# INFINITE DIMENSIONAL ENTANGLED MARKOV CHAINS 

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#### Abstract

We continue the analysis of nontrivial examples of quantum Markov processes. This is done by applying the construction of entangled Markov chains obtained from classical Markov chains with infinite state-space. The formula giving the joint correlations arises from the corresponding classical formula by replacing the usual matrix multiplication by the Schur multiplication. In this way, we provide nontrivial examples of entangled Markov chains on $\bigcup_{J \subset \mathbb{Z}} \bar{\otimes}_{J} F^{C^{*}}, F$ being any infinite dimensional type I factor, $J$ a finite interval of $\mathbb{Z}$, and the bar the von Neumann tensor product between von Neumann algebras. We then have new nontrivial examples of quantum random walks which could play a rôle in quantum information theory.

In view of applications to quantum statistical mechanics too, we see that the ergodic type of an entangled Markov chain is completely determined by the corresponding ergodic type of the underlying classical chain, provided that the latter admits an invariant probability distribution. This result parallels the corresponding one relative to the finite dimensional case.

Finally, starting from random walks on discrete ICC groups, we exhibit examples of quantum Markov processes based on type $\mathrm{II}_{1}$ von Neumann factors.


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## 1. INTRODUCTION

The recent development of quantum information raised the problem of finding a satisfactory quantum generalization of the classical random walks. The relevance of this problem for quantum information has been emphasized in the last past years, see e.g. [3]-[5] [8], [9], [14]-18], 24] for different solutions of this problem. However, these
proposals introduce some features which are not quite satisfactory from the mathematical point of view.

First of all, these constructions are based on special models and a general mathematical definition of quantum random walk seems to be lacking. Second, all these constructions are based on a quantum evolution, unitary in some cases, irreversible in others. Now, random walks are particular cases of Markov processes and it is well known that, while in the classical case a Markov evolution uniquely determines the law of the corresponding stochastic process, this is in general false in the quantum case. Finally, a desirable requirement for a quantum extension of a family of classical processes, is that there should exist a standard procedure to embed the original classical family into its quantum extension. In a satisfactory quantum generalization of the classical random walks, all these requirements should be precisely formulated and fulfilled.

We dealth with these problems in the previous paper [2], by defining a nontrivial quantum lifting of classical Markov chains by using the Schur multiplication (known also as the Hadamard multiplication, see e.g. [6]). In the above mentioned paper, we outlined this construction for general classical Markov chains, concentrating our attention to Markov chains with finite state-space.

In the present paper we study in detail the construction of entangled Markov chains based on classical Markov chains with infinite statespace. We refer the reader to [2] and the above quoted papers for a discussion about the motivations, the potential applications to quantum information, and further details.

We obtain nontrivial examples of random walks satisfying some natural requirements. Namely, they are quantum Markov chains such that their restrictions to at least one maximal Abelian subalgebra, are classical random walks. They are uniquely determined, up to arbitrary phases, by these classical restrictions. Finally, taking into account possible applications to information theory, they are purely generated (i.e. generated by isometries, see [11, 12], and [2] for a discussion about this point).

The entangled Markov chains so constructed, generate in a natural way, states on the $C^{*}$-infinite tensor product $\bigcup_{J \subset \mathbb{Z}} \bar{\otimes}_{J} F^{C^{*}}$ (denoted in the sequel $\bigotimes_{\mathbb{Z}} F$ by an abuse of notation), provided that the underlying classical Markov chains admit invariant distributions. Here, $F$ is any infinite dimensional type I factor, $J$ is a finite interval of $\mathbb{Z}$, and the
bar denotes the von Neumann tensor product between von Neumann algebras.

The present paper is organized as follows. Section 2 is devoted to prove that the Schur multiplication is well defined also in infinite dimensional case. Namely, the Schur multiplication generates a (commutative) multiplication also for infinite dimensional type I factors. In Section 3 we extend to the infinite dimensional case, the definition of entangled lifting of classical Markov chains by using the Schur multiplication. The ergodic properties of the entangled Markov chains with a stationary distribution are investigated in Section 4 As for finite dimensional case, we see that the ergodic properties of the entangled chain are determined by those of the underlying classical chain. Contrary to finite dimensional case, the problem concerning the pureness of strongly clustering infinite dimensional entangled Markov chains is left open. Yet, some very simple examples (see Section (5) of pure Markov chains go towards the conjecture that any strongly clustering entangled Markov chain should generate a pure state on $\bigotimes_{\mathbb{Z}} F$. Section 5 contains also the description of entangled Markov processes arising from classical random walks on discrete groups. Then, starting from random walks on discrete Infinite Conjugacy Class (ICC for short) groups, we provide examples of quantum random walks based on type $\mathrm{II}_{1}$ factors.

## 2. THE SChUR MULTIPLICATION IN THE INFINITE DIMENSIONAL CASE

Let $I$ be an index set. Consider the space $\stackrel{\circ}{\mathbb{M}}_{I}$ consisting of all the $I \times I$ matrices with complex entries. The subset of all bounded $I \times I$ matrices is denoted as $\mathbb{M}_{I}$. It gives the most general type $I W^{*}$-factor, as $I$ varies among all cardinalities. Define the map $\Phi: \mathbb{M}_{I} \mapsto \mathbb{M}_{I \times I}$ as

$$
\begin{equation*}
\Phi(A)_{(i, j)(k, l)}:=A_{i k} \delta_{i j} \delta_{k l} . \tag{2.1}
\end{equation*}
$$

Notice that the map $\Phi$ is identity preserving.
Let $A, B \in \mathbb{M}_{I}$. We can define the Schur multiplication as

$$
(A \diamond B)_{i j}:=A_{i j} B_{i j}
$$

Taking into account that

$$
(A \otimes B)_{(i, j)(k, l)}=A_{i k} B_{j l}
$$

we can extend the Schur multiplication to a map $m: \mathbb{M}_{I \times I} \mapsto \stackrel{\circ}{\mathbb{M}}_{I}$ by putting

$$
\begin{equation*}
m(X)_{i j}:=X_{(i, i)(j, j)} \tag{2.2}
\end{equation*}
$$

Lemma 2.1. Let $A \in \mathbb{M}_{I}$. Then $\Phi(A) \in \mathbb{M}_{I \times I}$.
Proof. We compute

$$
\begin{aligned}
|\langle\Phi(A) x, y\rangle| & \equiv\left|\sum \overline{y_{i j}} A_{i k} \delta_{i j} \delta_{k l} x_{k l}\right| \\
=\left|\sum \overline{y_{i i}} A_{i k} x_{k k}\right| & \leq\|A\|_{\mathbb{M}_{I}}\|x\|_{\ell^{2}(I \times I)}\|y\|_{\ell^{2}(I \times I)}
\end{aligned}
$$

which leads to $\|\Phi(A)\|_{\mathbb{M}_{I \times I}} \leq\|A\|_{\mathbb{M}_{I}}$.
From now on, we consider the map $\Phi$ as a bounded map from $\mathbb{M}_{I}$ to $\mathbb{M}_{I \times I}$.

Proposition 2.2. The map $\Phi: \mathbb{M}_{I} \mapsto \mathbb{M}_{I \times I}$ is an identity preserving *-morphism.
Proof. It is easily seen that $\Phi$ is $*-$ preserving and multiplicative.
Let $\rho \in L^{1}\left(\mathbb{M}_{I}\right)_{+}$. It is immediate to show that

$$
\begin{equation*}
\operatorname{Tr} \otimes \operatorname{Tr}(\Phi(\rho))=\operatorname{Tr}(\rho) \tag{2.3}
\end{equation*}
$$

that is the positive element $\Phi(\rho)$ is trace-class. Furthermore, if $\sigma \in$ $L^{1}\left(\mathbb{M}_{I \times I}\right)_{+}$, then

$$
\begin{equation*}
0 \leq \operatorname{Tr}(m(\sigma)) \leq \operatorname{Tr} \otimes \operatorname{Tr}(\sigma) \tag{2.4}
\end{equation*}
$$

that is the positive element $m(\sigma)$ is trace-class as well.
For the sake of completeness we report the following
Theorem 2.3. Let $\Phi: \mathbb{M}_{I} \mapsto \mathbb{M}_{I \times I}$ be the linear map given in (2.1).
Then $\Phi$ is a normal faithful identity preserving *-morphism of $\mathbb{M}_{I}$ into $\mathbb{M}_{I \times I}$.

Moreover if $p \geq 1$, $\Phi$ restricts itself to isometries of $L^{p}\left(\mathbb{M}_{I}\right)$ into $L^{p}\left(\mathbb{M}_{I \times I}\right)$.
Proof. Let $\rho \in L^{1}\left(\mathbb{M}_{I \times I}\right)_{+}$, and $\left\{X_{\alpha}\right\} \subset \mathbb{M}_{I}$ be a net converging to $X \in \mathbb{M}_{I}$ in the $*-$ weak topology. We have by (2.4),

$$
\begin{aligned}
\lim _{\alpha} \operatorname{Tr} & \otimes \operatorname{Tr}\left(\rho \Phi\left(X_{\alpha}\right)\right)
\end{aligned}=\lim _{\alpha} \operatorname{Tr}\left(m(\rho) X_{\alpha}\right), ~=\operatorname{Tr}(m(\rho) X)=\operatorname{Tr} \otimes \operatorname{Tr}(\rho \Phi(X)), ~ \$
$$

where $m$ is the Schur multiplication. Namely, $\Phi$ is normal. The first part follows by Proposition [2.2, taking into account that a normal $*-$ morphism between von Neumann factors is automatically faithful.

Let now $p \geq 1$, and $T \in L^{p}\left(\mathbb{M}_{I}\right)$. We get by the previous results,

$$
\begin{aligned}
\|\Phi(T)\|_{p}^{p} & \equiv \operatorname{Tr} \otimes \operatorname{Tr}\left(\left(\Phi(T)^{*} \Phi(T)\right)^{p / 2}\right) \\
& =\operatorname{Tr} \otimes \operatorname{Tr}\left(\Phi\left(\left(T^{*} T\right)^{p / 2}\right)\right) \\
& =\operatorname{Tr}\left(\left(T^{*} T\right)^{p / 2}\right) \equiv\|T\|_{p}^{p}
\end{aligned}
$$

The main properties of the Schur multiplication (2.2) are summarized in the following

Theorem 2.4. The Schur multiplication (2.2) defines a completely positive identity preserving normal map from $\mathbb{M}_{I \times I}$ into $\mathbb{M}_{I}$.

Moreover, $m=\left(\Phi \Gamma_{L^{1}\left(\mathbb{M}_{I}\right)}\right)^{*}$ where $\Phi$ is the map given in (2.1).
Proof. An easy computation gives $\left(\Phi \Gamma_{L^{1}\left(\mathbb{M}_{I}\right)}\right)^{*}=m$. Then, taking into account the properties of $\Phi$, it remains to check the normality. Let $\left\{X_{\alpha}\right\} \subset \mathbb{M}_{I \times I}$ be a net converging to $X \in \mathbb{M}_{I \times I}$ in the $*$-weak topology, and $\rho \in L^{1}\left(\mathbb{M}_{I}\right)_{+}$. We get by (2.3),

$$
\begin{aligned}
\lim _{\alpha} \operatorname{Tr}\left(\rho m\left(X_{\alpha}\right)\right) & =\lim _{\alpha} \operatorname{Tr} \otimes \operatorname{Tr}\left(\Phi(\rho) X_{\alpha}\right) \\
=\operatorname{Tr} \otimes \operatorname{Tr}(\Phi(\rho) X) & =\operatorname{Tr}(\rho m(X)),
\end{aligned}
$$

that is $m$ is normal.
Taking into account the above results, we have that $m \Gamma_{L^{1}\left(\mathbb{M}_{I \times I}\right)}$ is bounded and $\Phi=\left(m \Gamma_{L^{1}\left(\mathbb{M}_{I \times I}\right)}\right)^{*}$.

From the above considerations, the Schur multiplication $m$ can be defined as a completely positive identity preserving normal map from $F \bar{\otimes} F$ to $F$ for arbitrary type I factor $F$, provided that a complete system of matrix units $e:=\left\{e_{i j}\right\} \subset F$ is kept fixed.

## 3. INFINITE DIMENSIONAL ENTANGLED MARKOV CHAINS

Let $I$ be any index set which is kept fixed during the analysis. Consider a copy $\mathcal{M}_{j}$ of the algebra $\mathbb{M}_{I} \equiv \mathbb{M}_{I}(\mathbb{C})$ of all bounded $I \times I$ matrices with complex entries, together with a copy $\mathcal{D}_{j}$ of the maximal Abelian subalgebra $\ell^{\infty}(I)$ of $\mathbb{M}_{I}(\mathbb{C})$. For each finite subset $J \subset \mathbb{Z}$, we put

$$
\mathfrak{M}_{J}:=\bar{\bigotimes}_{j \in J} \mathcal{M}_{j}, \quad \mathfrak{D}_{J}:=\bar{\bigotimes}_{j \in J} \mathcal{D}_{j}
$$

where $\bar{\otimes}$ is the usual von Neumann tensor product between von Neumann algebras. If $F \subset G$, we consider the natural embedding $a_{F} \mapsto$ $a_{F} \otimes \mathbf{I}_{G \backslash F}$.

The local algebra

$$
\mathfrak{M}:={\overline{\left(\underset{\overrightarrow{J \uparrow \mathbb{Z}}}{\lim } \mathfrak{M}_{J}\right)}}^{C^{*}}
$$

is the $C^{*}$-inductive limit associated to the directed system $\left\{\mathfrak{M}_{J}\right\}_{J \subset \mathbb{Z}}$, $J$ finite subsets of $\mathbb{Z}$. For our purpouse, we consider also the maximal

Abelian subalgebra

$$
\mathfrak{D}:={\overline{\left(\lim _{\overrightarrow{J \uparrow \mathbb{Z}}} \mathfrak{D}_{J}\right)}}^{C}
$$

of $\mathfrak{M}$ made of the $C^{*}$-inductive limit associated to the directed system $\left\{\mathfrak{D}_{J}\right\}_{J \subset \mathbb{Z}}$ as before.

For general $C^{*}$-algebras $\mathfrak{A}, \mathfrak{B}$, a completely positive identity preserving linear map $\mathcal{E}: \mathfrak{A} \otimes \mathfrak{B} \mapsto \mathfrak{B}$ will be called in the sequel, a transition expectation.

Consider a completely positive normal map $P: \mathbb{M}_{I} \mapsto \mathbb{M}_{I}$. Such a map is said to be Schur identity preserving if $\mathcal{E}: \mathbb{M}_{I} \bar{\otimes} \mathbb{M}_{I} \mapsto \mathbb{M}_{I}$ given by

$$
\begin{equation*}
\mathcal{E}=m \circ(\mathrm{id} \otimes P) \tag{3.1}
\end{equation*}
$$

is identity preserving (where "o" stands for composition of maps). This condition means

$$
P(\mathbb{I})_{i i}=1, \quad i \in I
$$

Following [2], Definition 1, any such a $P$ is said to be an entangled Markov operator if $P(\mathbf{I}) \neq \mathbf{I}$.

We consider also unbounded entangled Markov operators $P: \mathbb{M}_{I} \mapsto \mathbb{M}_{I}$ such that the associated maps (3.1) are well defined (i.e. bounded) transition expectations, see below.

Let $P$ be an entangled Markov operator. Consider $\mathcal{E}$ as above. Put $\mathcal{E}_{A}(B):=\mathcal{E}(A \otimes B)$ and consider a "initial distribution" $\rho$ which is a positive normalized element in $L^{1}\left(\mathbb{M}_{I}\right)$ satisfying $\rho=\rho \circ \mathcal{E}_{\mathbf{I}} .{ }^{1}$ A state $\omega \in \mathcal{S}(\mathfrak{M})$ is uniquely determined by all the "finite dimensional distributions"

$$
\omega\left(A_{1} \otimes \cdots \otimes A_{n}\right):=\rho\left(\mathcal{E}_{A_{1}} \circ \cdots \circ \mathcal{E}_{A_{n}}(\mathbb{I})\right)
$$

Such a state is a translation invariant quantum Markov chain on $\mathfrak{M}$ generated by the triplet $\left(\mathbb{M}_{I}, \mathcal{E}, \rho\right)$ following the terminology of [12]. It generalizes the construction given in [2] to the infinite dimensional case. For further details about the quantum Markov chains, we refer to [1, 7, 11, 12], and the references cited therein.

Now we specialize the matter to the quantum Markov chains generated in a canonical way by classical Markov chains with an infinite state-space, that is to infinite dimensional entangled Markov chains.

[^0]Let $\Pi \in \mathbb{M}_{I}$ be a stochastic matrix. Define, for $A \in \mathbb{M}_{I}$,

$$
\begin{equation*}
P(A)_{i j}:=\sum_{k, l \in I} \sqrt{\Pi_{i k} \Pi_{j l}} A_{k l} . \tag{3.2}
\end{equation*}
$$

By Holder inequality, we have

$$
\begin{aligned}
& \left|P(A)_{i j}\right| \equiv\left|\sum_{k, l} \sqrt{\Pi_{i k}} A_{k l} \sqrt{\Pi_{j l}}\right| \\
& \leq\|A\|_{\mathbb{M}_{I}}\left(\sum_{k} \Pi_{i k}\right)^{1 / 2}\left(\sum_{k} \Pi_{j k}\right)^{1 / 2} \equiv\|A\|_{\mathbb{M}_{I}}
\end{aligned}
$$

that is $P(A) \in \stackrel{\circ}{M}_{I} .{ }^{2}$
We show that the transition expectation $\mathcal{E}$ is purely generated, following the terminology of [12].

Proposition 3.1. We have for the map $\mathcal{E}=m \circ(\mathrm{id} \otimes P)$, with $P$ as in (3.2),

$$
\mathcal{E}(A)=V^{*} A V, \quad A \in \mathbb{M}_{I \times I}
$$

where the isometry $V: \ell^{2}(I) \mapsto \ell^{2}(I \times I)$ is given by

$$
\begin{equation*}
V e_{i}=\sum_{j \in I} \sqrt{\Pi_{i j}} e_{i} \otimes e_{j} \tag{3.3}
\end{equation*}
$$

and $\left\{e_{i}\right\}_{i \in I}$ is the canonical basis of $\ell^{2}(I)$.
Hence, $\mathcal{E}$ extends to a completely positive, identity preserving normal map of $\mathbb{M}_{I} \bar{\otimes} \mathbb{M}_{I} \cong \mathbb{M}_{I \times I}$ into $\mathbb{M}_{I}$.

Proof. It is immediate to show that Formula (3.3) defines a bounded operator. The proof follows as

$$
V^{*} e_{i} \otimes e_{j}=\sqrt{\Pi_{i j}} e_{i}
$$

Let now $\pi:=\left\{\pi_{j}\right\}_{j \in I}$ be an invariant measure for $\Pi$. Define the matrix $Q(\pi) \in \stackrel{\circ}{\mathbb{M}}_{I}$ given by

$$
\begin{equation*}
Q(\pi)_{i j}:=\sum_{k \in I} \pi_{k} \sqrt{\Pi_{k i} \Pi_{k j}} . \tag{3.4}
\end{equation*}
$$

[^1]By Holder inequality, we get

$$
\begin{aligned}
& Q(\pi)_{i j} \equiv \sum_{k} \sqrt{\pi_{k} \Pi_{k i}} \sqrt{\pi_{k} \Pi_{k j}} \\
\leq & \left(\sum_{k} \pi_{k} \Pi_{k i}\right)^{1 / 2}\left(\sum_{k} \pi_{k} \Pi_{k j}\right)^{1 / 2} \equiv \sqrt{\pi_{i} \pi_{j}}
\end{aligned}
$$

that is, $Q(\pi)$ is well defined. Moreover, it is positive by construction. Furthermore, $Q(\pi)$ defines a bounded positive form on $\mathbb{M}_{I}$ if and only if $\pi$ defines a positive form on $\ell^{\infty}(I)$, with $\|Q(\pi)\|_{L^{1}\left(\mathbb{M}_{I}\right)}=\|\pi\|_{\ell^{1}(I)}$.

Proposition 3.2. $Q$ given in (3.4) maps the set of invariant measures for $\Pi$ into the set of normal semifinite weights on $\mathbb{M}_{I}$ invariant for $\mathcal{E}_{\mathbf{I}} \equiv \mathbb{I} \diamond P(\cdot)$, with $P$ given in (3.2).
$Q$ restricts itself to a one-to-one correspondence between the invariant probability distributions for $\Pi$ and the normal states invariant for $\mathcal{E}_{\mathbf{I}}$.

Proof. Taking into account the proof of Proposition 2.4 of [2], it is enough to prove semifiniteness and normality. Let $\pi$ be as above. Let $\mathfrak{m}_{Q(\pi)}$ be the definition-domain of the weight $Q(\pi)$ ([23], Definition VII.1.3). Then

$$
\mathfrak{m}_{0}:=\bigcup\left\{\mathbb{M}_{F} \mid F \text { finite subset of } I\right\} \subset \mathfrak{m}_{Q(\pi)}
$$

and $\mathfrak{m}_{0}^{\prime \prime}=\mathbb{M}_{I}$, that is $Q(\pi)$ is semifinite.
We write, with an abuse of notation, $Q(\pi)(A)=\pi(\mathcal{E}(\mathbf{I} \otimes A))$ as $\mathcal{E}(\mathbb{I} \otimes A) \in \ell^{\infty}(I)$. The normality follows by Fatou Lemma (i.e. $\pi$ is normal on $\left.\ell^{\infty}(I)\right)$, taking into account that $\mathcal{E}$ is a normal map.

Let $(\Pi, \pi)$ consist of a stochastic matrix as above, and an invariant probability measure for it, respectively. Put $\mathbb{M}:=\mathbb{M}_{I}$. The entangled Markov chain associated to ( $\Pi, \pi$ ) is the translation invariant locally normal state on $\mathfrak{M}$ generated by the triplet $(\mathbb{M}, \mathcal{E}, Q(\pi))$. It is immediate to show that the quantum chain reduces itself to the classical one, when restricted to $\mathfrak{D}$.

An invariant distribution (i.e. an invariant probability measure) does not always exist for a given infinite stochastic matrix, see e.g. Theorem 6.2.1 of [10]. However, by Proposition [3.2] one can define on $\mathfrak{M}$ a translation invariant locally normal weight starting from the triplet $(\mathbb{M}, \mathcal{E}, Q(\pi)), \pi$ being an invariant measure for $\Pi$ if the last exists. ${ }^{3}$

[^2]Notice that, by Proposition [3.1, an entangled Markov chain is always generated by an isometry (purely generated in the terminology of [12]).

As it was shown in [2], the entangled Markov chains are defined up to arbitrary phases. Namely, let $\mathbb{M}_{I}(\mathbb{T})$ be the set consisting of all the $I \times I$ matrices with entries in the unit circle $\mathbb{T}$. If $\chi \in \mathbb{M}_{l}(\mathbb{T})$, then the map $P_{\chi}$ defined as

$$
\begin{equation*}
P_{\chi}(A)_{i j}:=\sum_{k, l \in I} \overline{\chi_{i k}} \chi_{j l} \sqrt{\Pi_{i k} \Pi_{j l}} a_{k l} \tag{3.5}
\end{equation*}
$$

gives rise to an entangled Markov operator as well. The corresponding quantum measures $Q_{\chi}(\pi)$ associated to the entangled Markov operator in (3.5) are given by

$$
Q_{\chi}(\pi)_{i j}=\sum_{k \in I} \pi_{k} \chi_{k i} \overline{\chi_{k j}} \sqrt{\Pi_{k i} \Pi_{k j}}
$$

provided that the classical chain $\Pi$ admits the invariant measure $\pi$.

## 4. ERGODIC PROPERTIES

The present section is devoted to the investigation of the ergodic properties of infinite dimensional entangled Markov chains. The following analysis parallels the corresponding one of Section 3 of [2] relative to the finite dimensional case.

Taking into account Formula (3.15) of [10], we write

$$
\begin{equation*}
\Pi=p_{0} \Pi p_{0}+\sum_{\theta \in \Theta}\left(p_{0} \Pi p_{\theta}+p_{\theta} \Pi p_{\theta}\right)+\sum_{\lambda \in \Lambda}\left(p_{0} \Pi p_{\lambda}+p_{\lambda} \Pi p_{\lambda}\right) . \tag{4.1}
\end{equation*}
$$

Here, $\Theta, \Lambda$ label the recurrent-null, and the recurrent-positive classes of $\Pi$, respectively, and $p_{0}$ is the selfadjoint projection associated to the indices relative to the transient states of the classical chain under consideration. Furthermore, for each $\Pi_{\lambda}:=p_{\lambda} \Pi p_{\lambda}$,

$$
\Pi_{\lambda}=\sum_{j_{\lambda}=1}^{m_{\lambda}} p_{\lambda, j_{\lambda}} \Pi_{\lambda} p_{\lambda, j_{\lambda}+1}=: \sum_{j_{\lambda}=1}^{m_{\lambda}} p_{\lambda, j_{\lambda}} \Pi_{j_{\lambda}, j_{\lambda}+1} p_{\lambda, j_{\lambda}+1}
$$

with the convention that $m_{\lambda}+1=1$. Here, $m_{\lambda}$ is the (finite) period corresponding to indices ("states" in the terminology of classical Markov chains) of the ergodic class $\lambda$, with the convention that $m_{\lambda}=1$ in the aperiodic case. Notice that the indices (i.e. "states") corresponding to an ergodic class are at most denumerable. We suppose that $\Lambda$ is nonvoid. ${ }^{4}$ In this case, there exist stationary distributions for $\Pi$, and

[^3]by Proposition [3.2] stationary distributions for $\mathcal{E}$ given by (3.1). Let $\pi$ be any such a distribution. It has the form
$$
x=\sum_{\lambda \in \Lambda} \alpha_{\lambda} x_{\lambda}, \quad \sum_{\lambda \in \Lambda} \alpha_{\lambda}=1, \alpha_{\lambda} \geq 0, \lambda \in \Lambda
$$
$x_{\lambda}$ being the unique stationary distribution for the stochastic matrix $\Pi_{\lambda}$. Let $\Lambda_{\pi} \subset \Lambda$ be the set of the ergodic classes $\lambda$ of $\Pi$ such that $\alpha_{\lambda}>0$. Of course, the cardinality of $\Lambda_{\pi}$ is at most denumerable. If
\[

$$
\begin{equation*}
p:=\sum_{\lambda \in \Lambda} p_{\lambda} \tag{4.2}
\end{equation*}
$$

\]

is the support in $\ell^{\infty}(I)$ of $\pi$ considered in a natural way as an element of $\mathbb{M}$, define $\tilde{\mathcal{E}}: \mathbb{M} \otimes \mathbb{M}_{p} \mapsto \mathbb{M}_{p}$ given by

$$
\begin{equation*}
\tilde{\mathcal{E}}:=p \mathcal{E}\left\lceil_{\mathbb{M}_{p}}(\cdot) p .\right. \tag{4.3}
\end{equation*}
$$

Let $\eta$ be the action of the completely reducible part $p \Pi p$ on the set of projections $\left\{\left\{p_{\lambda, j_{1}}, \ldots, p_{\lambda, j_{m_{\lambda}}}\right\}\right\}_{\lambda \in \Lambda}, p$ being given in (4.2). Such an action leaves each ergodic component $\left\{p_{\lambda, j_{\lambda}}\right\}_{\lambda=1}^{m_{\lambda}}$ globally invariant, acting cyclically on it. Choose any projection, say $\bar{p}_{\lambda}:=p_{\lambda, \bar{j}}$, in each ergodic class $\lambda \in \Lambda_{\pi}$, where $\Lambda_{\pi}$ labels the ergodic components present in the stationary distribution $\pi$ as described above. Define for $\left\{A_{1}, \ldots, A_{n}\right\} \subset \mathbb{M}$,

$$
\begin{aligned}
& \varphi_{\lambda}\left(A_{1} \otimes \cdots \otimes A_{n}\right):=\pi\left(\bar{p}_{\lambda}\right)^{-1} Q(\pi)\left(\mathcal{E}_{A_{1}} \circ \cdots \circ \mathcal{E}_{A_{n}}\left(\eta^{-n} \bar{p}_{\lambda}\right)\right) \\
& \omega_{\lambda}\left(A_{1} \otimes \cdots \otimes A_{n}\right):=\pi\left(p_{\lambda}\right)^{-1} Q(\pi)\left(\mathcal{E}_{A_{1}} \circ \cdots \circ \mathcal{E}_{A_{n}}\left(p_{\lambda}\right)\right)
\end{aligned}
$$

Let $\omega$ be the entangled Markov chain on $\mathfrak{M}$ associated to the triplet $(\mathbb{M}, \mathcal{E}, Q(\pi))$. It is straightforward to verify that

$$
\begin{gather*}
\omega_{\lambda}=\frac{1}{m_{\lambda}} \sum_{k=1}^{m_{\lambda}} \varphi_{\lambda} \circ \tau^{k}, \\
\omega=\sum_{\lambda \in \Lambda_{\pi}} \frac{\pi\left(p_{\lambda}\right)}{m_{\lambda}} \sum_{k=1}^{m_{\lambda}} \varphi_{\lambda} \circ \tau^{k}, \tag{4.4}
\end{gather*}
$$

where $\tau$ is the one-step shift on the chain.
The states $\omega_{\lambda}, \varphi_{\lambda}$ describe the decomposition of $\omega$ into ergodic and completely ergodic components, respectively. ${ }^{5}$

[^4]Theorem 4.1. Let $(\Pi, \pi)$ consist of a stochastic matrix and a stationary distribution for it, with $\Lambda_{\pi} \neq \emptyset$. Consider the entangled Markov chain $\omega$ on $\mathfrak{M}$ generated by the triplet $(\mathbb{M}, \mathcal{E}, Q(\pi))$. The following assertions hold true.
(i) The state $\omega$ is ergodic w.r.t the spatial translations if and only if the set $\Lambda_{\pi}$ is a singleton.
(ii) The state $\omega$ is strongly clustering w.r.t the spatial translations if and only if the set $\Lambda_{\pi}$ is a singleton, and in addition, the corresponding block in the decomposition (4.1) of $\Pi$ is aperiodic.

Proof. It is immediate to verify that $\omega$ is given by (4.4). Furthermore, the states appearing in the r.h.s. of (4.4) give rise to different states when restricted to the Abelian subalgebra $\mathfrak{D}$ of $\mathfrak{M}$. So, they are mutually different. It is then enough to show that the $\omega_{\lambda}$ are ergodic w.r.t. the one step shift, and the $\varphi_{\lambda}$ are strongly clustering w.r.t. the $m_{\lambda}$-step shift, respectively.

Let $A=A_{1} \otimes \cdots \otimes A_{r}, B=B_{1} \otimes \cdots \otimes B_{s}$, we compute by applying Lemma 2.1 of [2],

$$
\begin{aligned}
& \frac{1}{n} \sum_{k=1}^{n} \omega_{\lambda}\left(A \tau^{k}(B)\right)=\pi\left(p_{\lambda}\right)^{-1} \\
& \times Q(\pi)\left(\mathcal{E}_{A_{1}} \circ \cdots \circ \mathcal{E}_{A_{r}} \circ\left(\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{E}_{\mathbf{I}}^{k}\right)\left(\mathcal{E}_{\mathbf{I}} \circ \tilde{\mathcal{E}}_{B_{1}} \circ \cdots \circ \tilde{\mathcal{E}}_{B_{s}}\left(p_{\lambda}\right)\right)\right),
\end{aligned}
$$

where $\tilde{\mathcal{E}}$ is given by (4.3). Define, componentwise, the element $v \in \ell^{1}(I)$ as

$$
v_{i}:=Q(\pi)\left(\mathcal{E}_{A_{1}} \circ \cdots \circ \mathcal{E}_{A_{r}}\left(e_{i i}\right)\right) .
$$

Define the element $D \in \ell^{\infty}(I)_{p_{\lambda}} \subset \mathbb{M}_{I}$ as

$$
D:=\mathcal{E}_{\mathbf{I}} \circ \tilde{\mathcal{E}}_{B_{1}} \circ \cdots \circ \tilde{\mathcal{E}}_{B_{s}}\left(p_{\lambda}\right) .
$$

We get by Dominated Convergence Theorem,

$$
\begin{aligned}
& \lim _{n} Q(\pi)\left(\mathcal{E}_{A_{1}} \circ \cdots \circ \mathcal{E}_{A_{r}} \circ\left(\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{E}_{\mathbf{I}}^{k}(D)\right)\right) \\
= & \lim _{n} Q(\pi)\left(\mathcal{E}_{A_{1}} \circ \cdots \circ \mathcal{E}_{A_{r}} \circ\left(\frac{1}{n} \sum_{k=0}^{n-1} \Pi_{\lambda}^{k} D\right)\right) \\
= & \lim _{n}\left\langle v,\left(\frac{1}{n} \sum_{k=0}^{n-1} \Pi_{\lambda}^{k}\right) D\right\rangle \\
= & \pi\left(p_{\lambda}\right)^{-1} Q(\pi)\left(\mathcal{E}_{B_{1}} \circ \cdots \circ \mathcal{E}_{B_{s}}\left(p_{\lambda}\right)\right) Q(\pi)\left(\mathcal{E}_{A_{1}} \circ \cdots \circ \mathcal{E}_{A_{r}}\left(p_{\lambda}\right)\right) .
\end{aligned}
$$

Here, $\langle\cdot, \cdot\rangle$ is the natural pairing between $\ell^{1}$ and $\ell^{\infty}$, and the last equality follows by Theorem 6.1 of [13].

Collecting together, we have

$$
\lim _{n} \frac{1}{n} \sum_{k=1}^{n} \omega_{\lambda}\left(A \tau^{k}(B)\right)=\omega_{\lambda}(A) \omega_{\lambda}(B)
$$

that is, $\omega_{\lambda}$ is ergodic. The mixing property w.r.t. the $m_{\lambda}$-step shift for the $\varphi_{\lambda}$ is proven in the same way, taking into account Theorem 6.38 of (13).

Now, the quantum chain is ergodic iff $\Lambda_{\pi}$ is a singleton (i.e. $\Lambda_{\pi}=$ $\left\{\lambda_{0}\right\}$ ), which corresponds to the case when there is only one summand in the first sum in (4.4). The quantum chain is strongly clustering when, in addition, also the second sum in (4.4) consists of one element, that is when, for the period, $m_{\lambda_{0}}=1$.

## 5. SOME APPLICATIONS

We are going to consider some interesting examples of infinite dimensional entangled Markov chains.

We begin with the classical chain $(\Pi, \pi), \pi$ being an invariant distribution for the stochastic matrix $\Pi$. Consider the collections $\left\{D^{(k)}\right\}_{k \in \mathbb{N} \backslash\{0\}}$ of the density matrices relative to the local algebras

$$
\mathfrak{M}_{[1, k]}:=\bar{\bigotimes}_{1 \leq j \leq k} \mathcal{M}_{j}
$$

and their translates, arising from the entangled Markov chain $(\mathbb{M}, \mathcal{E}, Q(\pi))$ associated to ( $\Pi, \pi$ ). Here, $\mathcal{E}$ is given in (3.1), with the entangled operator $P$ given in (3.2).

Proposition 5.1. We have for the collections $\left\{D^{(k)}\right\}_{k \in \mathbb{N} \backslash\{0\}}$ of the density matrices,

$$
\begin{equation*}
D_{\left(i_{1}, \ldots, i_{k}\right)\left(j_{1}, \ldots, j_{k}\right)}^{(k)}=\operatorname{Tr}_{\mathbb{M}}\left(\mathcal{E}_{D_{\pi}} \circ \mathcal{E}_{e_{i_{1} j_{1}}} \circ \cdots \circ \mathcal{E}_{e_{i_{k} j_{k}}}(\mathbb{I})\right) \tag{5.1}
\end{equation*}
$$

where $D_{\pi}:=\sum_{k \in I} \pi_{k} e_{k k}$ is the diagonal embedding of $\pi$ in $L^{1}(\mathbb{M})$, and the $e_{i j}$ are the canonical matrix units of $\mathbb{M}$.

Proof. A simple computation.
We specialize the situation to the entangled Markov chain generated by the the aperiodic stochastic projection $Q$ with matrix elements $q_{i j}=$
$\pi_{j}>0, i \in \mathbb{N}$ and $\pi \in \ell^{1}$. We obtain for the sequences of density matrices given in (5.1),

$$
D^{(k)}=\underbrace{D^{(1)} \otimes \cdots \otimes D^{(1)}}_{k \text {-times }}
$$

with $D_{i j}^{(1)}=\sqrt{\pi_{i} \pi_{j}}$. The state $\omega$ on $\mathfrak{M}$ associated to these examples of entangled Markov chains, is then an infinite product vector state based on the vector $\left\{\sqrt{\pi_{i}}\right\}_{i \in \mathbb{N}} \in \ell^{2}([19])$. Namely, $\omega$ is a pure state, see [21], Proposition 4.4.4. ${ }^{6}$

This very simple example, together with the result in Theorem 3.4 of [2] relative to the pureness of finite dimensional entangled Markov chains, allows us to conjecture that strongly clustering infinite dimensional entangled Markov chains generate pure states on $\mathfrak{M}$. This might be proved by studying the algebraic properties of the sequence of the ranges of the $D^{(k)}$ (see [11]), taking into account (5.1). For this aim, we report another useful formula for the density matrices $D^{(k)}$. Define the matrix $\Gamma \in \mathbb{M}_{I \times I}$ as

$$
\Gamma_{(i, j)(k, l)}:=\sqrt{\Pi_{i j} \Pi_{k l}} .
$$

We get

$$
D^{(k)}=\underbrace{m \otimes \cdots \otimes m}_{k \text {-times }}(Q(\pi) \otimes \underbrace{\Gamma \otimes \cdots \otimes \Gamma}_{(k-1) \text {-times }} \otimes P(\mathbb{I}))
$$

where $m$ is the Schur multiplication, $P$ is the entangled Markov operator associated to the stochastic matrix $\Pi$ via (3.2), and $Q(\pi)$ is the trace-class matrix associated to the stationary distribution $\pi$ via (3.4).

We pass to the entangled Markov processes based on random walks on discrete groups. We note that most of such Markov chains generate merely locally normal weights on $\mathfrak{M}$. For some basic facts about random walks on groups, see e.g. [20].

Let $G$ be a discrete group and $\mu$ a probability measure on it which is kept fixed during the analysis. The right and left random walks on $G$ are given by the doubly stochastic transition matrices $\Pi^{r}, \Pi^{l}$ respectively, with

$$
\Pi_{g h}^{r}:=\mu\left(g^{-1} h\right), \quad \Pi_{g h}^{l}:=\mu\left(g h^{-1}\right)
$$

[^5]Let $P^{r}, P^{l}$ be the corresponding entangled Markov operators obtained by (3.2)..$^{7}$ An easy computation gives rise for $g \in G$,

$$
\begin{equation*}
\operatorname{ad}(\lambda(g)) \circ P^{r}=P^{r} \circ \operatorname{ad}(\lambda(g)), \operatorname{ad}(\rho(g)) \circ P^{l}=P^{l} \circ \operatorname{ad}(\rho(g)), \tag{5.2}
\end{equation*}
$$

where $\lambda, \rho$ are the left and right translations on $G$. Here, (5.2) follows by the corresponding equivariance properties of $\Pi^{r}, \Pi^{l}$. Denote by $R(G), L(G)$ the von Neumann algebras generated by the right and left translations on $G$.

Proposition 5.2. We have for the entangled Markov operators $P^{r}, P^{l}$, and for the transition expectations $\mathcal{E}^{r}, \mathcal{E}^{l}$ constructed as in (3.1),

$$
\begin{aligned}
& P^{r}(R(G)) \subset R(G), \quad P^{l}(L(G)) \subset L(G) \\
& \mathcal{E}^{r}(R(G) \bar{\otimes} R(G)) \subset R(G), \quad \mathcal{E}^{l}(L(G) \bar{\otimes} L(G)) \subset L(G)
\end{aligned}
$$

Proof. Taking into account that $\rho(g)_{x y}=\delta_{x, y g^{-1}}, \lambda(g)_{x y}=\delta_{x, g y}$, we easily obtain

$$
P^{r}(R(G)) \subset L(G)^{\prime} \equiv R(G), \quad P^{l}(L(G)) \subset R(G)^{\prime} \equiv L(G)
$$

The proof follows as, for the Schur multiplication,

$$
m(R(G) \bar{\otimes} R(G)) \subset R(G), \quad m(L(G) \bar{\otimes} L(G)) \subset L(G)
$$

We end by noticing that, starting with a discrete ICC group, one can construct entangled Markov processes on $\bigcup_{J \subset \mathbb{Z}} \bar{\otimes}_{J} R C^{C^{*}}(J$ runs on all finite subsets of $\mathbb{Z}), R$ being the algebra $L(G)$ generated by left translations (equally well the algebra $R(G)$ generated by right translations). Then, we provide entangled Markov processes based on type $\mathrm{II}_{1}$ factors.

We hope to return elsewhere on all the questions left open in this section.

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[^6]
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[^0]:    ${ }^{1}$ In the infinite dimensional case, an invariant distribution does not always exist in general.

[^1]:    ${ }^{2}$ The entangled operator $P$ given in (3.2) is not bounded in general. However, we define the entangled Markov chain directly in terms of the transition expectation given in (3.1). This allows us to treat entangled Markov chains arising from any stochastic matrix without affecting our analysis, see Proposition 3.1

[^2]:    ${ }^{3}$ An invariant measure for the stochastic matrix $\Pi$ always exists if the latter contains recurrent indices ("states" in the terminology of classical Markov chains), see e.g. [10], Theorem 6.2.25 and Theorem 5.3.14.

[^3]:    ${ }^{4}$ This is always the case in the finite dimensional case, where the class of recurrent-null states is void by finiteness, see [22] for further details.

[^4]:    ${ }^{5}$ The state $\varphi_{\lambda}$ is only $m_{\lambda}$-step translation invariant, $m_{\lambda}$ being the period of the component $\lambda$, and keeps track of the localization (modulo a period), see [2], Section 5 for the precise way to define $\varphi_{\lambda}$ on $\mathfrak{M}$.

[^5]:    ${ }^{6}$ This non generic situation exhibits a low degree of entanglement. These examples are then not suitable for possible applications to quantum information theory.

[^6]:    ${ }^{7} \mathrm{As} \Pi^{r}, \Pi^{l}$ are doubly stochastic, the associated entangled operators $P^{r}, P^{l}$ are bounded.

