$. The constructions are combinatorial in nature and are obtained by using ordered orthogonal arrays and BCH codes.

Construction 26. In [45] Niederreiter and Xing provide a new propagation rule for nets. In particular, if $b \geq 2$, $h \geq 1$, $s \geq 1$, and $0 \leq u \leq m$ are integers, then every (u, m, s)-net in base b^h is a (t, hm, s)-net in base b with $t = \min(hu + (h-1)(s-1), hm)$.

3. NEW (t, s)-SEQUENCE CONSTRUCTIONS

A family of new methods for the construction of (t, s)-sequences was introduced by Niederreiter and Xing in 1995 and 1996. The basic idea of these methods, namely the use of algebraic curves over finite fields or, equivalently, of algebraic function fields over finite fields, was already sketched earlier in Niederreiter [40, 41]. Altogether, the family of these new methods consists of four different construction principles. In all cases, we first obtain digital (t, s)-sequences in a prime-power base q, and then by the usual direct-product construction in the digital method we get digital (t, s)-sequences in an arbitrary base b.

The first construction principle was introduced in Niederreiter and Xing [44]. It extends the method of Niederreiter listed as Construction 9 in [35] from the rational function field to general algebraic function fields over a finite field F_q. For certain choices of the algebraic function field this yields better sequence parameters than Construction 9. Dramatic improvements were obtained by the second construction principle described in Niederreiter and Xing [45]. Here the heart of the matter is to work with algebraic curves over F_q with many F_q-rational points or, equivalently, with global function fields with many places of degree 1. The third construction principle developed in Xing and Niederreiter [59] offers more freedom of choice since it works well for global function fields containing many places of small degree (not necessarily degree 1), and in some cases it can produce even better sequence parameters than the second construction principle. The fourth construction principle presented in Niederreiter and Xing [46] is relatively easy to apply and yields competitive values of t at least for small dimensions s. A detailed survey of these construction principles and tables of sequence parameters can be found in [47].

4. BOUNDS FOR THE EXISTENCE OF (t, m, s)-NETS

Let $b \geq 2$ and $t \geq 0$ be integers. As noted by Niederreiter in [38], for $m = t$ and $m = t + 1$ a (t, m, s)-net in base b always exists for each $s \geq 1$. Mullen and Whittle [37] showed that for $m \geq t + 2$ a (t, m, s)-net in base b can exist only if

$$s \leq \frac{b^{t+2} - 1}{b - 1}. \tag{1}$$

The values of this upper bound for s (and the corresponding lower bound for t, when m, s and b are prescribed) were tabulated in [35].

A substantial improvement to this bound was obtained by Lawrence as a consequence of the combinatorial equivalence between the existence of a (t, m, s)-net in base b and a corresponding generalized orthogonal array. Before discussing this new bound, we recall the basic notion of an orthogonal array, which is due to Rao [49] (see [15] for more details).

Let k, N, d be positive integers with $k \geq d$ and let $b \geq 2$ be an integer. A $k \times N$ matrix A with entries from a set of b elements is called an *orthogonal array* (N, k, b, d) *with* N *columns*, k *rows*, b *levels*, *strength* d *and index* λ if any $d \times N$ submatrix of A contains all possible $d \times 1$ columns with the same frequency $\lambda = N/b^d$. We define $f(N, b, d)$ to be the maximum value of k for which an orthogonal array (N, k, b, d) exists.

We note that the transposed array is sometimes used, as for example in [15]. We now state the *orthogonal array bound for nets*, due to Lawrence [27, Thm. 6.1; 29].

Theorem. *Let $s \geq 1$, $b \geq 2$, $t \geq 0$ and m be integers with $m \geq t+2$. Then a (t, m, s)-net in base b can exist only if*

$$s \leq \min_{t+2 \leq h \leq m} f(b^h, b, h - t).$$

We note that since $f(b^{t+2}, b, 2) \leq (b^{t+2} - 1)/(b - 1)$, this bound generalizes the preceding bound (1) of Mullen and Whittle and can represent a significant improvement when $m \geq t+3$, particularly when t is large. For example, it is known that $f(2^{48}, 2, 26) = 52$ and it follows that for $m \geq 48$, a $(22, m, s)$-net in base 2 can exist only if $s \leq 52$; the corresponding bound from (1) is $2^{24} - 1$.

The difficulty with calculating numerical values of this orthogonal array bound for nets is that exact values of the quantities $f(b^h, b, h - t)$ are not known in all cases. However, very good upper bounds for these quantities are available in many cases, as we now explain.

A general upper bound on k, the number of rows in an orthogonal array (N, k, b, d), was obtained by Rao [49] and expressed in the form of certain inequalities which the parameters of any orthogonal array must satisfy, see [49, 26, 27, 15]. Let $R(N, b, d)$ be the maximum number of factors or rows, k, such that the existence of an orthogonal array (N, k, b, d) is not denied by Rao's inequalities. Then $f(N, b, d) \leq R(N, b, d)$. Another upper bound on $f(N, b, d)$ can be obtained from linear programming, following the work of Delsarte [9], and produces results which are always at least as good as Rao's bound (see [15]). We write $LP(N, b, d)$ for the maximum number of rows, k, such that the existence of an orthogonal array (N, k, b, d) is not denied by the linear programming bound. Then

$$f(N, b, d) \leq LP(N, b, d) \leq R(N, b, d).$$

The linear programming bound actually gives a lower bound on the number of columns, N, in an orthogonal array (N, k, b, d) for fixed values of k, b, and d, from which corresponding upper bounds on the number of rows can be inferred. Thus for example, by solving an appropriate linear programming problem we find that in an orthogonal array $(N, 30, 2, 4)$ we must have $N > 2^9$. It follows that no orthogonal array $(2^9, 30, 2, 4)$ exists and hence that $LP(2^9, 2, 4) \leq 29$; in fact equality holds. We note that for certain parameter ranges there is also an explicit lower bound for N, the *dual Plotkin bound*, which is due to Bierbrauer [2]. In the binary case, this explicit bound is equivalent to the LP bound, see [3]. Lawrence has inverted the dual Plotkin bound and obtained explicit upper bounds for the quantities $f(N, b, d)$ for the same parameter ranges, see [29].

We computed the upper bounds for s which follow from the orthogonal array bound for nets and appear in the s tables for bases $b = 2, 3, 5$ by using the following procedure. First, we easily computed the values of $R(b^h, b, h-t)$ for all necessary values of the parameters (up to $h = 50$) for $b = 2, 3, 5$. Next, we calculated the upper bounds for $f(b^h, b, h-t)$ where possible, from the explicit bounds due to Lawrence. The linear programming bounds for certain parameter values were then obtained using Delsarte's method, which was implemented in two ways. If the number of rows did not exceed about 40, satisfactory results were obtained very easily using the AMPL mathematical programming language [14] in combination with the CPLEX optimization system [4]. However, for larger numbers of rows (up to 100) this procedure did not yield results that were accurate to the nearest integer, and we instead used the simplex algorithm that is implemented in the MAPLE programming language [7]. This was very much slower, but did produce exact answers in these parameter situations, that is, for orthogonal arrays with 100 or fewer rows. Consequently, values for $LP(b^h, b, h-t)$ could only be inferred when these values did not exceed 99. The best available upper bound for each term $f(b^h, b, h-t)$ was then used in the computation of the orthogonal array bound for nets, and the resulting upper bound for s was entered into the appropriate cell in the corresponding s table in base b (see Section 6 for an explanation of the entries in the tables). We note that for $b = 2$, we determined all values of $LP(b^h, b, h-t) \leq 99$ (for all possible choices of $h \leq 50$ and t) which could affect any entry in the binary table of upper bounds for s; in fact all entries affected satisfy $t \leq 27$. However, in the case $b = 3$ the values of $LP(b^h, b, h-t)$ not exceeding 99 were determined only for $t \leq 10$, and for $b = 5$ these were computed only for $t \leq 5$.

It is a consequence of the foregoing that for $b = 2, 3, 5$ many of the upper bounds for s which appear in these tables for $m \geq t + 4$ and which exceed 99 might perhaps be reduced with knowledge of the linear programming bound for certain orthogonal arrays (with more than 100 rows). Furthermore, some additional upper bounds for s with $t \geq 11$ in case $b = 3$

and $t \geq 6$ in case $b = 5$ might also be reduced in a similar manner. However, we observe that the equality

$$f(b^h, b, h-t) = LP(b^h, b, h-t)$$

is known to hold in many cases, and so further improvement in many of the upper bounds on s which do not exceed 99 will not be possible solely by using the orthogonal array bound. We refer the reader to [26, 29] for further discussion of these issues.

It is of special interest to note that the orthogonal array bound for nets cannot always be attained and so is not best possible, in general. In particular, it follows from this bound that a $(1, 5, s)$-net in base 2 can exist only if $s \leq 6$. However, Lawrence [30] used properties of orthogonal arrays and generalized orthogonal arrays, together with further combinatorial arguments to show that there is no (1, 5, 6)-net in base 2. It follows that if $m \geq 5$, a $(1, m, s)$-net in base 2 can exist only if $s \leq 5$; this upper bound is reflected in the $t = 1$ row of the s table base 2.

Following Lawrence [26, 29], we next observe that these necessary conditions for the existence of a (t, m, s)-net in base b which have been conveniently expressed and discussed above in terms of upper bounds on s, may be equivalently expressed in the form of lower bounds on t, as we now explain. In general, each s bound implies the nonexistence of a specific net, which in turn implies a corresponding t bound (and conversely).

More specifically, let t', m be fixed integers with $m \geq t'+2$ and suppose that a (t', m, s)-net in base b can possibly exist only if $s \leq s'$. Then no $(t', m, s'+1)$-net exists and it follows that a $(t, m, s'+1)$-net can possibly exist only if $t \geq t'+1$. For example, from the s table in base 2 we see that a binary $(5, 13, s)$-net can possibly exist only if $s \leq 15$. Then no binary (5, 13, 16)-net exists. We conclude that a binary $(t, 13, 16)$-net can possibly exist only if $t \geq 6$, as is reflected in the corresponding cell of the t table base 2. In this way, the bounds in the t tables were derived from the bounds in the s tables.

Furthermore, these bounds for the existence of (t, m, s)-nets also immediately imply corresponding bounds for the existence of (t, s)-sequences, in light of the standard result that the existence of a (t, s)-sequence in base b implies the existence of a $(t, m, s+1)$-net in base b for all integers $m \geq t$. For example, the nonexistence of a (1, 5, 6)-net in base 2 implies the nonexistence of a (1, 5)-sequence in base 2.

We conclude this section by observing that if one considers the subclass of digital (t, m, s)-nets in base b, necessary conditions for existence which are more restrictive than the orthogonal array bound have been obtained in certain instances. We refer to Schmid and Wolf [53] for a series of examples of necessary conditions, some of which are optimal, for the existence of certain digital nets.

5. RELATED RESULTS

Since the writing of [35] there has been a lot of other work related to (t, m, s)-nets and (t, s)-sequences. For the sake of completeness we include a brief discussion of this work.

An important aspect is the combinatorial theory of (t, m, s)-nets which leads not only to some of the constructions listed in Section 2, but also to the bounds for the existence of nets which were discussed in the previous section. The crucial concepts in this combinatorial theory are orthogonal arrays, generalized orthogonal arrays and orthogonal hypercubes, and (as indicated in Section 2) certain types of these combinatorial objects turn out to be equivalent to (t, m, s)-nets. We refer to the work of Lawrence [26, 27, 29], Lawrence and Mullen [32], Mullen and Schmid [36], and Schmid [51]. A survey of combinatorial constructions of

(t, m, s)-nets using mutually orthogonal latin squares and mutually orthogonal hypercubes is given in Mullen [34]. A lower bound on the parameter t in (t, s)-sequences was derived in Niederreiter and Xing [46] from a combinatorial result of Lawrence [26].

The article of Larcher, Lauss, Niederreiter, and Schmid [20] contains existence results for (t, m, s)-nets and tables of net parameters obtained from Construction 13 in [35]. An official ACM software package implementing the (t, s)-sequences of Niederreiter (see Construction 9 in [35]) was announced in Bratley, Fox, and Niederreiter [5]. Another construction of (t, s)-sequences yielding the same parameters as that of Niederreiter was discussed in Tezuka [54, Chapter 6] and Tezuka and Tokuyama [55]. An investigation of general digital constructions of (t, m, s)-nets and (t, s)-sequences with the use of finite rings was carried out in Larcher, Niederreiter, and Schmid [22].

A generalization of the concept of a (t, s)-sequence was introduced and analyzed in Larcher and Niederreiter [21]. This article contains also a study of Kronecker-type sequences, that is, of nonarchimedean analogs of classical Kronecker sequences. Further work on Kronecker-type sequences and their connections with generalized (t, s)-sequences was carried out by Larcher [17] and Niederreiter [42].

A new way of obtaining upper bounds for the star discrepancy of digital (t, m, s)-nets was developed by Larcher [19], and this method was applied in Larcher [18] to obtain probabilistic results on the star discrepancy of sequences constructed by the digital method. Faure and Chaix [13] proved a lower bound of the best possible order of magnitude for the star discrepancy of a special (0,2)-sequence in base 2. Rote and Tichy [50] gave bounds for the dispersion of (t, m, s)-nets. The volume discrepancy of $(0, s)$-sequences was studied by Xiao [56] and Xiao and Faure [58]. The article of Xiao [57] is devoted to geometric properties of $(0, m, 2)$-nets. Further investigations of randomized (t, m, s)-nets were carried out by Hickernell [16] and Owen [48].

Earlier in this decade it was realized that (t, m, s)-nets provide very effective node sets for the numerical integration of multivariate Walsh series by quasi-Monte Carlo methods (see [35] for references). This line of research was continued by Larcher, Niederreiter, and Schmid [22], Larcher and Schmid [23], Larcher, Schmid, and Wolf [24], and Larcher and Wolf [25]. Related applications to the numerical integration of multivariate Haar series were studied by Entacher [12]. Error bounds for the calculation of volumes of Jordan-measurable sets by quasi-Monte Carlo methods with (t, m, s)-nets were given by Lécot [33]. Applications of the theory of (t, m, s)-nets to the analysis of pseudorandom numbers can be found in Larcher [19] and Niederreiter [43].

6. TABLES

For base $b = 2$ we provide updated versions of the t and s tables from [35]. For bases $b = 3$ and $b = 5$, we not only provide an updated version of the corresponding t table from [35], but we also provide s tables for these bases. We provide these new s tables not given in [35] since for many if not most combinatorial constructions, the s parameter is very important. In an effort to keep the tables to a manageable length, we have however not included any tables in base $b = 10$ since in this base there have been few if any new results other than those which arise by use of a product construction from bases 2 and 5.

In an effort to reduce the length of this article, we have only provided here, for the base $b = 2$, a t table for the range $1 \leq m, s \leq 18$ and we have included an s table only for the range $0 \leq t \leq 18$ and $1 \leq m \leq 18$. These rather small tables are included here for purposes of illustration only.

T Table Base 2

m/s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
2	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	6	6	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
3	0	0	0	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2
	0	0	0	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2
	2	6	6	6	11	2	12	5	5	5	5	5	5	5	5	5	5	5
4	0	0	0	1	1	1	1	2	2	2	2	2	2	2	2	3	3	3
	0	0	0	1	1	1	1	2	2	2	2	2	2	2	2	3	3	3
	2	6	6	6	13	13	17	2	2	2	2	2	2	2	12	5	5	5
5	0	0	0	1	1	2	2	2	2	2	2	2	2	2	2	3	3	3
	0	0	0	1	1	2	2	2	2	2	2	2	2	2	2	3	3	3
	2	6	6	6	18	16	13	13	13	13	17	17	17	17	17	2	2	2
6	0	0	0	1	1	2	2	2	3	3	3	3	3	3	3	3	3	3
	0	0	0	1	1	2	2	2	3	3	3	3	3	3	3	3	3	3
	2	6	6	6	18	13	20	20	16	16	17	17	17	17	17	17	17	17
7	0	0	0	1	1	2	2	2	3	3	3	3	3	3	3	4	4	4
	0	0	0	1	1	2	2	3	3	3	3	4	4	4	4	4	4	4
	2	6	6	6	18	13	21	13	20	20	20	13	17	17	17	17	17	17
8	0	0	0	1	1	2	2	2	3	3	3	3	3	3	3	4	4	4
	0	0	0	1	1	2	3	3	3	3	4	4	4	4	4	4	4	5
	2	6	6	6	18	18	18	21	3	24	13	20	20	20	2	2	24	17
9	0	0	0	1	1	2	2	2	3	3	3	4	4	4	4	4	4	4
	0	0	0	1	1	2	3	3	3	4	4	4	4	4	5	5	5	5
	2	6	6	6	18	18	18	2	21	3	3	2	2	24	20	20	20	20
10	0	0	0	1	1	2	2	2	3	3	3	4	4	4	4	5	5	5
	0	0	0	1	1	2	3	4	4	4	4	5	5	5	5	5	5	5
	2	6	6	6	18	18	18	18	16	21	3	3	16	16	2	2	2	24
11	0	0	0	1	1	2	2	2	3	3	3	4	4	4	4	5	5	5
	0	0	0	1	1	2	3	4	4	4	4	5	5	5	6	6	6	6
	2	6	6	6	18	18	18	18	2	2	21	3	2	24	17	17	16	16
12	0	0	0	1	1	2	2	2	3	3	3	4	4	4	5	5	5	5
	0	0	0	1	1	2	3	4	4	5	5	5	5	6	6	6	6	7
	2	6	6	6	18	18	18	18	24	16	16	21	24	3	2	2	24	17
13	0	0	0	1	1	2	2	2	3	3	3	4	4	4	5	6	6	6
	0	0	0	1	1	2	3	4	5	5	5	6	6	6	6	6	7	7
	2	6	6	6	18	18	18	18	18	2	24	16	21	3	2	24	16	2
14	0	0	0	1	1	2	2	2	3	3	3	4	4	4	5	6	6	6
	0	0	0	1	1	2	3	4	5	5	6	6	6	6	7	7	7	7
	2	6	6	6	18	18	18	18	18	24	2	2	2	21	3	16	2	2
15	0	0	0	1	1	2	2	2	3	3	3	4	4	4	5	6	6	6
	0	0	0	1	1	2	3	4	5	6	6	6	7	7	7	7	8	8
	2	6	6	6	18	18	18	18	18	18	2	24	16	16	21	24	3	16
16	0	0	0	1	1	2	2	2	3	3	3	4	4	4	5	6	6	6
	0	0	0	1	1	2	3	4	5	6	6	7	7	7	8	8	8	9
	2	6	6	6	18	18	18	18	18	18	24	16	2	24	16	21	3	3
17	0	0	0	1	1	2	2	2	3	3	3	4	4	4	5	6	6	6
	0	0	0	1	1	2	3	4	5	6	7	8	8	8	8	8	8	9
	2	6	6	6	18	18	18	18	18	18	16	2	2	2	2	2	21	3
18	0	0	0	1	1	2	2	2	3	3	3	4	4	4	5	6	6	6
	0	0	0	1	1	2	3	4	5	6	8	9	9	9	9	9	9	9
	2	6	6	6	18	18	18	18	18	18	18	15	16	16	16	16	16	21

S Table Base 2

t/m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	999	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
	999	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	6
1	999	999	7	7	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	999	999	7	7	5	5	5	5	5	5	5	5	5	5	5	5	5	5
	4	5	12	17	18	18	18	18	18	18	18	18	18	18	18	18	18	18
2		999	999	15	15	8	7	6	6	6	6	6	6	6	6	6	6	6
		999	999	15	15	8	8	8	8	8	8	8	8	8	8	8	8	8
		4	5	12	17	20	21	18	18	18	18	18	18	18	18	18	18	18
3			999	999	31	31	11	10	9	7	7	7	7	7	7	7	7	7
			999	999	31	31	15	15	11	11	11	11	11	11	11	11	11	11
			4	5	12	17	20	24	21	18	18	18	18	18	18	18	18	18
4				999	999	63	63	17	14	11	11	9	8	8	8	8	8	8
				999	999	63	63	20	20	15	15	14	14	14	14	14	14	14
				4	5	12	17	24	24	3	21	24	18	18	18	18	18	18
5					999	999	127	127	23	18	14	13	11	10	9	9	9	9
					999	999	127	127	29	29	23	23	15	15	15	15	15	15
					4	5	12	17	24	24	24	24	24	24	18	18	18	18
6						999	999	255	255	32	24	17	16	14	12	11	10	10
						999	999	255	255	42	42	26	26	20	20	18	18	18
						4	5	12	17	24	24	24	24	21	24	24	18	18
7							999	999	511	511	47	36	22	20	16	14	11	10
							999	999	511	511	63	63	34	34	25	25	20	20
							4	5	12	17	24	24	24	24	24	24	16	16
8								999	999	998	998	65	45	26	22	17	17	11
								999	999	998	998	89	89	44	44	33	33	24
								4	5	12	17	25	24	24	24	3	21	18
9									999	999	998	998	65	45	32	24	18	18
									999	999	998	998	127	127	56	56	40	40
									4	5	12	17	23	16	24	24	3	21
10										999	999	998	998	128	64	32	24	20
										999	999	998	998	180	180	71	71	48
										4	5	12	17	24	17	16	16	3
11											999	999	998	998	129	127	32	24
											999	999	998	998	255	255	91	91
											4	5	12	17	25	22	16	16
12												999	999	998	998	257	131	32
												999	999	998	998	361	361	116
												4	5	12	17	25	22	16
13													999	999	998	998	257	139
													999	999	998	998	511	511
													4	5	12	17	25	22
14														999	999	998	998	510
														999	999	998	998	723
														4	5	12	17	25
15															999	999	998	998
															999	999	998	998
															4	5	12	17
16																999	999	998
																999	999	998
																4	5	12
17																	999	999
																	999	999
																	4	5
18																		999
																		999
																		4

The full tables for each of the bases 2, 3, and 5 for both t and s can be found on the *Journal of Combinatorial Designs* web page located at http://www.emba.uvm.edu/∼colbourn/jcd/T-table-base2.html. In particular the t table for each base covers the range $1 \leq m$, $s \leq 50$ while the s tables cover the range $0 \leq t \leq 50$ and $1 \leq m \leq 50$. In addition to saving journal space, by putting the tables on the *Journal of Combinatorial Designs* web page, we hope to be able to regularly update the tables as new results are obtained.

As in [35], in the t table base 2, at the intersection of row m and column s, we list three numbers. The top number is a lower bound for the smallest value of t for which a (t, m, s)-net in base 2 can possibly exist (see Section 4 for a discussion of these bounds). The middle value is the smallest value of t which arises from any known construction and the bottom value is a tag to that construction (see Section 2 for a discussion of the known constructions). Similar statements hold for the t tables in bases 3 and 5.

In the s table base 2, at the intersection of row t and column m we also provide three numbers. The top number is the largest value of s for which it is known how to construct a (t, m, s)-net in base 2, while the bottom number is a tag to that construction (see Section 2). The middle number is an upper bound for the largest value of s for which a (t, m, s)-net in base 2 can possibly exist (see Section 4). As in [35], due to space limitations we have restricted the range of values to at most three digits. As a result we have used 999 to represent the fact that the value of s is arbitrarily large; for example as in the cases of (t, t, s) and $(t, t+1, s)$-nets in base b. If $s \geq 998$, but is bounded, we denote this by 998. This occurs for example in the case of an $(8, 10, s)$-net in base 2 where $s = 2^{10} - 1$, which of course exceeds 998. Similar statements hold for the s tables in bases 3 and 5.

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