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Periodic solutions of second-order differential equations with multiple delays

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Abstract

By using the critical point theory and S^1 index theory, we obtain a new result for the existence and multiplicity of periodic solutions for a class of second-order delay differential equations $x''(t) = f(x(t))-[f(x(t-1))+f(x(t-2))+...+f(x(t-(N-1)))]$.

Keywords: delay differential equations, multiple periodic so1utiois, critical point theory, index theory

1 Introduction

Inspired by the excellent study in [1], many authors [2-16] studied the following differential delay equations

$$x'(t) = -\left[f\left(x(t-1)\right) + f\left(x(t-2)\right) + \cdots + f\left(x\left(t-(N-1)\right)\right)\right], \tag{1.1}$$

where $f \in C(\mathbb{R}, \mathbb{R})$,and $N \geq 2$ is an integer.

Kaplan and Yorke [2] introduced a technique of couple system which allows them to reduce the search for periodic solutions of a differential delay equation to the problem of finding periodic solutions for a related system of ordinary differential equations. They study periodic solutions of (1.1) with $N = 2$, $f \in C(\mathbb{R}, \mathbb{R})$ is odd, $xf(x) > 0$ for $x \neq 0$ and f satisfies some suitable conditions near 0 and ∞. More precisely, if the solution $x(t)$ of (1.1) with $N = 2$ satisfies $x(t) = -x(t-2)$, let

$$x_1(t) = x(t),\ x_2(t) = x(t-1), \tag{1.2}$$

then $X(t) = (x_1(t), x_2(t))^T$ satisfies

$$X'(t) = A_2 \nabla H(X), \tag{1.3}$$

where, $A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

i.e., A_2 is a skew symmetric matrix, and

$$H(X) = \int_0^{x_1} f(s)ds + \int_0^{x_2} f(s)ds. \tag{1.4}$$

$\nabla H(X)$ is the gradient of H.

In fact, by direct computation, one has the following proposition:

Proposition 1.1. *(i) Any solution $x(t)$ of (1.1) with $N = 2$, and $x(t) = -x(t - 2)$ will give a solution of (1.3) $X(t) = (x_1(t), x_2(t))^T$ by (1.2). Moreover, $X(t)$ has the following symmetric structure*

$$x_1(t) = -x_2(t-1),\ x_2(t) = x_1(t-1), \tag{1.5}$$

(ii) Any solution $X(t) = (x_1(t), x_2(t))^T$ of (1.3) with the symmetric structure (1.5) will give a solution of (1.1) by letting $x(t) = x_1(t)$. Moreover, $x(t + 2) = -x(t)$.

Kaplan and Yorke proved that (1.3) has periodic solutions with the symmetric structure (1.5), which give the Kaplan-Yorke type periodic solutions of (1.1) with period 4, i.e., $x(t)$ satisfying $x(t) = -x(t - 2)$. They further conjectured that similar result should be true for the general case $N \geq 2$, i.e., under similar conditions for f, (1.1) has a $2N$-periodic Kaplan-Yorke type periodic solution $x(t)$, i.e., $x(t)$ satisfying $x(t) = -x(t - N)$.

Li and He [6-8], in an attempt to reuse Kaplan and Yorke's original idea, applied Lyapunov Center Theorem and some known results about convex Hamiltonian systems [17, Theorem 7.2] to obtain 4-periodic solutions of (1.4). But those 4-periodic solutions obtained by [17, Theorem 7.2] give no information about the symmetric structure (1.5) or the minimal period. The solutions of (1.3), which will not generate noncontact solutions of (1.1), see [14, Remark 3.3].

Herz [12] study (1.1) with $N = 2$ by Lyapunov direct method. And, Jekel and Johnston [13], proved the existence of a $2N$-periodic Kaplan-Yorke type periodic solution for (1.1) by Kaplan-Yorke original method and homotopic method.

Fei [14,15] applied the pseudo-index theory [17-20] to obtain periodic solution in a subspace, which surely have the required symmetric structure (1.5) and give solutions to (1.1).

In recent years, Guo and Yu [16] considered (1.1) with $N = 2$ by variational methods directly, and they obtain the Kaplan-Yorke type periodic solutions. That is to say that they do not necessarily transform the existence problem of (1.1) to existence problems for related systems (1.3). Afterwards, Cheng and Hu [21] studied (1.1) with $N = 2$ by Guo-Yu's method in [16]. Guo [22] studied the following second-order differential delay equation by Guo-Yu's method in [16]

$$x''(t) = -f(x(t-r)), \tag{1.6}$$

they obtained the multiplicity results for periodic solutions, but the solutions are not Kaplan- Yorke type.

The authors [23,24] considered the Kaplan-Yorke type periodic solutions of (1.1) with $N = 2$ by Maslov-type index [25] and Morse theory [26], respectively.

Recently, some researchers [27-31] have begun to study the existence of solutions for second-order differential delay equation by using a variational method. However, to the best of authors' knowledge, the study of Kaplan-Yorke type periodic solutions of second- order differential delay equation using a variational method has received considerably less attention. We find the method apply in [6-15], such as the structure of variational does not directly apply to second-order differential delay equation.

Motivated by the study in [7-16,21-24,27-32], in this article we are concerned with the existence of Kaplan-Yorke type periodic solutions of the following second-order differential delay equations

$$x''(t) = f(x(t)) - \left[f(x(t-1)) + f(x(t-2)) + \cdots + f(x(t-(N-1)))\right], \tag{1.7}$$

where $f \in C(\mathbb{R},\mathbb{R})$ is odd, and $N \geq 2$ is an integer

If $x(t) = x(t+N)$, let

$$x_1(t) = x(t), x_2(t) = x(t-1), \ldots, x_N(t) = x(t-(N-1)), \tag{1.8}$$

then $z(t) = (x_1(t), x_2(t),\ldots, x_N(t))^T$ satisfies

$$z''(t) = A_N \nabla H(z), \tag{1.9}$$

where $A_N = \begin{pmatrix} 1 & -1 & \cdots & -1 \\ -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & 1 \end{pmatrix}$

i.e., A_N is a $n \times n$ symmetric matrix, and

$$H(z) = \int_0^{x_1} f(s)\,ds + \int_0^{x_2} f(s)\,ds + \cdots + \int_0^{x_N} f(s)\,ds. \tag{1.10}$$

In fact, by direct computation, one has the following proposition.

Proposition 1.2. *(i) Any solution $x(t)$ of (1.7) with $x(t) = x(t - N)$ will give a solution of (1.9) $X(t) = (x_1(t), x_2(t), . . . , x_N(t))^T$ by (1.8). Moreover, $X(t)$ has the following symmetric structure*

$$\begin{aligned} &x_1(t) = x_N(t-1), \; x_2(t) = x_1(t-1), \\ &x_3(t) = x_2(t-1), \ldots, x_N(t) = x_{N-1}(t-1). \end{aligned} \tag{1.11}$$

(ii) Any solution $X(t) = (x_1(t), x_2(t), . . . , x_N(t))^T$ of (1.9) with the symmetric structure (1.11) will give a solution of (1.7) by letting $x(t) = x_1(t)$. Moreover, $x(t + N) = x(t)$.

Throughout this article, we always assume that:

($f1$) $f \in C(\mathbb{R}, \mathbb{R})$ is odd and $0 < \alpha, \beta < +\infty$

$$\lim_{x\to 0} \frac{f(x)}{x} = \alpha, \quad \lim_{x\to\infty} \frac{f(x)}{x} = \beta;$$

($f\,2^{\pm}$) $|f(x) - \beta x|$ is bounded and $G_\beta(x) \to \pm\infty$, as $|x| \to \infty$;

($f\,3^{\pm}$) $\pm G\alpha(x) > 0$ for $|x| > 0$ being small,

where $F(x) = \int_0^x f(s)\,ds$, and

$$G_\beta(x) = F(x) - \frac{1}{2}\beta x^2, \; G_\alpha(x) = F(x) - \frac{1}{2}\alpha x^2. \tag{1.12}$$

Similarly to the argument in [14,20,33], for given a number $\alpha \in \mathbb{R}$, and $N \geq 2$, $k \geq 1$ being two integer, we set:

$$i(\alpha, N) = \left\{\text{numbers of elements of } \left\{k \middle| \left(\tfrac{2\pi}{N}(2k-1)\right)^2 - 2\alpha < 0,\ (2k-1 \bmod N) \neq 0.\right\}\right\}$$

and

$$v(\alpha, N) = \left\{\text{numbers of elements of } \left\{k \middle| \left(\tfrac{2\pi}{N}(2k-1)\right)^2 - 2\alpha = 0,\ (2k-1 \bmod N) \neq 0.\right\}\right\}.$$

For convenience, denote #(1.7) = the number of geometrically different nonconstant periodic solutions of (1.7) which satisfy $x(t - N/2) = -x(t)$, $\forall t \in R$.

Our main result reads as:

Theorem 1.1. *Suppose f satisfies (f 1) and $N \geq 2$ being an integer in (1.7). We have the following conclusions:*

(i) #(1.7) ≥ $i(\alpha, N)$ - $i(\beta, N)$ provided $v(\beta, N) = 0$ or (f 2-) holds.
(ii) #(1.7) ≥ $i(\alpha, N)$ - $i(\beta, N)$ + $v(\alpha, N)$ provided (f 3+)holds and either $v(\beta, N) = 0$ or (f 2$^-$) holds.
(iii) #(1.7) ≥ $i(\beta, N)$ + $v(\beta, N)$ - $i(\alpha, N)$ provided (f 3-) and (f 2+) holds.

2 Variational structure

For $S^1 = \mathbb{R}/(N\mathbb{Z})$,let $E = H^1(S^1, \mathbb{R}^N)$. Then E is a Hilbert space with norm $||\cdot||$ and inner product$\langle,\rangle$, and E consists of those $z(t)$ in $L^2(S^1, \mathbb{R}^N)$ whose Fourier series

$$z(t) = a_0 + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{2\pi}{N}mt\right) + b_m \sin\left(\frac{2\pi}{N}mt\right)\right),$$

satisfies

$$\|z\|^2 = N|a_0|^2 + \frac{N}{2}\sum_{m=1}^{\infty}\left(1 + \beta_m^2\right)\left(|a_m|^2 + |b_m|^2\right) < \infty,$$

where a_m, $b_m \in \mathbb{R}^N$ and $\beta_m = \frac{2\pi m}{N}$.

We can define an operator

$$\langle L_0 z, y\rangle = \int_0^N \left(A_N^{-1}\dot{z}, \dot{y}\right)dt, \tag{2.1}$$

on E. By direct computation, L_0 is a bounded self-adjoint linear operator on E and

$$L_0 z(t) = \sum_{m=1}^{\infty}\frac{\beta_m^2}{1+\beta_m^2}A_N^{-1}\left(a_m \cos\beta_m t + b_m \sin\beta_m t\right). \tag{2.2}$$

By (f1), one can show that $H(z) \in C^1(\mathbb{R}^{\mathbb{N}}, \mathbb{R})$ and satisfies

$$|H(z)| \leq d_1|z|^2 + d_2, \quad \forall z \in \mathbb{R}^{\mathbb{N}},$$

where d_1, d_2 >0. By using similar arguments as in [14,21], we know that

$$\varphi(z) = \frac{1}{2}\langle L_0 z, z\rangle - \int_0^N H(z)\,dt \in C^1(E, \mathbb{R}) \tag{2.3}$$

and critical points of ϕ in E are classic solutions of (1.9).

Let T_N be the $N \times N$ matrix given by

$$T_N = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

For $z(t) \in E$, define

$$\delta z(t) = T_N z(t-1). \tag{2.4}$$

Then we have $\delta^N z(t) = z(t - N)$, and $G = \{\delta, \delta^2, \ldots, \delta^N\}$ is a compact group action over E. Moreover, if $\delta z(t) = z(t)$ holds, $z(t)$ has the symmetric structure (1.11).

Lemma 2.1. *Denote*$SE = \{z \in E : \delta z(t) = z(t),\ z(t - \frac{N}{2}) = -z(t)\}$ *Then we have*

$$\begin{aligned} SE = \Bigg\{ & z(t) = \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{2\pi}{N}(2m-1)t\right) + b_m \sin\left(\frac{2\pi}{N}(2m-1)t\right)\right) : \\ & \begin{pmatrix} a_m \\ b_m \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} u_m \\ w_m \end{pmatrix}, \begin{pmatrix} -w_m \\ u_m \end{pmatrix} \right\} \Bigg\}, \end{aligned} \tag{2.5}$$

where $\theta_m = \frac{2\pi}{N}(2m-1)$ *and*

$$\begin{aligned} u_m &= (1, \cos\theta_m, \ldots, \cos(N-1)\theta_m)^T, \\ w_m &= (0, \sin\theta_m, \ldots, \sin(N-1)\theta_m)^T. \end{aligned} \tag{2.6}$$

Proof. For any, $z(t) = a_0 + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{2\pi}{N}mt\right) + b_m \sin\left(\frac{2\pi}{N}mt\right)\right) \in SE$, we must have $z(t - N/2) = -z(t)$, which implies that

$$a_0 = -a_0,\ a_m = (-1)^{m+1} a_m,\ b_m = (-1)^{m+1} b_m,\ \text{i.e.,}\ a_m = b_m = 0$$

for even m.

this means that for any $z \in SE$,

$$z(t) = \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{2\pi}{N}(2m-1)t\right) + b_m \sin\left(\frac{2\pi}{N}(2m-1)t\right)\right).$$

Note that $\delta z(t) = z(t) \Leftrightarrow T_N\, z(t - 1) = z(t)$, i.e.,

$$\begin{aligned} & \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{2\pi}{N}(2m-1)t\right) + b_m \sin\left(\frac{2\pi}{N}(2m-1)t\right)\right) \\ & = \sum_{m=1}^{\infty} \left(T_N a_m \cos\left(\frac{2\pi}{N}(2m-1)(t-1)\right) + T_N b_m \sin\left(\frac{2\pi}{N}(2m-1)(t-1)\right)\right) \end{aligned}$$

This implies that for $k \geq 1$

$$\begin{cases} a_m = T_N a_m \cos\theta_m - T_N b_m \sin\theta_m \\ b_m = T_N a_m \sin\theta_m + T_N b_m \cos\theta_m \end{cases} \tag{2.7}$$

If we introduce complex vector $C_m = a_m + ib_m$, the above (2.7) becomes

$$C_m = e^{i\theta_m} T_N C_m.$$

Note that

$$\det(T_N - \lambda I_N) = \lambda^N - 1,$$

the eigenvalues of T_N are

$$\lambda = e^{-i\frac{2j\pi}{N}}, j = 1, 2, 3, \dots,$$

Equation (2.7) implies that C_m must be the eigenvector associated with the eigenvalue

$$\lambda = e^{-i\theta_m},$$

We obtain that

$$C_m = (1, e^{i\theta_m}, \dots, e^{i(N-1)\theta_m})^T, \tag{2.8}$$

$$\begin{cases} u_m = \mathrm{Re}(C_m) = (1, \cos\theta_m, \cos 2\theta_m, \dots, \cos(N-1)\theta_m)^T, \\ w_m = \mathrm{Im}(C_m) = (0, \sin\theta_m, \sin 2\theta_m, \dots, \sin(N-1)\theta_m)^T. \end{cases} \tag{2.9}$$

By direct computation, iC_m is also the eigenvector associated with the eigenvalue$\lambda = e^{-i\theta_m}$.

Therefore, we have the conclusion. The proof is complete. □

Lemma 2.2. *Let ϕ be given in (2.3) and $\phi\,|_{SE}$ be the restriction of ϕ on SE. Then critical points of $\phi\,|_{SE}$ over SE are critical points of ϕ over E.*

Proof. By (1.9) and direct computation, we have

$$A_N T_N = T_N A_N,\ H(T_N z) = H(z),\ \nabla H(T_N z) = T_N \nabla H(z).$$

Combining these with (2.3) and the fact that any $z(t) \in E$ is N-periodic, one can easily verify that

$$\varphi(\delta z) = \varphi(z), \quad \varphi'(\delta z) = \delta\varphi'(z),$$

i.e., ϕ is G-invariant, and ϕ', is G-equivariant. The conclusion follows directly. □

For any $\alpha \in \mathbb{R}$, define an operator L_α by extending the bilinear form

$$\langle L_\alpha z, y\rangle = \langle L_0 z, y\rangle - \int_0^N (\alpha z, y)dt, \tag{2.10}$$

on E. Moreover, L_α is G-equivariant. By direct computation, L_α is a bounded self-adjoint linear operator on E and if $z(t) \in SE$

$$L_\alpha z(t) = \sum_{m=1}^{\infty} \frac{1}{1+\beta_m^2}(\beta_m^2 A_N^{-1} - \alpha I)(a_m \cos\beta_m t + b_m \sin\beta_m t). \tag{2.11}$$

For $m \geq 1$, denote

$$SE(m) = \left\{ z(t) = a_m \cos\left(\frac{2\pi}{N}(2m-1)t\right) + b_m \sin\left(\frac{2\pi}{N}(2m-1)t\right) : \begin{pmatrix} a_m \\ b_m \end{pmatrix} \text{satisfies}(2.5) \right\}.$$

Then $SE = \oplus_{j=1}^{\infty} SE(j)$. Since $A_N = I_N - (T_N + T_N^2 + \cdots + T_N^{N-1})$, and $A_N^T = A_N$, so $SE(m)$ is the eigen-subspace of L_0 corresponding to eigenvalue λ_m, here $\lambda_m = 2 - N$, *as* ($2m$ - 1 mod N) = 0; $\lambda_m = 2$, *as* ($2m$ - 1 mod N) ≠ 0. Denote by M -(·), M +(·) and $M^0(\cdot)$ the positive definite, negative definite and null subspaces of the self-adjoint linear operator defining it, respectively.

Lemma 2.3. *For* $k \geq 1$, $\gamma_k^\alpha = (\frac{2\pi}{N}(2k-1))^2 \lambda_k^{-1} - \alpha$. *Then* L_α*, as an operator on SE, has the following properties on SE.*

$$\begin{aligned} M^-(L_\alpha) &= \bigoplus_{k=1,\gamma_k^\alpha<0}^{\infty} SE(k), \\ M^+(L_\alpha) &= \bigoplus_{k=1,\gamma_k^\alpha>0}^{\infty} SE(k), \\ M^0(L_\alpha) &= \bigoplus_{k=1,\gamma_k^\alpha=0}^{\infty} SE(k). \end{aligned} \tag{2.12}$$

Proof. For $m \geq 1$ and $z_m = a_m \cos\left(\frac{2\pi}{N}(2m-1)t\right) + b_m \sin\left(\frac{2\pi}{N}(2m-1)t\right)$, consider the eigenvalue problem

$$L_\alpha z_m = \lambda_m^\alpha z_m.$$

By (2.7) and (2.11), we have

$$\begin{aligned} &\frac{1}{1+\theta_m^2}(\theta_m^2 A_N^{-1} - \alpha I)a_m = \lambda_m^\alpha a_m, \\ &\frac{1}{1+\theta_m^2}(\theta_m^2 A_N^{-1} - \alpha I)b_m = \lambda_m^\alpha b_m. \end{aligned}$$

Since a_m, b_m are the eigenvector of A_N corresponding to eigenvalue λ_m, then $\lambda_m^\alpha = (1+\theta_m^2)^{-1}(\theta_m^2\lambda_m^{-1} - \alpha)$. Therefore L_α is positive definite, negative definite, or null on $SE(k)$ if and only if $\gamma_k^\alpha = \left(\frac{2\pi}{N}(2k-1)\right)^2 \lambda_k^{-1} - \alpha$ is positive, negative, or zero, respectively.

This implies (2.12) directly. □

For $S^1 = \mathbb{R}/(N\mathbb{Z})$, there is a natural S^1-action over SE, defined by

$$T(\theta)z(t) = z(t+\theta), \forall \theta \in S^1, \forall z \in SE.$$

It is easy to see that ϕ is S^1-invariant, ϕ' is S^1-equivariant, and

$$\mathrm{Fix}(S^1) = \{u \in SE : T(\theta)u = u, \forall \theta \in S^1\} = \{0\}.$$

By directly applying [20, Theorem 2.4] to ϕ over SE, we have the following lemma.

Lemma 2.4. *Assume there exist two closed* S^1*-invariant linear subspaces,* SE^+ *and* SE^-*, of SE and r >0 such that (a)* (SE+ + SE^-) *is closed and of finite codimension in SE,*

(b) $L(SE^-) \subseteq SE^-$ *with* $L = L_\alpha$ *or* $L = L_\beta$*,*

(c) there exist $c_0 \in \mathbb{R}$, $c_0 > -\infty$ *such that*

$$\inf_{z\in SE^+} \varphi(z) = c_0,$$

(d) there exists $c_\infty \in \mathbb{R}$ *such that*

$$\varphi(z) \leq c_\infty < \varphi(0), \forall z \in (SE^- \cap S_r) = \{z \in SE^- : z = r\},$$

(e) ϕ *satisfies (PS)c condition for* $c_0 \leq c \leq c_\infty$*, i.e., every sequence* $\{z_m\} \subseteq SE$ *with* $\phi(z_m) \to c$ *and* $\phi'(z_m) \to 0$ *possesses a convergent subsequence. Then* ϕ *possesses at*

least$\frac{1}{2}[\dim(SE^- \cap SE^+) - \operatorname{codim}_{SE}(SE^- + SE^+)]$ *geometrically different critical orbits in* $\phi^{-1}([c_0, c_\infty])$.

3 Proof of main results

Proof of Theorem 1.1 As we already proved in Lemma 2.2, critical points of ϕ over SE are critical points of ϕ over E. Hence they are nonconstant classic N -periodic solutions of (1.7) with the symmetric structure (1.11). By Proposition 1.2, they give solutions of (1.7) with the property $x(t - N/2) = -x(t)$. Therefore, we can seek critical points of ϕ *on SE* directly.

Set:

$$\psi_\beta(z) = \int_0^N \left[H(z) - \left(\frac{1}{2}\beta z, z\right)\right]dt, \quad \psi_\alpha(z) = \int_0^N \left[H(z) - \left(\frac{1}{2}\alpha z, z\right)\right]dt$$

Then

$$\varphi(z) = \frac{1}{2}\langle L_\alpha z, z\rangle - \psi_\alpha(z), \quad \varphi(z) = \frac{1}{2}\langle L_\beta z, z\rangle - \psi_\beta(z)$$

Case (i): If $i(\alpha, N) > i(\beta, N)$. We shall carry out the proof in several steps.

Step 1: let

$$\psi_\alpha(0) = 0, \quad \frac{\|\psi'_\alpha(z)\|}{\|z\|} \to 0, \ as\ \|z\| \to 0,$$

Then Lemma 2.4(a)and (b) hold with $L = L_\alpha$. By (f1), using the same argument as [18, Lemma 5.5], it is easy to show that

$$\psi_\alpha(0) = 0, \quad \frac{\|\psi'_\alpha(z)\|}{\|z\|} \to 0, \ as\ \|z\| \to 0,$$

We denote

$$\lambda^- = \min\{|\gamma_m^\alpha|(1 + \theta_m^2)^{-1} \mid \gamma_m^\alpha < 0, \ m \in \mathbb{N}\},$$

and

$$\lambda^+ = \min\{(1 + \theta_m^2)^{-1}\gamma_m^\beta \mid \gamma_m^\beta > 0, \ m \in \mathbb{N}\}.$$

Since $\psi_\alpha(0) = 0$, $\frac{||\psi'_\alpha(z)||}{||z||} \to 0$, as $||z|| \to 0$, for $\varepsilon = \frac{\lambda^-}{4}$, there exists a constant $r > 0$ such that for any $z \in H^1(S^1, \mathbb{R}^N)$

$$\|\psi'_\alpha(z)\| \le \frac{\lambda^-}{4}\|z\|, \ \ as\ \|z\| \le r.$$

Furthermore,

$$\begin{aligned} |\psi_\alpha(z)| &= |\psi_\alpha(z) - \psi_\alpha(0)| \\ &= \left|\int_0^1 \frac{\partial \psi_\alpha(\tau z)}{\partial \tau} d\tau\right| \\ &\le \int_0^1 |\langle \psi'_\alpha(\tau z), z\rangle| d\tau \\ &\le \frac{\lambda^-}{4}\|z\|^2. \end{aligned}$$

Therefore, we have for $z \in (E^{-} \cap S_r) = \{z \in E^{-} \mid \|z\| = r\}$

$$\begin{aligned}\varphi(z) &= \frac{1}{2}\langle L_\alpha z, z\rangle - \psi_\alpha(z)\\ &\leq \frac{-\lambda^{-}}{2}\|z\|^2 + \frac{\lambda^{-}}{4}\|z\|^2\\ &= \frac{-\lambda^{-}}{4}\|z\|^2\\ &= \frac{-\lambda^{-}}{4}r^2 < 0.\end{aligned}$$

Thus (d) of Lemma 2.4 holds.

If $M^{0}(L\beta) = \{0\}$, by (f1), using the same argument as [18, Lemma 5.5], it is easy to show that

$$\frac{\left\|\psi'_\alpha(z)\right\|}{\|z\|} \to 0, \quad as\ \|z\| \to \infty,$$

it follows that given $\varepsilon = \frac{\lambda^{+}}{2} > 0$, there exists $r > 0$, such that

$$\left\|\psi'_\alpha(z)\right\| \leq \frac{\lambda^{+}}{2}\|z\|, \quad \|z\| > r.$$

Moreover, there is $d_1 > 0$ such that

$$\left\|\psi'_\alpha(z)\right\| \leq d_1, \quad \|z\| \leq r,$$

thus

$$\left\|\psi'_\alpha(z)\right\| \leq \frac{\lambda^{+}}{2}\|z\| + d_1, \forall z \in H^1(S^1, \mathbb{R}^N).$$

Using the above formula we get

$$|\nabla H(z) - \beta z| \leq \frac{\lambda^{+}}{2}|z| + d_1.$$

Since for any $z \in SE^{+} = M^{+}(L\beta) \oplus M^{0}(L\beta)$

$$\begin{aligned}\left|\psi_\beta(z)\right| &= \left|\psi_\beta(z) - \psi_\beta(0)\right|\\ &= \left|\int_0^1 \frac{\partial \psi_\beta(\tau z)}{\partial \tau} d\tau\right|\\ &= \left|\int_0^1 \langle \psi'_\beta(\tau z), z\rangle d\tau\right|\\ &\leq \int_0^1 \frac{\lambda^{+}\tau}{2} z^2 + d_1 \cdot z d\tau\\ &\leq \frac{\lambda^{+}}{4} z^2 + d_1 z.\end{aligned}$$

Then for any $z \in SE^+ = M^+(L\beta) \oplus M^0(L\beta) = M^+(L\beta) \oplus \{0\}$

$$\begin{aligned}\varphi(z) &= \frac{1}{2}\langle L_\beta z, z\rangle - \psi_\beta(z)\\ &\geq \frac{\lambda^+}{2}z^2 - \frac{\lambda^+}{4}z^2 - d_1 z\\ &= \frac{\lambda^+}{4}z^2 - d_1 z.\end{aligned}$$

Thus there exists $c_0 > -\infty$ such that (c) of Lemma 2.4 holds.

If $M^0(L_\beta) \neq \{0\}$, by $(f\,2^-)$ and (1.12), for any $M > 0$, there exists a constant $d_2 > 0$ such that

$$G_\beta(z) \leq -d_2, \quad |z| > M. \tag{3.1}$$

Let

$$\Omega_1 = \{t \in [0, N] \,|\, |z| > M\} \tag{3.2}$$

and

$$\Omega_2 = \{t \in [0, N] \,|\, |z| \leq M\}. \tag{3.3}$$

Since $F(z) \in C^2(\mathbb{R}^N, \mathbb{R})$, by (3.3), there exists a constant $d_3 > 0$ such that

$$\int_{\Omega_2} [G_\beta(z(t)), z(t)]\, dt \leq d_3, \quad |z| \leq M. \tag{3.4}$$

By (3.1) - (3.4), we have

$$\psi_\beta(z) = \int_0^N [H(z) - (\frac{1}{2}\beta z, z)]dt \leq d_3$$

Then we have for $z = z^+ + z^0 \in M^+(L_\beta) \oplus M^0(L_\beta)$

$$\begin{aligned}\varphi(z) &= \frac{1}{2}\langle L_\beta z, z\rangle - \psi_\beta(z)\\ &= \frac{1}{2}\langle L_\beta z^+, z^+\rangle - \psi_\beta(z)\\ &\geq -d_3.\end{aligned}$$

Thus ϕ is bounded from below on E^+. Therefore, condition (c) of Lemma 2.4 is verified.

Step 2: Using the same argument as [22,25] and [26, Lemma 4.2], one can prove that ϕ satisfies $(PS)c$ condition for any $c \in \mathbb{R}$ under the condition either $(f\,1)$ with $\nu(\beta, N) = 0$ holds or $(f\,2^\pm)$ holds.

Step 3: By Lemma (2.4), ϕ has at least $\sigma = \frac{1}{2}[\dim(SE^- \cap SE^+) - \mathrm{codim}_{SE}(SE^- + SE^+)]$ geometrically different critical orbits in $\phi^{-1}([c_0, c_\infty])$.

Now by Lemma (2.3), it is easy to show that

$$\begin{aligned}\sigma &= \frac{1}{2}[\dim(SE^- \cap SE^+) - \mathrm{codim}_{SE}(SE^- + SE^+)]\\ &= \frac{1}{2}[\dim M^-(L_\alpha) - \dim M^-(L_\beta)]\\ &= \frac{1}{2}[2i(\alpha, N) - 2i(\beta, N)].\end{aligned}$$

This means #(1.7) ≥ $i(\alpha, N)$-$i(\beta, N)$, and this completes the proof of case (i).

For case (ii), (iii), using the same idea and similar arguments, one can show that the conclusions hold. We omit the details. The proof is complete. □

Example. Consider the following equation

$$x''(t) = f(x(t)) - f(x(t-1)), \tag{3.5}$$

and its coupled system

$$x''(t) = f(x) - f(y), y''(t) = -f(x) + f(y). \tag{3.6}$$

with $f \in C(\mathbb{R}, \mathbb{R})$ being odd and

$$f(x) = \alpha x + \alpha_0 x^{\frac{1}{3}}, \ for|x| \ \geq 100$$
$$f(x) = \beta x + \beta_0 x^3, \ for|x| \ \leq 1.$$

Here $N = 2$, and by Definition

$$i(\alpha, 2) = \sharp\{k \mid \ (\pi(2k-1))^2 - 2\alpha < 0, \ k = 1, 2, \ldots\},$$

and

$$\nu(\alpha, 2) = \sharp\{k \mid (\pi(2k-1))^2 - 2\alpha = 0, \ k = 1, 2, \ldots\}.$$

Let $\alpha = 16$, $\beta = 1$. Then it is easy to see that

$$i(16, 2) = 2, \ \nu(16, 2) = 0, \ i(1, 2) = 0, \ \nu(1, 2) = 0$$

By Theorem 1.1, Equation (3.5) has at least $i(16, 2)$ - $i(1, 2) = 2$ geometrically different nonconstant 2-periodic solutions which satisfy $x(t - 1) = -x(t)$.

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Authors' contributions
All authors carried out the proof and authors conceived of the study. All authors read and approved the final manuscript.

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