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Infinitely many solutions for p-biharmonic equation with general potential and concave-convex nonlinearity in $\mathbb{R}^N$

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Abstract

In this paper, we study the existence of multiple solutions to a class of p-biharmonic elliptic equations, $\Delta_p^2 u - \Delta_p u + V(x)|u|^{p-2}u = \lambda h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u$, $x \in \mathbb{R}^N$, where $1 < m < p < q < p_* = \frac{pN}{N-2p}$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is a p-biharmonic operator and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The potential function $V(x) \in C(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) > 0$. By variational methods, we obtain the existence of infinitely many solutions for a p-biharmonic elliptic equation in $\mathbb{R}^N$.

Keywords: p-biharmonic; elliptic equation; variational methods; mountain pass lemma

1 Introduction

In this paper, we are interested in the existence of solutions to the following p-biharmonic elliptic equation:

$$\Delta_p^2 u - \Delta_p u + V(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $2 < 2p < N$, $f(x,u) = \lambda h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u$, $1 < m < p < q < p_* = \frac{pN}{N-2p}$, $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is a p-biharmonic operator and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The potential function $V(x) \in C(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) > 0$.

Recently, the nonlinear biharmonic equation in an unbounded domain has been extensively investigated, we refer the reader to [1–9] and the references therein. For the whole space $\mathbb{R}^N$ case, the main difficulty of this problem is the lack of compactness for the Sobolev embedding theorem. In order to overcome this difficulty, the authors always assumed the potential $V(x)$ has some special characteristic. For example, in [4], Yin and Wu studied the following fourth-order elliptic equation:

$$\begin{cases} \Delta^2 u - \Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u(x) \in H^2(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where the potential $V(x)$ satisfied

(V_0): $V \in C(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) > 0$, and for each $M > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N \leq M\} < +\infty$.

This assumption guarantees that the embedding $H^2 \hookrightarrow L^s(\mathbb{R}^N)$ is compact for each $s \in [2, \frac{2N}{N-4})$ and obeys the coercivity condition: $V(x) \to \infty$ as $|x| \to \infty$. Hence, under various sets of assumptions on the nonlinear term $f(x,t)$ (subcriticality, superquadraticity, *etc.*), the authors proved the existence of infinitely many solutions to problem (1.2) by using the variational techniques in a standard way. In [2], Liu *et al.* considered the following fourth-order elliptic equation:

$$\begin{cases} \Delta^2 u - \Delta u + \lambda V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u(x) \in H^2(\mathbb{R}^N), \end{cases} \tag{1.3}$$

where the potential $V(x)$ satisfied a weaker condition than (V_0), that is,

(V_1): $V \in C(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) > 0$, there exists some $M > 0$, $\text{meas}\{x \in \mathbb{R}^N \leq M\} < +\infty$.

Under the assumption (V_1), the compactness of the embedding is lost and this renders variational techniques more delicate. With the aid of the parameter $\lambda > 0$, they proved that the variational functional satisfies (PS) condition, and then they showed the existence and multiplicity results of problem (1.3). A natural question is whether the existence results still holds if we assume a more general potential $V(x)$ than (V_0), (V_1), namely,

(V): $V(x) \in C(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) > 0$.

In the present paper, we will answer this interesting question. We consider the existence of solutions to the p-biharmonic problem (1.1) with a more general potential $V(x)$. To prove that the (PS) sequence weakly converges to a critical point of the corresponding functional, we adapt ideas developed by [10–12] and then by variational methods, we establish the existence of infinitely many high-energy solutions to problem (1.1) with a concave-convex nonlinearity, *i.e.*, $f(x,u) = \lambda h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u$, $1 < m < p < q < p_* = \frac{pN}{N-2p}$. To the best of our knowledge, little has been done for p-biharmonic problems with this type of nonlinearity. Here, we give our assumptions on the weight functions $h_1(x)$ and $h_2(x)$:

(H_1) $h_1 \in L^\sigma(\mathbb{R}^N)$ with $\sigma = \frac{p}{p-m}$;
(H_2) $h_2(x) \geq 0$ $(\not\equiv 0)$, $h_2(x) \in L^\infty(\mathbb{R}^N)$.

The main result in this paper is as follows.

Theorem 1.1 *Let* $2 < 2p < N$, $1 < m < p < q < p_* = \frac{pN}{N-2p}$. *Assume* (V), (H_1), *and* (H_2) *hold. Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda \in [0, \lambda_0]$, *problem* (1.1) *admits infinitely many high-energy solutions in* $\mathbb{R}^N$.

This paper is organized as follows. In Section 2, we build the variational framework for problem (1.1) and establish a series of lemmas, which will be used in the proof of Theorem 1.1. In Section 3, we prove Theorem 1.1 by the mountain pass theorem [13].

2 Preliminaries

In order to apply the variational setting, we assume the solutions of (1.1) belong to the following subspace of $\mathcal{D}^{2,p}(\mathbb{R}^N)$:

$$E = \left\{ u \in \mathcal{D}^{2,p}(\mathbb{R}^N) \,\Big|\, \int_{\mathbb{R}^N} |\Delta u|^p + |\nabla u|^p + V(x)|u|^p \, dx < \infty \right\} \tag{2.1}$$

endowed with the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} \left(|\Delta u|^p + |\nabla u|^p + V(x)|u|^p \right) dx \right)^{1/p}, \tag{2.2}$$

where $\mathcal{D}^{2,p}(\mathbb{R}^N) = \{u \in L^{p_*}(\mathbb{R}^N) | \Delta u \in L^p(\mathbb{R}^N)\}$, $\|\cdot\|_s$ means the norm in $L^s(\mathbb{R}^N)$.

We denote by S_* the Sobolev constant, that is,

$$S_* = \inf_{u \in \mathcal{D}^{2,p} \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\Delta u|^p \, dx}{(\int_{\mathbb{R}^N} |u|^{p_*} \, dx)^{p/p_*}} \tag{2.3}$$

and

$$S_* \left(\int_{\mathbb{R}^N} |u|^{p_*} \, dx \right)^{p/p_*} \le \int_{\mathbb{R}^N} |\Delta u|^p \, dx, \quad \forall u \in \mathcal{D}^{2,p}(\mathbb{R}^N), \tag{2.4}$$

where S_* is obtained by a positive and radially symmetric function; see for instance [14].

Definition 2.1 A function $u \in E$ is said to be a weak solution of (1.1) if, for any $\varphi \in E$, we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\Delta u|^{p-2} \Delta u \Delta \varphi + |\nabla u|^{p-2} \nabla u \varphi + V|u|^{p-2} u \varphi \right) dx \\ &\quad = \int_{\mathbb{R}^N} \left(\lambda h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u \right) \varphi \, dx. \end{aligned} \tag{2.5}$$

Let $J(u) : E \to \mathbb{R}$ be the energy functional associated with problem (1.1) defined by

$$J(u) = \frac{1}{p}\|u\|_E^p - \frac{\lambda}{m} \int_{\mathbb{R}^N} h_1|u|^m \, dx - \frac{1}{q} \int_{\mathbb{R}^N} h_2|u|^q \, dx. \tag{2.6}$$

From the embedding inequality (2.4) and the assumptions in Theorem 1.1, we see the functional $J \in C^1(E, \mathbb{R})$ and its Gateaux derivative is given by

$$\begin{aligned} J'(u)\varphi = &\int_{\mathbb{R}^N} \left(|\Delta u|^{p-2} \Delta u \Delta \varphi + |\nabla u|^{p-2} \nabla u \varphi + V(x)|u|^{p-2} u \varphi \right) dx \\ &- \int_{\mathbb{R}^N} \left(\lambda h_1(x)|u|^{m-2}u + h_2(x)|u|^{q-2}u \right) \varphi \, dx. \end{aligned} \tag{2.7}$$

To prove the existence of infinitely many solutions to problem (1.1), we need to prove that the functional J defined by (2.6) satisfies the (PS) condition. Recall that a sequence $\{u_n\}$ in E is called a $(PS)_c$ sequence of J if

$$J(u_n) \to c, \qquad J'(u_n) \to 0 \quad \text{in } E^* \text{ as } n \to \infty. \tag{2.8}$$

The functional J satisfies the (PS) condition if any $(PS)_c$ sequence possesses a convergent subsequence in E.

Lemma 2.1 *Assume* (V), (H_1), *and* (H_2) *hold. If* $\{u_n\} \subset E$ *is a* $(PS)_c$ *sequence of* J, *then* $\{u_n\}$ *is bounded in* E.

Proof It follows from Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{R}^N} |h_1||u_n|^m\,dx &\le V_0^{-\frac{m}{p}}\left(\int_{\mathbb{R}^N}|h_1|^\sigma\,dx\right)^{\frac{1}{\sigma}}\left(\int_{\mathbb{R}^N}V|u_n|^p\,dx\right)^{\frac{m}{p}}\\ &\le a_1\|u_n\|_E^m,\end{aligned}\tag{2.9}$$

where $a_1 = V_0^{-m/p}\|h_1\|_\sigma$. Choose $t\in(0,1)$ such that $q = pt + (1-t)p_*$, then

$$\begin{aligned}\int_{\mathbb{R}^N} h_2|u_n|^q\,dx &\le \left(\int_{\mathbb{R}^N}V|u_n|^p\,dx\right)^t\left(\int_{\mathbb{R}^N}|u_n|^{p_*}h_2^{-\frac{1}{1-t}}V^{-\frac{t}{1-t}}\,dx\right)^{1-t}\\ &\le a_2\left(\int_{\mathbb{R}^N}V|u_n|^p\,dx\right)^t\|\Delta u_n\|_p^{(1-t)p_*}\le a_2\|u_n\|_E^q,\end{aligned}\tag{2.10}$$

where $a_2 = S_*^{-p_*(q-p)/p(p_*-p)}V_0^{-t}\|h_2\|_\infty$. Thus,

$$\begin{aligned}c+1+\|u_n\|_E &\ge J(u_n) - q^{-1}J'(u_n)u_n\\ &\ge \left(\frac{1}{p}-\frac{1}{q}\right)\|u\|_E^p - \lambda\left(\frac{1}{m}-\frac{1}{q}\right)\int_{\mathbb{R}^N}|h_1||u|^m\,dx\\ &\ge \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|_E^p - \lambda\left(\frac{1}{m}-\frac{1}{q}\right)a_1\|u_n\|_E^m.\end{aligned}\tag{2.11}$$

Since $1 < m < p < q$, we conclude that $\|u\|_E$ is bounded and the proof is complete. □

In the following, we shall show that $\{u_n\}$ has a convergent subsequence in E. Since the sequence $\{u_n\}$ given by (2.8) is a bounded sequence in E, there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $v\in E$ such that $\|u_n\|_E\le M$, $\|v\|_E\le M$, and

$$\begin{aligned}&u_n \rightharpoonup v\quad\text{weakly in } E,\\ &u_n\to v\quad\text{in } L^s_{\mathrm{loc}}(\mathbb{R}^N), 1<s<p_*,\\ &u_n(x)\to v(x)\quad\text{a.e. in } \mathbb{R}^N.\end{aligned}\tag{2.12}$$

Lemma 2.2 *Assume* (V), (H_1), *and* (H_2) *hold. If the sequence* $\{u_n\}$ *is bounded in E satisfying* (2.12), *then*

(i) $\lim_{n\to\infty}\int_{\mathbb{R}^N}h_1(x)|u_n|^m\,dx = \int_{\mathbb{R}^N}h_1(x)|v|^m\,dx$, $\lim_{n\to\infty}\int_{\mathbb{R}^N}h_1(x)|u_n-v|^m\,dx = 0$;
(ii) $\lim_{n\to\infty}\int_{\mathbb{R}^N}h_2(x)|u_n|^q\,dx = \int_{\mathbb{R}^N}h_2(x)|v|^q\,dx$, $\lim_{n\to\infty}\int_{\mathbb{R}^N}h_2(x)|u_n-v|^q\,dx = 0$.

Proof (i) In fact, from $h\in L^\sigma(\mathbb{R}^N)$ and (2.12), we obtain, for any $r>0$,

$$\int_{B_r}h_1(x)|u_n|^m\,dx\to\int_{B_r}h_1(x)|v|^m\,dx\quad\text{as } n\to\infty,\tag{2.13}$$

where and in the sequel $B_r = \{x\in\mathbb{R}^N : |x| < r\}$, $B_r^c = \mathbb{R}^N\setminus\overline{B}_r$. On the other hand, we see from the Hölder inequality that

$$\begin{aligned}\int_{B_r^c}|h_1||u_n|^m\,dx &\le V_0^{-\frac{m}{p}}\left(\int_{B_r^c}|h_1|^\sigma\,dx\right)^{\frac{1}{\sigma}}\left(\int_{B_r^c}V|u_n|^p\,dx\right)^{\frac{m}{p}}\\ &\le V_0^{-\frac{m}{p}}\|h_1\|_{L^\sigma(B_r^c)}\|u_n\|_E^m\le V_0^{-\frac{m}{p}}M^m\|h_1\|_{L^\sigma(B_r^c)}\to 0\end{aligned}\tag{2.14}$$

as $r \to \infty$. By Fatou's lemma, we see that, as $n \to \infty$,

$$\int_{B_r^c} |h_1||v|^m\,dx \le \liminf_{n\to\infty} \int_{B_r^c} |h_1||u_n|^m\,dx \le V_0^{-\frac{m}{p}} M^m \|h_1\|_{L^\sigma(B_r^c)} \to 0. \tag{2.15}$$

Then, the application of (2.13)-(2.15) gives the first limit of (i). Furthermore, by the Brezis-Lieb lemma in [15], we have the second limit of (i).

(ii) To prove the conclusion (ii), we follow the argument used in [10–12]. Here, we give a detailed proof for the reader's convenience.

Since $p < p < p_*$, it easy to see that, for any small $\varepsilon > 0$, there exist $S_0 > s_0 > 0$ such that $|s|^q < \varepsilon|s|^p$ if $|s| \le s_0$ and $|s|^q \le \varepsilon|s|^{p_*}$, if $|s| \ge S_0$. This shows that

$$|s|^q \le \varepsilon\bigl(|s|^p + |s|^{p_*}\bigr) + \chi_{[s_0,S_0]}\bigl(|s|\bigr)|s|^q, \quad \forall s \in \mathbb{R}. \tag{2.16}$$

Denote $A_n = \{x \in \mathbb{R}^N; s_0 \le |u_n(x)| \le S_0\}$. It follows from (2.16), (2.4), and (2.12) that

$$\begin{aligned}\int_{B_r^c} |h_2||u_n|^q\,dx &\le \|h_2\|_\infty \int_{B_r^c} \bigl(\varepsilon\bigl(|u_n|^p + |u_n|^{p_*}\bigr) + \chi_{[s_0,S_0]}\bigl(|u_n|\bigr)|u_n|^q\bigr)\,dx \\ &\le \varepsilon\|h_2\|_\infty \left(\int_{\mathbb{R}^N} V_0^{-1}V(x)|u_n|^p\,dx + S^{-\frac{p_*}{p}}\|\Delta u\|_p^{p_*}\right) \\ &\quad + S_0^q\|h_2\|_\infty \operatorname{meas}\bigl(A_n \cap B_r^c\bigr) \\ &\le M_1\varepsilon + S_0^q\|h_2\|_\infty \operatorname{meas}\bigl(A_n \cap B_r^c\bigr)\end{aligned} \tag{2.17}$$

with some constant $M_1 > 0$, and

$$|s_0|^{p_*}|A_n| \le \int_{\mathbb{R}^N} |u_n|^{p_*}\,dx \le M_1, \quad \forall n \in \mathbb{N}, \tag{2.18}$$

where $|A_n| = \operatorname{meas}(A_n)$. Equation (2.18) implies that $\sup_{n\in\mathbb{N}} |A_n| \le M_1|s_0|^{-p_*} < \infty$, so it is easy to see that

$$\lim_{r\to\infty} \operatorname{meas}\bigl(A_n \cap B_r^c\bigr) = 0, \quad \text{for all } n \in \mathbb{N}. \tag{2.19}$$

In the following, we show that $\lim_{r\to\infty} \operatorname{meas}(A_n \cap B_r^c) = 0$ uniformly in $n \in \mathbb{N}$.

In fact, it follows from (2.12) that $v \in L^p(\mathbb{R}^N)$ and $u_n(x) \to v(x)$ a.e. $\mathbb{R}^N$. Therefore, for any small $\varepsilon > 0$, there exists $r_0 > 1$ such that $r \ge r_0$,

$$\int_{B_r^c} |v|^p\,dx \le \varepsilon.$$

For this ε, we choose $t_1 = r_0$, $t_j \uparrow \infty$ such that $D_j = B_{t_j}^c \setminus \overline{B}_{t_{j+1}}^c$, $B_{r_0}^c = \bigcup_{j=1}^\infty D_j$ and

$$\int_{D_j} |v|^p\,dx \le \frac{\varepsilon}{2^j}, \quad \forall j \in \mathbb{N}.$$

Obviously, for every fixed $j \in N$, D_j is a bounded domain and $D_j \cap D_i = \emptyset$ $(j \ne i)$. Furthermore, $s_0 \le |u_n| \le S_0$ in $D_j \cap A_n$. By Fatou's lemma, we have, for every $j \in \mathbb{N}$,

$$\limsup_{n\to\infty} \int_{D_j\cap A_n} |u_n|^p\,dx \le \int_{D_j} \limsup_{n\to\infty} |u_n|^p\,dx \le \int_{D_j} |v|^p\,dx \le \frac{\varepsilon}{2^j}.$$

Then, for $s_1 = 2^{1-q}s_0^q$, we obtain

$$\begin{aligned}
s_1 \limsup_{n\to\infty} \left|A_n \cap B_{r_0}^c\right| &\leq \limsup_{n\to\infty} \int_{B_{r_0}^c \cap A_n} |u_n|^p \, dx \\
&= \limsup_{n\to\infty} \sum_{j=1}^{\infty} \int_{D_j \cap A_n} |u_n|^p \, dx \\
&\leq \sum_{j=1}^{\infty} \limsup_{n\to\infty} \int_{D_j \cap A_n} |u_n|^p \, dx \\
&\leq \sum_{j=1}^{\infty} \int_{D_j} |v|^p \, dx \leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon. \qquad (2.20)
\end{aligned}$$

Notice that, for any $r \geq r_0$ and $n \in \mathbb{N}$, we have $(A_n \cap B_r^c) \subset (A_n \cap B_{r_0}^c)$. Therefore, the application of (2.19) and (2.20) yields $\lim_{r\to\infty} |A_n \cap B_r^c| = 0$ uniformly in $n \in \mathbb{N}$. Thus, for any $\varepsilon > 0$, there exists $r_0 \geq 1$ such that $\operatorname{meas}(A_n \cap B_r^c) < \frac{\varepsilon}{S_0^q \|h_2\|_\infty}$, for $r \geq r_0$. Then it follows from (2.17) that

$$\int_{B_r^c} h_2 |u_n|^q \, dx \leq \max\{M_1, 1\}\varepsilon, \quad \forall n \in \mathbb{N}, r \geq r_0 \qquad (2.21)$$

and

$$\int_{B_r^c} h_2 |v|^q \, dx \leq \liminf_{n\to\infty} \int_{B_r^c} h_2 |u_n|^q \, dx \leq \max\{M_1, 1\}\varepsilon, \quad r \geq r_0. \qquad (2.22)$$

Moreover, we derive from (2.12) that

$$\int_{B_r} h_2(x) |u_n|^q \, dx \to \int_{B_r} h_2(x) |v|^q \, dx. \qquad (2.23)$$

Therefore, using (2.21) and (2.22), and the application of Brezis-Lieb lemma in [15] we conclude the second limit of (ii). Then the proof is complete. □

Lemma 2.3 *Let* $\{u_n\}$ *be a* $(PS)_c$ *sequence satisfying* (2.12), *then* $u_n \to v$ *in* E, *that is, the functional* J *satisfies the* (PS) *condition.*

Proof Denote

$$\begin{aligned}
P_n &= J'(u_n)(u_n - v) \\
&= \int_{\mathbb{R}^N} \big(|\Delta u_n|^{p-2} \Delta u_n \Delta(u_n - v) + |\nabla u_n|^{p-2} \nabla u_n \nabla(u_n - u) \\
&\quad + V(x)|u_n|^{p-2} u_n (u_n - v)\big) \, dx \\
&\quad - \int_{\mathbb{R}^N} \big(\lambda h_1(x)|u_n|^{m-2} u_n + h_2(x)|u_n|^{q-2} u_n\big)(u_n - v) \, dx.
\end{aligned}$$

Then the fact $J'(u_n) \to 0$ in E^* shows that $P_n \to 0$ as $n \to \infty$. Moreover, the fact $u_n \rightharpoonup v$ in E implies $Q_n \to 0$, where

$$Q_n = \int_{\mathbb{R}^N} \big(|\Delta v|^{p-2} \Delta v \Delta(u_n - v) + |\nabla u|^{p-2} \nabla u \nabla(u_n - u) + V(x)|v|^{p-2} v (u_n - v)\big) \, dx.$$

It follows from the Hölder inequality and the limit (i) in Lemma 2.2 that

$$\int_{\mathbb{R}^N} |h_1(x)||u_n|^{m-1}|u_n - v|\,dx \le \left(\int_{\mathbb{R}^N} |h_1(x)||u_n - v|^m\,dx\right)^{\frac{1}{m}} \left(\int_{\mathbb{R}^N} |h_1(x)||u_n|^m\,dx\right)^{\frac{m-1}{m}} \to 0. \tag{2.24}$$

Similarly, we can derive from the limit (ii) in Lemma 2.2 that

$$\int_{\mathbb{R}^N} h_2(x)|u_n|^{q-1}|u_n - v|\,dx \le \left(\int_{\mathbb{R}^N} h_2(x)|u_n - v|^q\,dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} h_2(x)|u_n|^q\,dx\right)^{\frac{q-1}{q}} \to 0. \tag{2.25}$$

Then (2.24) and (2.25) show that as $n \to \infty$

$$\begin{aligned} o_n(1) &= P_n - Q_n \\ &= \int_{\mathbb{R}^N} \Big(\big(|\Delta u_n|^{p-2}\Delta u_n - |\Delta v|^{p-2}\Delta v\big)\Delta(u_n - v) \\ &\quad + \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v\big)\nabla(u_n - v) \\ &\quad + V(x)\big(|u_n|^{p-2}u_n - |v|^{p-2}v\big)(u_n - v)\Big)\,dx. \end{aligned} \tag{2.26}$$

Then we have $\|u_n - v\|_E \to 0$ as $n \to \infty$. Thus $J(u)$ satisfies the (PS) condition on E and the proof is completed. □

3 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. We assume that all conditions in the theorem hold. The proof mainly relies on the mountain pass theorem.

Lemma 3.1 ([13]) *Let E be an infinite dimensional real Banach space, $J \in C^1(E, \mathbb{R})$ be even and satisfies the (PS) condition and $J(0) = 0$. Assume $E = Y \oplus Z$, Y is finite dimensional, and J satisfies*:

- (J_1) *There exist constants $\rho, \alpha > 0$ such that $J(u) \le \alpha$ on $\partial B_\rho \cap Z$.*
- (J_2) *For each finite dimensional subspace $E_0 \subset E$, there is an $R_0 = R_0(E_0)$ such that $J(u) \le 0$ on $E_0 \setminus B_{R_0}$, where $B_r = \{u \in E : \|u\|_E < r\}$.*

Then J possesses an unbounded sequence of critical values.

Proof of Theorem 1.1 Clearly, the functional J defined by (2.6) is even in E. By Lemma 2.2 in Section 2, the functional satisfies the (PS) condition. Next, we prove that J satisfies (J_1) and (J_2). From (2.9) and (2.10), it follows that

$$J(u) \ge \frac{1}{p}\|u\|_E^p - \lambda\frac{a_1}{m}\|u\|_E^m - \frac{a_2}{q}\|u\|_E^q, \quad u \in E.$$

Denote

$$\phi(z) = z^p\left(\frac{1}{p} - \lambda\frac{a_1}{m}z^{m-p} - \frac{a_2}{q}z^{q-p}\right), \quad z > 0.$$

Then, there exist $\lambda_0, z_1, \alpha > 0$ such that $\phi(z_1) \geq \alpha$ for any $\lambda \in [0, \lambda_0]$. Let $\rho = z_1$, we have $J(u) \geq \alpha$ with $\|u\|_E = \rho$ and $\lambda \in [0, \lambda_0]$. So the condition (J_1) is satisfied.

We now verify (J_2). For any finite dimensional subspace $E_0 \subset E$, we assert that there is a constant $R_0 > \rho$ such that $J < 0$ on $E_0 \setminus B_{R_0}$. Otherwise, there exists a sequence $\{u_n\} \subset E_0$ such that $\|u\|_n \to \infty$ and $J(u_n) \geq 0$. Hence

$$\frac{1}{p}\|u_n\|_E^p \geq \frac{\lambda}{m}\int_{R^N} h_1|u_n|^m\,dx + \frac{1}{q}\int_{\mathbb{R}^N} h_2\|u_n\|^q\,dx. \tag{3.1}$$

Set $\omega_n = \frac{u_n}{\|u_n\|_E}$. Then up to a sequence, we can assume $\omega_n \rightharpoonup \omega$ in E, $\omega_n \to \omega$ a.e. in $\mathbb{R}^N$. Denote $\Omega = \{x \in \mathbb{R}^N : \omega(x) \neq 0\}$. Assume $|\Omega| > 0$. Clearly, $u_n(x) \to \infty$ in Ω. It follows from (2.8) and (2.9) that

$$\|u\|_E^{-p}\int_\Omega |h_1||u_n|^m\,dx \leq a_1\|u_n\|_E^{m-p} \to 0 \quad \text{as } n \to \infty.$$

On the other hand, we derive

$$\|u\|_E^{-p}\int_\Omega |h_2||u_n|^q\,dx \leq a_2\|u_n\|_E^{q-p} \to \infty \quad \text{as } n \to \infty.$$

Therefore, multiplying (3.1) by $\|u\|_E^{-p}$ and passing to the limit as $n \to \infty$ show that $\frac{1}{p} \geq \infty$. This is impossible. So $|\Omega| = 0$ and $\omega(x) = 0$ a.e. on $\mathbb{R}^N$. By the equivalence of all norms in E_0, there exists a constant $\beta > 0$ such that

$$\int_{\mathbb{R}^N} |h_2||u|^q\,dx \geq \beta^q\|u\|_E^q, \quad \forall u \in E_0 \quad \text{and} \quad \int_{\mathbb{R}^N} |h_2||u_n|^q\,dx \geq \beta^q\|u_n\|_E^q, \quad \forall n \in \mathbb{N}.$$

Hence

$$0 < \beta^q \leq \limsup_{n\to\infty}\int_{\mathbb{R}^N} |h_2||\omega_n|^q\,dx \leq \int_{\mathbb{R}^N}\limsup_{n\to\infty}\bigl(|h_2||\omega_n|^q\bigr)\,dx = \int_{\mathbb{R}^N}\bigl(|h_2||\omega|^q\bigr)\,dx = 0.$$

This is a contradiction. So there exists a constant R_0 such that $J < 0$ on $E_0 \setminus B_{R_0}$. Therefore, the existence of infinitely many solutions $\{u_n\}$ for problem (1.1) follows from Lemma 3.1 and we finish the proof of Theorem 1.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this study equally. All authors read and approved the final vision of the manuscript.

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