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On the stability of a mixed type functional equation in generalized functions

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Abstract

We reformulate the following mixed type quadratic and additive functional equation with n-independent variables

$$2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{\substack{1 \le i,j \le n \\ i \ne j}} f(x_i - x_j) = (n+1)\sum_{i=1}^{n} f(x_i) + (n-1)\sum_{i=1}^{n} f(-x_i)$$

as the equation for the spaces of generalized functions. Using the fundamental solution of the heat equation, we solve the general solution and prove the Hyers-Ulam stability of this equation in the spaces of tempered distributions and Fourier hyperfunctions.

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1. Introduction

In 1940, Ulam [1] raised a question concerning the stability of group homomorphisms as follows:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In 1941, Hyers [2] firstly presented the stability result of functional equations under the assumption that G_1 and G_2 are Banach spaces. In 1978, Rassias [3] generalized Hyers' result to the unbounded Cauchy difference. After that stability problems of various functional equations have been extensively studied and generalized by a number of authors (see [4-7]). Among them, Towanlong and Nakmahachalasint [8] introduced the following functional equation with n-independent variables

$$2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{\substack{1 \le i,j \le n \\ i \ne j}} f(x_i - x_j) = (n+1)\sum_{i=1}^{n} f(x_i) + (n-1)\sum_{i=1}^{n} f(-x_i), \quad (1.1)$$

where n is a positive integer with $n \geq 2$. For real vector spaces X and Y, they proved that a function $f : X \to Y$ satisfies (1.1) if and only if there exist a quadratic function $q : X \to Y$ satisfying

$$q(x+y) + q(x-y) = 2q(x) + 2q(y)$$

and an additive function $a : X \to Y$ satisfying

$$a(x+y) = a(x) + a(y)$$

such that

$$f(x) = q(x) + a(x)$$

for all $x \in X$. For this reason, equation (1.1) is called the mixed type quadratic and additive functional equation. We refer to [9-14] for the stability results of other mixed type functional equations.

In this article, we consider equation (1.1) in the spaces of generalized functions such as the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions and the space $\mathcal{F}'(\mathbb{R})$ of Fourier hyperfunctions. Making use of similar approaches in [15-20], we reformulate equation (1.1) and the related inequality for the spaces of generalized functions as follows:

$$2u \circ A + \sum_{\substack{1 \le i,j \le n, \\ i \ne j}} u \circ B_{ij} = (n+1)\sum_{i=1}^{n} u \circ P_i + (n-1)\sum_{i=1}^{n} u \circ Q_i, \quad (1.2)$$

$$\left\| 2u \circ A + \sum_{\substack{1 \le i,j \le n, \\ i \ne j}} u \circ B_{ij} - (n+1)\sum_{i=1}^{n} u \circ P_i - (n-1)\sum_{i=1}^{n} u \circ Q_i \right\| \le \varepsilon, \quad (1.3)$$

where A, B_{ij}, P_i and Q_i are the functions defined by

$$\begin{aligned} &A(x_1, \ldots, x_n) = x_1 + \cdots + x_n, \\ &B_{ij}(x_1, \ldots, x_n) = x_i - x_j, \quad 1 \le i,j \le n, i \ne j, \\ &P_i(x_1, \ldots, x_n) = x_i, \quad 1 \le i \le n, \\ &Q_i(x_1, \ldots, x_n) = -x_i, \quad 1 \le i \le n. \end{aligned}$$

Here $\circ$ denotes the pullback of generalized functions and the inequality $||v|| \le \varepsilon$ in (1.3) means that $|\langle v, \varphi \rangle| \le \varepsilon ||\varphi||_{L^1}$ for all test functions ϕ.

In order to solve the general solution of (1.2) and prove the Hyers-Ulam stability of (1.3), we employ the heat kernel method stated in section 2. In section 3, we prove that every solution u in $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$, resp.) of equation (1.2) is of the form

$$u = ax^2 + bx$$

for some a, $b \in \mathbb{C}$. Subsequently, in section 4, we prove that every solution u in $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$, resp.) of the inequality (1.3) can be written uniquely in the form

$$u = ax^2 + bx + \mu(x),$$

where μ is a bounded measurable function such that $\|\mu\|_{L^\infty} \le \frac{n^2+n-3}{n^2+n-2}\varepsilon$.

2. Preliminaries

In this section, we introduce the spaces of tempered distributions and Fourier hyperfunctions. We first consider the space of rapidly decreasing functions which is a test function space of tempered distributions.

Definition 2.1. [21]*The space* $\mathcal{S}(\mathbb{R})$*denotes the set of all infinitely differentiable functions* $\phi : \mathbb{R} \to \mathbb{C}$ *such that*

$$\|\varphi\|_{\alpha,\beta} = \sup_x |x^\alpha D^\beta \varphi(x)| < \infty$$

for all nonnegative integers α, β.

In other words, $\phi(x)$ as well as its derivatives of all orders vanish at infinity faster than the reciprocal of any polynomial. For that reason, we call the element of $\mathcal{S}(\mathbb{R})$ as the rapidly decreasing function. It can be easily shown that the function $\phi(x) = \exp(-ax^2)$, $a > 0$, belongs to $\mathcal{S}(\mathbb{R})$, but $\psi(x) = (1 + x^2)^{-1}$ is not a member of $\mathcal{S}(\mathbb{R})$. Next we consider the space of tempered distributions which is a dual space of $\mathcal{S}(\mathbb{R})$.

Definition 2.2. [21]*A linear functional u on* $\mathcal{S}(\mathbb{R})$*is said to be a tempered distribution if there exists constant* $C \ge 0$ *and nonnegative integer N such that*

$$|\langle u, \varphi\rangle| \le C \sum_{\alpha,\beta \le N} \sup_x |x^\alpha D^\beta \varphi| \tag{2.1}$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$*. The set of all tempered distributions is denoted by* $\mathcal{S}'(\mathbb{R})$.

For example, every $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, defines a tempered distribution by virtue of the relation

$$\langle f, \varphi\rangle = \int f(x)\varphi(x)dx, \ \ \varphi \in \mathcal{S}(\mathbb{R}).$$

Note that tempered distributions are generalizations of L^p-functions. These are very useful for the study of Fourier transforms in generality, since all tempered distributions have a Fourier transform, but not all distributions have one. Imposing the growth condition on $|| \cdot ||_{\alpha,\beta}$ in (2.1) a new space of test functions has emerged as follows.

Definition 2.3. [22]*We denote by* $\mathcal{F}(\mathbb{R})$*the set of all infinitely differentiable functions* ϕ *in* $\mathbb{R}$ *such that*

$$\|\varphi\|_{A,B} = \sup_{x,\alpha,\beta} \frac{|x^\alpha D^\beta \varphi(x)|}{A^{|\alpha|}B^{|\beta|}\alpha!\beta!} < \infty \tag{2.2}$$

for some positive constants A, B depending only on ϕ.

It can be verified that the seminorm (2.2) is equivalent to

$$\|\varphi\|_{h,k} = \sup_{x,\alpha} \frac{|D^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty$$

for some constants h, $k > 0$.

Definition 2.4. [22]*The strong dual space of* $\mathcal{F}(\mathbb{R})$*is called the Fourier hyperfunctions. We denote the Fourier hyperfunctions by* $\mathcal{F}'(\mathbb{R})$.

It is easy to see the following topological inclusions:

$$\mathcal{F}(\mathbb{R}) \hookrightarrow \mathcal{S}(\mathbb{R}), \quad \mathcal{S}'(\mathbb{R}) \hookrightarrow \mathcal{F}'(\mathbb{R}). \tag{2.3}$$

Taking the relations (2.3) into account, it suffices to consider the space $\mathcal{F}'(\mathbb{R})$. In order to solve the general solution and the stability problem of (1.2) in the space $\mathcal{F}'(\mathbb{R})$, we employ the fundamental solution of the heat equation called the heat kernel,

$$E_t(x) = E(x,\ t) = \begin{cases} (4\pi t)^{-1/2} \exp(-x^2/4t) & , x \in \mathbb{R},\ t > 0, \\ 0 & , x \in \mathbb{R},\ t \le 0. \end{cases}$$

Since for each $t > 0$, $E(\cdot,\ t)$ belongs to the space $\mathcal{F}(\mathbb{R})$, the convolution

$$\tilde{u}(x,t) = (u * E)(x,t) = \langle u_y, E_t(x-y)\rangle, \quad x \in \mathbb{R}, \quad t > 0$$

is well defined for all $u \in \mathcal{F}'(\mathbb{R})$. We call $\tilde{u}$ as the Gauss transform of u. Semigroup property of the heat kernel

$$(E_t * E_s)(x) = E_{t+s}(x)$$

holds for convolution. It is useful to convert equation (1.2) into the classical functional equation defined on upper-half plane. We also use the following famous result called heat kernel method, which states as follows.

Theorem 2.5. [23]*Let* $u \in \mathcal{S}'(\mathbb{R})$. *Then its Gauss transform* $\tilde{u}$*is a* C^∞*-solution of the heat equation*

$$(\partial/\partial t - \Delta)\tilde{u}(x,t) = 0$$

satisfying

(i) There exist positive constants C, M and N such that

$$\left|\tilde{u}(x,t)\right| \le Ct^{-M}(1+|x|)^N \text{ in } \mathbb{R} \times (0,\ \delta). \tag{2.4}$$

(ii) $\tilde{u}(x,t) \to u$ *as* $t \to 0^+$ *in the sense that for every* $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int \tilde{u}(x,t)\varphi(x)dx.$$

Conversely, every C^∞*-solution* $U(x,\ t)$ *of the heat equation satisfying the growth condition* (2.4) *can be uniquely expressed as* $U(x,t) = \tilde{u}(x,t)$*for some* $u \in \mathcal{S}'(\mathbb{R})$.

Similarly, we can represent Fourier hyperfunctions as initial values of solutions of the heat equation as a special case of the results as in [24]. In this case, the condition (i) in the above theorem is replaced by the following:

For every $\varepsilon > 0$ there exists a positive constant C_ε such that

$$\left|\tilde{u}(x,t)\right| \le C_\varepsilon \exp(\varepsilon(|x| + 1/t)) \text{ in } \mathbb{R} \times (0,\ \delta).$$

3. General solution in $\mathcal{F}'(\mathbb{R})$

We are now going to solve the general solution of (1.2) in the space of $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$, resp.). In order to do so, we employ the heat kernel mentioned in the previous section. Convolving the tensor product $E_{t_1}(x_1)\ldots E_{t_n}(x_n)$ of the heat kernels on both sides of (1.2) we have

$$\begin{aligned}
&\left[(u \circ A) * (E_{t_1}(x_1) \dots E_{t_n}(x_n))\right](\xi_1, \dots, \xi_n) \\
&\quad = \langle u \circ A,\, E_{t_1}(\xi_1 - x_1) \dots E_{t_n}(\xi_n - x_n)\rangle \\
&\quad = \left\langle u, \int \cdots \int E_{t_1}(\xi_1 - x_1 + x_2 + \cdots + x_n) E_{t_2}(\xi_2 - x_2) \dots E_{t_n}(\xi_n - x_n)\, dx_2 \dots dx_n \right\rangle \\
&\quad = \left\langle u, \int \cdots \int E_{t_1}(\xi_1 + \cdots + \xi_n - x_1 - \cdots - x_n) E_{t_2}(x_2) \dots E_{t_n}(x_n)\, dx_2 \dots dx_n \right\rangle \\
&\quad = \langle u, (E_{t_1} * \dots * E_{t_n})(\xi_1 + \cdots + \xi_n - x_1)\rangle \\
&\quad = \langle u, E_{t_1 + \cdots + t_n}(\xi_1 + \cdots + \xi_n)\rangle \\
&\quad = \tilde{u}(\xi_1 + \cdots + \xi_n, t_1 + \cdots + t_n), \\
&[(u \circ B_{ij}) * (E_{t_1}(x_1) \dots E_{t_n}(x_n))](\xi_1, \dots, \xi_n) = \tilde{u}(\xi_i - \xi_j,\, t_i + t_j), \\
&[(u \circ P_i) * (E_{t_1}(x_1) \dots E_{t_n}(x_n))](\xi_1, \dots, \xi_n) = \tilde{u}(\xi_i,\, t_i), \\
&[(u \circ Q_i) * (E_{t_1}(x_1) \dots E_{t_n}(x_n))](\xi_1, \dots, \xi_n) = \tilde{u}(-\xi_i,\, t_i),
\end{aligned}$$

where $\tilde{u}$ is the Gauss transform of u. Thus, (1.2) is converted into the following classical functional equation

$$\begin{aligned}
&2\tilde{u}\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_i\right) + \sum_{\substack{1 \le i, j \le n, \\ i \ne j}} \tilde{u}(x_i - x_j,\, t_i + t_j) \\
&\quad = (n+1)\sum_{i=1}^{n} \tilde{u}(x_i, t_i) + (n-1)\sum_{i=1}^{n} \tilde{u}(-x_i,\, t_i)
\end{aligned}$$

for all $x_1, \dots, x_n \in \mathbb{R}$, $t_1, \dots, t_n > 0$. We here need the following lemma which will be crucial role in the proof of main theorem.

Lemma 3.1. *A continuous function* $f : \mathbb{R} \times (0, \infty) \to \mathbb{C}$ *satisfies the functional equation*

$$\begin{aligned}
&2f\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_i\right) + \sum_{\substack{1 \le i, j \le n \\ i \ne j}} f(x_i - x_j,\, t_i + t_j) \\
&\quad = (n+1)\sum_{i=1}^{n} f(x_i, t_i) + (n-1)\sum_{i=1}^{n} f(-x_i,\, t_i)
\end{aligned} \tag{3.1}$$

for all $x_1, \dots, x_n \in \mathbb{R}$, $t_1, \dots, t_n > 0$ *if and only if there exist constants* $a, b, c \in \mathbb{C}$ *such that*

$$f(x, t) = ax^2 + bx + ct$$

for all $x \in \mathbb{R}$, $t > 0$.

Proof. Putting $(x_1, \dots, x_n) = (0, \dots, 0)$ in (3.1) yields

$$f\left(0, \sum_{i=1^n} t_i\right) + \sum_{1 \le i < j \le n} f(0,\, t_i + t_j) = n\sum_{i=1}^{n} f(0, t_i) \tag{3.2}$$

for all $t_1, \dots, t_n > 0$. In view of (3.2) we see that

$$c := \lim_{t \to 0^+} f(0, t)$$

exists. Letting $t_1 = \cdots = t_n \to 0^+$ in (3.2) gives $c = 0$. Setting $(x_1, x_2, x_3, \ldots, x_n) = (x, y, 0, \ldots, 0)$ and letting $t_1 = t, t_2 = s, t_3 = \cdots = t_n \to 0^+$ in (3.1) we have

$$\begin{aligned} &2f(x+y,t+s) + f(x-y,\ t+s) + f(-x+y,t+s) \\ &\quad = 3f(x,t) + 3f(y,s) + f(-x,t) + f(-y,s) \end{aligned} \tag{3.3}$$

for all $x, y \in \mathbb{R}$, $t, s > 0$. Replacing x and y with $-x$ and $-y$ in (3.3) yields

$$\begin{aligned} &2f(-x-y,t+s) + f(-x+y,\ t+s) + f(x-y,t+s) \\ &\quad = 3f(-x,t) + 3f(-y,s) + f(x,t) + f(y,s) \end{aligned} \tag{3.4}$$

for all $x, y \in \mathbb{R}$, $t, s > 0$. We now define the even part and the odd part of the function f by

$$f_e(x,t) = \frac{f(x,t)+f(-x,t)}{2}, \quad f_o(x,t) = \frac{f(x,t)-f(-x,t)}{2}$$

for all $x \in \mathbb{R}$, $t > 0$. Adding (3.3) to (3.4) we verify that f_e satisfies

$$f_e(x+y,t+s) + f_e(x-y,\ t+s) = 2f_e(x,t) + 2f_e(y,s) \tag{3.5}$$

for all $x, y \in \mathbb{R}$, $t, s > 0$. Similarly, taking the difference of (3.3) and (3.4) we see that f_o satisfies

$$f_o(x+y,t+s) = f_o(x,t) + f_o(y,s) \tag{3.6}$$

for all $x, y \in \mathbb{R}$, $t, s > 0$. It follows from (3.5), (3.6) and given the continuity that f_e and f_o are of the forms

$$f_e(x,t) = ax^2 + c_1 t, \quad f_o(x,t) = bx + c_2 t$$

for some constants $a, b, c_1, c_2 \in \mathbb{C}$. Finally we have

$$f(x,t) = f_e(x,t) + f_o(x,t) = ax^2 + bx + ct,$$

where $c = c_1 + c_2$.

Conversely, if $f(x, t) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{C}$, then it is obvious that f satisfies equation (3.1). □

According to the above lemma, we solve the general solution of (1.2) in the space of $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$, resp.) as follows.

Theorem 3.2. *Every solution u in $\mathcal{F}'(\mathbb{R})$(or $\mathcal{S}'(\mathbb{R})$, resp.) of equation* (1.2) *has the form*

$$u = ax^2 + bx,$$

for some $a, b \in \mathbb{C}$.

Proof. Convolving the tensor product $E_{t_1}(x_1) \ldots E_{t_n}(x_n)$ of the heat kernels on both sides of (1.2) we have

$$\begin{aligned} &2\tilde{u}\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_i\right) + \sum_{\substack{1 \le i,j \le n, \\ i \ne j}} \tilde{u}(x_i - x_j,\ t_i + t_j) \\ &\quad = (n+1)\sum_{i=1}^{n} \tilde{u}(x_i, t_i) + (n-1)\sum_{i=1}^{n} \tilde{u}(-x_i,\ t_i) \end{aligned} \tag{3.7}$$

for all $x_1, \ldots, x_n \in \mathbb{R}$, $t_1, \ldots, t_n > 0$. It follows from Lemma 3.1 that the solution $\tilde{u}$ of equation (3.7) has the form

$$\tilde{u}(x,t) = ax^2 + bx + ct \tag{3.8}$$

for some $a, b, c \in \mathbb{C}$. Letting $t \to 0^+$ in (3.8), we finally obtain the general solution of (1.2). □

4. Stability in $\mathcal{F}'(\mathbb{R})$

In this section, we are going to state and prove the Hyers-Ulam stability of (1.3) in the space of $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$, resp.).

Lemma 4.1. *Suppose that* $f : \mathbb{R} \times (0,\infty) \to \mathbb{C}$ *is a continuous function satisfying*

$$\left| 2f\left(\sum_{i=1}^n x_i, \sum_{i=1}^n t_i\right) + \sum_{\substack{1 \le i,j \le n \\ i \ne j}} f(x_i - x_j,\ t_i + t_j) - (n+1)\sum_{i=1}^n f(x_i, t_i) - (n-1)\sum_{i=1}^n f(-x_i,\ t_i) \right| \le \varepsilon \tag{4.1}$$

for all $x_1, \ldots, x_n \in \mathbb{R}$, $t_1, \ldots, t_n > 0$, *then there exists the unique function* $g : \mathbb{R} \times (0, \infty) \to \mathbb{C}$ *satisfying equation* (3.1) *such that*

$$|f(x,t) - g(x,t)| \le \frac{n^2+n-3}{n^2+n-2}\varepsilon$$

for all $x \in \mathbb{R}$, $t > 0$.

Proof. Putting $(x_1, \ldots, x_n) = (0, \ldots, 0)$ in (4.1) yields

$$\left| f\left(0,\ \sum_{i=1}^n t_i\right) + \sum_{1 \le i < j \le n} f(0,\ t_i + t_j) - n\sum_{i=1}^n f(0, t_i) \right| \le \frac{\varepsilon}{2} \tag{4.2}$$

for all $t_1, \ldots, t_n > 0$. In view of (4.2) we see that

$$c := \limsup_{t \to 0^+} f(0,t)$$

exists. Letting $t_1 = \cdots = t_n \to 0^+$ in (4.2) gives

$$|c| \le \frac{\varepsilon}{n^2+n-2}. \tag{4.3}$$

Setting $(x_1, x_2, x_3, \ldots, x_n) = (x, x, 0, \ldots, 0)$ and letting $t_1 = t_2 = t$, $t_3 = \cdots = t_n \to 0^+$ in (4.1) we have

$$\left| f(2x,\ 2t) + f(0,2t) - 3f(x,\ t) - f(-x,\ t) - \frac{c(n^2+n-6)}{2} \right| \le \frac{\varepsilon}{2} \tag{4.4}$$

for all $x \in \mathbb{R}$, $t > 0$. Replacing x by $-x$ in (4.4) yields

$$\left| f(-2x,\ 2t) + f(0,2t) - 3f(-x,\ t) - f(x,\ t) - \frac{c(n^2+n-6)}{2} \right| \le \frac{\varepsilon}{2} \tag{4.5}$$

for all $x \in \mathbb{R}$, $t > 0$. Let f_e and f_o be even and odd part of f defined in Lemma 3.1, respectively. Using the triangle inequality in (4.4) and (4.5) we get the inequalities

$$\left| \frac{g_e(2x, 2t)}{4} - g_e(x, t) + \frac{g_e(0, 2t)}{4} \right| \le \frac{\varepsilon}{8}, \tag{4.6}$$

$$\left| \frac{f_o(2x, 2t)}{2} - f_o(x, t) \right| \le \frac{\varepsilon}{4} \tag{4.7}$$

for all $x \in \mathbb{R}$, $t > 0$, where $g_e(x, t) := f_e(x, t) + \frac{c(n^2+n-6)}{4}$.

We first consider the even case. Using the iterative method in (4.6) we obtain

$$\left| \frac{g_e(2^k x, 2^k t)}{4^k} - g_e(x, t) + \sum_{j=1}^{k} \frac{g_e(0, 2^j t)}{4^j} \right| \le \frac{\varepsilon}{6} \tag{4.8}$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}$, $t > 0$. Letting $t_1 = t$, $t_2 = s$, $t_3 = \cdots = t_n \to 0^+$ in (4.2) we have

$$\left| g_e(0, t + s) - g_e(0, t) - g_e(0, s) \right| \le \frac{\varepsilon}{4} \tag{4.9}$$

for all $t, s > 0$. We verify from (4.9) that

$$h(t) := \lim_{k \to \infty} \frac{g_e(0, 2^k t)}{2^k}$$

converges and is the unique function satisfying

$$h(t + s) = h(t) + h(s), \tag{4.10}$$

$$|h(t) - g_e(0, t)| \le \frac{\varepsilon}{4} \tag{4.11}$$

for all $t, s > 0$. Combining (4.10) and (4.11) we get

$$\left| (1 - 2^{-k})h(t) - \sum_{i=1}^{k} \frac{g_e(0, 2^k t)}{4^k} \right| \le \frac{\varepsilon}{12} \tag{4.12}$$

for all $k \in \mathbb{N}$, $t > 0$. Adding (4.8) to (4.12) we have

$$\left| \tilde{g}_e(x, t) - \frac{\tilde{g}_e(2^k x, 2^k t)}{4^k} \right| \le \frac{\varepsilon}{4} \tag{4.13}$$

for all $k \in \mathbb{N}$, $x \in \mathbb{R}$, $t > 0$, where $\tilde{g}_e(x, t) := g_e(x, t) - h(t)$. From (4.1) and (4.13) we verify that

$$G_e(x, t) := \lim_{k \to \infty} \frac{\tilde{g}_e(2^k x, 2^k t)}{4^k}$$

is the unique function satisfying equation (3.1) and the inequality

$$|\tilde{g}_e(x, t) - G_e(x, t)| \le \frac{\varepsilon}{4} \tag{4.14}$$

for all $x \in \mathbb{R}$, $t > 0$. If we define a function $q(x, t) := G_e(x, t) + h(t)$, then q also satisfies (3.1). By Lemma 3.1 and evenness of q we have

$$q(x,t) = ax^2 + c_1 t$$

for some a, $c_1 \in \mathbb{C}$. It follows from (4.3) and (4.14) that

$$|f_e(x,t) - ax^2 - c_1 t| \le \frac{n^2+n-4}{2(n^2+n-2)}\varepsilon \tag{4.15}$$

for all $x \in \mathbb{R}$, $t > 0$.

Next, we consider the odd case. From (4.7), in the similar manner, we verify that

$$F_o(x,t) := \lim_{k\to\infty} \frac{f_o(2^k x, 2^k t)}{2^k}$$

is the unique function satisfying equation (3.1) and the inequality

$$\left|F_o(x,\ t) - f_o(x,\ t)\right| \le \frac{\varepsilon}{2} \tag{4.16}$$

for all $x \in \mathbb{R}$, $t > 0$. By Lemma 3.1 and oddness of F_o we have

$$F_o(x,t) = bx + c_2 t$$

for some b, $c_2 \in \mathbb{C}$.

Therefore, from (4.15) and (4.16), we obtain

$$\begin{aligned} &|f(x,\ t) - (ax^2 + bx + ct)| \\ &\quad \le |f_e(x,\ t) - (ax^2 + c_1 t)| + |f_0(x,\ t) - \ (bx + c_2 t)| \\ &\quad \le \frac{n^2+n-3}{n^2+n-2}\varepsilon \end{aligned}$$

for all $x \in \mathbb{R}$, $t > 0$, where $c = c_1 + c_2$. □

From the above lemma we immediately prove the Hyers-Ulam stability of (1.3) in the space of $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$, resp.) as follows.

Theorem 4.2. *Suppose that u in* $\mathcal{F}'(\mathbb{R})$(or $\mathcal{S}'(\mathbb{R})$, *resp.) satisfies the inequality* (1.3), *then there exists the unique quadratic additive function* $q(x) = ax^2 + bx$ *such that*

$$\|u - q(x)\| \le \frac{n^2+n-3}{n^2+n-2}\varepsilon. \tag{4.17}$$

Proof. Convolving the tensor product $E_{t_1}(x_1)\ldots E_{t_n}(x_n)$ of the heat kernels on both sides of (1.3) we verify that the inequality (1.3) is converted into

$$\left| 2\tilde{u}\left(\sum_{i=1}^n x_i, \sum_{i=1}^n t_i\right) + \sum_{\substack{1 \le i,j \le n \\ i \ne j}} \tilde{u}(x_i - x_j,\ t_i + t_j) - (n+1)\sum_{i=1}^n \tilde{u}(x_i, t_i) - (n-1)\sum_{i=1}^n \tilde{u}(-x_i,\ t_i) \right| \le \varepsilon$$

for all $x_1, \ldots, x_n \in \mathbb{R}$, $t_1, \ldots, t_n > 0$. According to Lemma 4.1, there exists the unique function $g(x, t) = ax^2 + bx + ct$ such that

$$|\tilde{u}(x,t) - g(x,t)| \leq \frac{n^2+n-3}{n^2+n-2}\varepsilon \tag{4.18}$$

for all $x \in \mathbb{R}$, $t > 0$. Letting $t \to 0^+$ in (4.18) finally we have the stability result (4.17). □

Remark 4.3. The above norm inequality $\|u - q(x)\| \leq \frac{n^2+n-3}{n^2+n-2}\varepsilon$ implies that $u - q(x)$ belongs to $(L^1)' = L^\infty$. Thus, every solution u of the inequality (4.17) in $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$, resp.) can be rewritten uniquely in the form

$$u = q(x) + \mu(x),$$

where μ is a bounded measurable function such that $\|u\|_{L^\infty} \leq \frac{n^2+n-3}{n^2+n-2}\varepsilon$.

Competing interests
The author declares that they have no competing interests.

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