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Fixed point results for generalized mappings

Farhan Golkarmanesh[1*], Abdullah E Al-Mazrooei[2], Vahid Parvaneh[3] and Abdul Latif[2]

*Correspondence:
fgolkarmanesh@yahoo.com
[1]Department of Mathematics, Sanandaj Branch, Islamic Azad University, Sanandaj, Iran
Full list of author information is available at the end of the article

Abstract

In this paper, first we establish fixed point results for weak asymptotic pointwise contraction type mappings in metric spaces. Then we study the existence of fixed points for weak asymptotic pointwise nonexpansive type mappings in *CAT*(0) spaces. Our results improve and extend some corresponding known results in the literature.

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1 Introduction

The notion of asymptotic pointwise contraction was introduced by Kirk [1] as follows.

Let (M,d) be a metric space. A mapping $T:M\to M$ is called an asymptotic pointwise contraction if there exists a function $\alpha:M\to[0,1)$ such that, for each integer $n\geq 1$,

$$d\bigl(T^nx,T^ny\bigr)\leq\alpha_n(x)d(x,y)\quad\text{for each }x,y\in M,$$

where $\alpha_n\to\alpha$ pointwise on M.

Moreover, Kirk and Xu [2] proved that if C be a weakly compact convex subset of a Banach space E and $T:C\to C$ an asymptotic pointwise contraction, then T has a unique fixed point $v\in C$ and for each $x\in C$ the sequence of Picard iterates $\{T^nx\}$ converges in norm to v.

Rakotch [3] proved that if M be a complete metric space and $f:M\to M$ satisfies $d(f(x),f(y))\leq\alpha(d(x,y))d(x,y)$, for all $x,y\in M$, where $\alpha:[0,\infty)\to[0,1)$ is monotonically decreasing, then f has a unique fixed point z and $\{f^n(x)\}$ converges to z, for each $x\in M$.

Boyd and Wong [4] proved that if M be a complete metric space and $f:M\to M$ satisfies $d(f(x),f(y))\leq\psi(d(x,y))d(x,y)$, for all $x,y\in M$, where $\psi:[0,\infty)\to[0,\infty)$ is upper semicontinuous from the right and satisfies $0\leq\psi(t)<t$ for $t>0$, then f has a unique fixed point z and $\{f^n(x)\}$ converges to z, for each $x\in M$.

Using the diameter of an orbit, Walter [5] obtained a result that may be stated as follows:

Let (M,d) be a complete metric space and let $T:M\to M$ be a mapping with bounded orbits. If there exists a continuous, increasing function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ for which $\varphi(r)<r$ for every $r>0$ and

$$d(Tx,Ty)\leq\varphi\bigl(\operatorname{diam}\bigl(O_T(x,y)\bigr)\bigr)\quad\text{for every }x,y\in M,$$

where $O_T(x,y)=\{T^nx\}\cup\{T^ny\}$, then T has a unique fixed point x_0. Moreover, $\{T^nx\}$ converges to x_0, for each $x\in M$.

In [6], the authors proved coincidence results for asymptotic pointwise nonexpansive mappings. Kirk [7], proved that if M be a complete metric space and $T: M \to M$ satisfies $d(T^n x, T^n y) \leq \phi_n(d(x,y))$, for all $x, y \in M$, where $\phi_n : [0,\infty) \to [0,\infty)$, $\phi_n \to \phi$ uniformly on the range of d and ϕ is continuous with $\phi(s) < s$ for all $s > 0$, then T has a unique fixed point z and $\{T^n(x)\}$ converges to z, for each $x \in M$.

References [3, 4, 6–11] present simple and elegant proofs of fixed point results for pointwise contraction, asymptotic pointwise contraction, and asymptotic nonexpansive mappings.

Very recently, Saeidi [11] introduced the concept of (weak) asymptotic pointwise contraction type mappings.

Let (M,d) be a metric space. A mapping $T: M \to M$ is said to be of asymptotic pointwise contraction type (resp. of weak asymptotic pointwise contraction type) if T^N is continuous for some integer $N \geq 1$ and there exists a function $\alpha : M \to [0,1)$ such that, for each x in M,

$$\limsup_{n\to\infty} \sup_{y\in M} \bigl\{ d\bigl(T^n x, T^n y\bigr) - \alpha_n(x) d(x,y) \bigr\} \leq 0 \tag{1.1}$$

$$\Bigl(\text{resp. } \liminf_{n\to\infty} \sup_{y\in M} \bigl\{ d\bigl(T^n x, T^n y\bigr) - \alpha_n(x) d(x,y) \bigr\} \leq 0\Bigr), \tag{1.2}$$

where $\alpha_n \to \alpha$ pointwise on M. Taking

$$r_n(x) = \sup_{y\in M} \bigl\{ d\bigl(T^n x, T^n y\bigr) - \alpha_n(x) d(x,y) \bigr\} \in \mathbb{R}^+ \cup \{\infty\}$$

it can easily be seen from (1.1) (resp. (1.2)) that

$$\lim_{n\to\infty} r_n(x) = 0 \tag{1.3}$$

$$\Bigl(\text{resp. } \liminf_{n\to\infty} r_n(x) \leq 0\Bigr) \tag{1.4}$$

for all $x \in M$ and

$$d\bigl(T^n x, T^n y\bigr) \leq \alpha_n(x) d(x,y) + r_n(x). \tag{1.5}$$

It is easy to see that the class of asymptotic pointwise contraction type mappings contains the class of an asymptotic pointwise contraction mappings, but the converse is not true [11]. Furthermore, if C is a nonempty weakly compact subset of a Banach space E and $T: C \to C$ a mapping of weak asymptotic pointwise contraction type, then T has a unique fixed point $v \in C$ and, for each $x \in C$, the sequence of Picard iterates $\{T^n x\}$ converges in norm to v (see [11]). Golkarmanesh and Saeidi [12] obtained some results for this mappings in modular spaces.

In this paper, motivated by [1, 9–11], we combine the above results and obtain fixed point results for classes of mappings that extend the notions of asymptotic contraction and asymptotic pointwise contraction introduced by Kirk [1] and Saeidi [11].

2 Preliminaries

Let M be a metric space and $\mathcal{F}$ a family of subsets of M. We say that $\mathcal{F}$ defines a convexity structure of M if it contains the closed ball and is stable by intersection. For instance

$\mathcal{A}(M)$, the class of the admissable subsets of M, defines a convexity structure on any metric space M. Recall that a subset of M is admissable if it is a nonempty intersection of closed balls.

At this point we introduce some notation which will be used throughout the remainder of this work. For a subset A of a metric space M, set:

$r_x(A) = \sup\{d(x,y) : y \in A\}$;
$R(A) = \inf\{r_x(A) : x \in A\}$;
$\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\}$;
$C_A(A) = \{x \in A : r_x(A) = R(A)\}$;
$\operatorname{cov}(A) = \bigcap\{B : B \text{ is a closed ball and } B \supseteq A\}$.

$\operatorname{diam}(A)$ is called the diameter of A, $R(A)$ is called the Chebyshev radius of A, $C_A(A)$ is called the Chebyshev center of A and $\operatorname{cov}(A)$ is called the cover of A.

Definition 1 ([9]) Let $\mathcal{F}$ be a convexity structure on M.

(i) We will say that $\mathcal{F}$ is compact if any family $(A_\alpha)_{\alpha\in\Gamma}$ of elements of $\mathcal{F}$, has a nonempty intersection provided $\bigcap_{\alpha\in F} A_\alpha \neq \emptyset$ for any finite subset $F \subset \Gamma$.

(ii) We will say that $\mathcal{F}$ is normal if for any $A \in \mathcal{F}$, not reduced to one point, we have $R(A) < \operatorname{diam}(A)$.

(iii) We will say that $\mathcal{F}$ is uniformly normal if there exists $c \in (0,1)$ such that, for any $A \in \mathcal{F}$, not reduced to one point, we have $R(A) \leq c(\operatorname{diam}(A))$. It is easy to check that $c \geq 1/2$.

Let M be a metric space and $\mathcal{F}$ a convexity structure. We will say that a function $\Phi : M \to M$ is $\mathcal{F}$-convex if $\{x : \Phi(x) \leq r\} \in \mathcal{F}$ for any $r \geq 0$. Also we define a type to be a function $\Phi : M \to [0,\infty]$ such that

$$\Phi(u) = \limsup_{n\to\infty} d(x_n, u),$$

where (x_n) is a bounded sequence in M. Types are very useful in the study of the geometry of Banach spaces and the existence of fixed point of mappings. We will say that a convexity structure $\mathcal{F}$ on M is T-stable if types are $\mathcal{F}$-convex. We have the following lemma.

Lemma 1 ([9]) *Let M be a metric space and $\mathcal{F}$ a compact convexity structure on M which is T-stable. Then, for any type Φ, there exists $x_0 \in M$ such that*

$$\Phi(x_0) = \inf\bigl\{\Phi(x) : x \in M\bigr\}.$$

Hussain and Khamsi [9] and Nicolae [10] proved the following results in metric spaces.

Theorem 1 ([9]) *Let M be a bounded metric space. Assume that there exists a convexity structure $\mathcal{F}$ which is compact and T-stable. $T : M \to M$ be an asymptotic pointwise contraction. Then T has a unique fixed point x_0. Moreover, the orbit $\{T^n x\}$ converges to x_0 for each $x \in M$.*

Theorem 2 ([10]) *Let M be a bounded metric space, $T : M \to M$ and suppose that there exists a convexity structure $\mathcal{F}$ which is compact and T-stable. Assume that*

$$d\bigl(T^n x, T^n y\bigr) \leq \alpha_n(x) r_x\bigl(O_T(y)\bigr) \quad \textit{for every } x, y \in M,$$

where $\alpha_n : M \to \mathbb{R}$ for each $n \in \mathbb{N}$ and the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converges pointwise to a function $\alpha : M \to [0,1)$. Then T has a unique fixed point x_0. Moreover, the orbit $\{T^n x\}$ converges to x_0, for each $x \in M$.

3 Fixed point results for asymptotic pointwise contractive type mappings in metric spaces

In this section, we generalize the results obtained by Hussain and Khamsi [9] and Nicolae [10] for the wider class of weak asymptotic pointwise contraction type mappings.

Theorem 3 *Let (M,d) be a bounded metric space. Assume that there exists a convexity structure $\mathcal{F}$ which is compact and T-stable. Let $T : M \to M$ be a weak asymptotic pointwise contraction type mapping. Then T has a unique fixed point x_0. Moreover, the orbit $\{T^n x\}$ converges to x_0 for each $x \in M$.*

Proof Fix $x \in M$ and define a function f by

$$f(u) = \limsup_{n\to\infty} d\big(T^n x, u\big), \quad u \in M.$$

Since $\mathcal{F}$ is compact and T-stable, there exists $x_0 \in M$ such that

$$f(x_0) = \inf\big\{f(x); x \in M\big\}.$$

Let us show that $f(x_0) = 0$. Indeed, for any $m \geq 1$ we have

$$\begin{aligned} f\big(T^m x_0\big) &= \limsup_{n\to\infty} d\big(T^n x, T^m x_0\big) \\ &= \limsup_{n\to\infty} d\big(T^{m+n} x, T^m x_0\big) \\ &= \limsup_{n\to\infty} d\big(T^m\big(T^n x\big), T^m x_0\big) \\ &\leq \limsup_{n\to\infty} \alpha_m(x_0) d\big(T^n x, x_0\big) + r_m(x_0) \\ &= \alpha_m(x_0) f(x_0) + r_m(x_0), \end{aligned}$$

which implies

$$f(x_0) = \inf\big\{f(x); x \in C\big\} \leq f\big(T^m x_0\big) \leq \alpha_m(x_0) f(x_0) + r_m(x_0). \tag{3.1}$$

Now, by (1.3) and (3.1), we obtain

$$f(x_0) \leq \liminf_{m\to\infty} \big[\alpha_m(x_0) f(x_0) + r_m(x_0)\big] = \alpha(x_0) f(x_0),$$

which forces $f(x_0) = 0$ as $\alpha(x_0) < 1$. Hence, $d(T^n x, x_0) \to 0$ as $n \to \infty$. From this and the continuity of T^N, for some $N \geq 1$, it follows that

$$T^N x_0 = T^N\Big(\lim_{n\to\infty} T^n x\Big) = \lim_{n\to\infty} T^{n+N} x = x_0,$$

namely, x_0 is a fixed point of T^N. Now, repeating the above proof for x_0 instead of x, we deduce that $\{T^n x_0\}$ is convergent to an element of M. But $T^{kN}x_0 = x_0$ for all $k \geq 1$. Hence, $T^n x_0 \to x_0$. We show that $Tx_0 = x_0$. For this purpose, consider an arbitrary $\epsilon > 0$. Then there exists a $k_0 > 0$ such that $d(T^n x_0, x_0) < \epsilon$ for all $n > k_0$. So, choosing a natural number $k > k_0/N$, we obtain

$$d(Tx_0, x_0) = d\bigl(T\bigl(T^{kN}x_0\bigr), x_0\bigr) = d\bigl(T^{kN+1}x_0, x_0\bigr) < \epsilon.$$

Since the choice of $\epsilon > 0$ is arbitrary, we get $Tx_0 = x_0$.

It is easy to verify that T has only one fixed point. Indeed, if $a, b \in M$ are two fixed points of T, then we have

$$d(a,b) = d\bigl(T^n a, T^n b\bigr) \leq \alpha_n(a)d(a,b) + r_n(a).$$

Taking lim inf in the above inequality, it follows that $d(a,b) \leq \alpha(a)d(a,b)$. Since $\alpha(a) < 1$, we immediately get $a = b$. □

In the following, we present an example in a bounded metric space which shows that a mapping of asymptotic pointwise contraction type is not necessary an asymptotic pointwise contraction.

Example 1 Let $M = \prod_{i=1}^n I_i$ ($I_i = [0,1]$), equipped with the Euclidean norm. Then M is a bounded metric space. For each $(x_1, x_2, \ldots, x_n) \in M$, define

$$T(x_1, x_2, \ldots, x_n) = \bigl(f(x_2), f(x_3), \ldots, f(x_n), 0\bigr),$$

where $f : [0,1] \to [0,1]$ is some discontinuous function with $f(0) = 0$. We deduce that T is discontinuous, and then it would not be an asymptotic pointwise contraction. But we see that $T^n x = 0$ for all $x \in M$, and so T is of asymptotic pointwise contraction type.

Theorem 4 *Let (M,d) be a bounded metric space, $T : M \to M$ and suppose there exists a convexity structure $\mathcal{F}$ which is compact and T-stable and T^N is continuous for some integer $N \geq 1$. Assume*

$$\liminf_{n\to\infty} \sup_{y\in M} \bigl\{d\bigl(T^n x, T^n y\bigr) - \alpha_n(x) r_x\bigl(O_T(y)\bigr)\bigr\} \leq 0 \quad \text{for every } x, y \in M,$$

where $\alpha_n : M \to \mathbb{R}$ for each $n \in N$ and the sequence $\{\alpha_n\}$ converges pointwise to a function $\alpha : M \to [0,1)$. Then T has a unique fixed point z. Moreover, the orbit $\{T^n x\}$ converges to z for each $x \in M$.

Proof Taking

$$\gamma_n(x) = \sup_{y\in M} \bigl\{d\bigl(T^n x, T^n y\bigr) - \alpha_n(x) r_x\bigl(O_T(y)\bigr)\bigr\} \in \mathbb{R},$$

it can easily be seen that

$$\liminf_{n\to\infty} \gamma_n(x) \leq 0, \tag{3.2}$$

for all $x \in M$, and

$$d(T^n x, T^n y) \leq \alpha_n(x) r_x(O_T(y)) + \gamma_n(x). \tag{3.3}$$

Fix $x \in M$ and define a function f by

$$f(u) = \limsup_{n\to\infty} d(T^n x, u), \quad u \in M.$$

Since $\mathcal{F}$ is compact and T-stable, there exists $z \in M$ such that

$$f(z) = \inf\{f(x) : x \in M\}.$$

Let us prove that $f(z) = 0$. Indeed, for any $m \geq 1$ we have

$$\begin{aligned} f(T^m z) &= \limsup_{n\to\infty} d(T^n x, T^m z) \\ &= \limsup_{n\to\infty} d(T^{m+n} x, T^m z) \\ &= \limsup_{n\to\infty} d(T^m(T^n x), T^m z) \\ &\leq \limsup_{n\to\infty} \alpha_m(z) r_z(O_T(T^n x)) + \gamma_m(z) \\ &= \alpha_m(z) \limsup_{n\to\infty} r_z(O_T(T^n x)) + \gamma_m(z), \end{aligned}$$

which implies

$$f(z) = \inf\{f(x); x \in C\} \leq f(T^m z) \leq \alpha_m(z) \limsup_{n\to\infty} r_z(O_T(T^n x)) + \gamma_m(z). \tag{3.4}$$

By (3.2) we have $\liminf_{n\to\infty} \gamma_m(z) \leq 0$, thus, for the subsequence $\{\gamma_{m_k}(z)\}$ of $\{\gamma_m(z)\}$, we have

$$\lim_{k\to\infty} \gamma_{m_k}(z) \leq 0. \tag{3.5}$$

Now, by (3.4) and (3.5) we obtain

$$f(z) \leq \liminf_{k\to\infty} \Big[\alpha_{m_k}(z) \limsup_{n\to\infty} r_z(O_T(T^n x)) + \gamma_{m_k}(z)\Big] = \alpha(z) f(z),$$

which forces $f(z) = 0$ as $\alpha(z) < 1$. Hence, $d(T^n x, z) \to 0$ as $n \to \infty$. From this and the continuity of T^N, for some $N \geq 1$, it follows that

$$T^N z = T^N\Big(\lim_{n\to\infty} T^n x\Big) = \lim_{n\to\infty} T^{n+N} x = z;$$

namely, z is a fixed point of T^N. Now, repeating the above proof for z instead of x, we deduce that $T^n z$ is convergent to a member of M. But $T^{kN} z = z$ for all $k \geq 1$. Hence, $T^n z \to z$. We show that $Tz = z$; for this purpose, consider an arbitrary $\epsilon > 0$. Then there exists a

$k_0 > 0$ such that $d(T^n z, z) < \epsilon$ for all $n > k_0$. So, by choosing a natural number $k > k_0/N$, we obtain

$$d(Tz, z) = d\bigl(T\bigl(T^{kN}x_0\bigr), z\bigr) = d\bigl(T^{kN+1}z, z\bigr) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get $Tz = z$.

Assume that T has two fixed points $a, b \in M$, then, for each $n \in \mathbb{N}$,

$$d(a, b) = d\bigl(T^n a, T^n b\bigr) \le \alpha_n(a) r_a\bigl(O_T(b)\bigr) + \gamma_n(a) = \alpha_n(a) d(a, b) + \gamma_n(a).$$

Taking the lim inf in the above inequality, it follows that

$$d(a, b) \le \alpha(a) d(a, b).$$

Since $\alpha(a) < 1$, we immediately get $a = b$. □

4 Fixed point results for weak asymptotic pointwise nonexpansive type mappings in metric spaces

In this section we introduce weak asymptotic pointwise nonexpansive type mappings in metric spaces and we extend the results found of [9].

Definition 2 Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be of asymptotic pointwise nonexpansive type (resp. weak asymptotic pointwise nonexpansive type) if T^N is continuous for some integer $N \ge 1$ and there exists a sequence $\alpha_n : M \to [0, +\infty)$ such that

$$\limsup_{n\to\infty} \sup_{y\in M} \bigl\{ d\bigl(T^n x, T^n y\bigr) - \alpha_n(x) d(x, y) \bigr\} \le 0 \tag{4.1}$$

$$\Bigl(\text{resp. } \liminf_{n\to\infty} \sup_{y\in M} \bigl\{ d\bigl(T^n x, T^n y\bigr) - \alpha_n(x) d(x, y) \bigr\} \le 0\Bigr). \tag{4.2}$$

where $\limsup_{n\to\infty} \alpha_n(x) \le 1$. Taking

$$r_n(x) = \sup_{y\in M} \bigl\{ d\bigl(T^n x, T^n y\bigr) - \alpha_n(x) d(x, y) \bigr\} \in \mathbb{R}^+ \cup \{\infty\}$$

it can easily be seen from (4.1) (resp. (4.2)) that

$$\lim_{n\to\infty} r_n(x) = 0 \tag{4.3}$$

$$\Bigl(\text{resp. } \liminf_{n\to\infty} r_n(x) \le 0\Bigr) \tag{4.4}$$

for all $x \in M$ and

$$d\bigl(T^n x, T^n y\bigr) \le \alpha_n(x) d(x, y) + r_n(x). \tag{4.5}$$

A metric space (M, d) is said to be a length space if each two points of M are joined by a rectifiable path (that is, a path of finite length) and the distance between any two points is taken to be the infimum of the lengh of all rectifiable paths joining them. In this case, d is said to be a length metric (otherwise, known an inner metric or intrinsic metric). In the case that there is no rectifiable path joins two points of the space, the distance between them is said to be ∞.

A geodesic path joining $x \in M$ to $y \in M$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to M such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. (M,d) is said to be a geodesic space if every two points of (M,d) are joined by a geodesic. M is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in M$, which we will denote by $[x,y]$, called the segment joining x to y.

A geodesic metric space $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (M,d) consists of three points in M (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle $\Delta(x_1,x_2,x_3)$ in (M,d) is a triangle $\bar{\Delta}(x_1,x_2,x_3) := \Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in M_k^2 such that $d_{\mathbb{R}^2}(\bar{x}_i,\bar{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$. If $k > 0$ it is further assumed that perimeter of $\Delta(x_1,x_2,x_3)$ is less than $2D_k$, where D_k denotes the diameter of M_k^2. Such a triangle always exists.

A geodesic metric space is said to be a $CAT(k)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(k)$ comparison axiom.

$CAT(k)$: Let Δ be a geodesic triangle in M and $\bar{\Delta} \subset M_k^2$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(k)$ inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x,y) \leq d(\bar{x},\bar{y}).$$

Complete $CAT(0)$ spaces are often called Hadamard spaces. These spaces are of particular relevance to this study.

Finally, we observe that if x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1,y_2]$, which we denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d\left(x, \frac{y_1 \oplus y_2}{2}\right)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2.$$

Let M be a complete $CAT(0)$ space and $x, y \in M$, then for any $\alpha \in [0,1]$ there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x,y]$ such that

$$d\big(z, \alpha x \oplus (1-\alpha)y\big) \leq \alpha d(z,x) + (1-\alpha)d(z,y), \tag{4.6}$$

for any $z \in M$.

Let M be a complete $CAT(0)$ space. A subset $C \subset M$ is convex if for any $x, y \in C$ we have $[x,y] \subset C$. Any type function achieves its infimum, *i.e.*, for any bounded sequence $\{x_n\}$ in a $CAT(0)$ space M, there exists $\omega \in M$ such that $f(\omega) = \inf\{f(x) : x \in M\}$, where

$$f(x) = \limsup_{n\to\infty} d(x_n, x).$$

Theorem 5 *Let M be a complete $CAT(0)$ metric space. Let C be a bounded closed nonempty convex subset of M. If $T : C \to C$ is a weak asymptotic pointwise nonexpansive type, then the fixed point set* $\mathrm{Fix}(T)$ *is closed and convex.*

Proof Fix $x \in C$ and define a function f by

$$f(u) = \limsup_{n\to\infty} d\big(T^n x, u\big), \quad u \in C.$$

Let $x_0 \in C$ be such that $f(x_0) = \inf\{f(x); x \in M\} = f_0$. According to the proof of Theorem 3 we have $f(T^n x_0) \leq \alpha_n(x_0) f_0 + r_n(x_0)$, for any $n \geq 1$. The $CAT(0)$ inequality implies

$$d\left(T^n(x), \frac{T^m x_0 \oplus T^h x_0}{2}\right)^2 \leq \frac{1}{2} d\big(T^n x, T^m x_0\big)^2 + \frac{1}{2} d\big(T^n x, T^h x_0\big)^2 - \frac{1}{4} d\big(T^m x_0, T^h x_0\big)^2.$$

If we let n go to infinity, we get

$$f_0^2 \leq f\left(\frac{T^m x_0 \oplus T^h x_0}{2}\right) \leq \frac{1}{2} f\big(T^m x_0\big)^2 + \frac{1}{2} f\big(T^h x_0\big)^2 - \frac{1}{4} d\big(T^m x_0, T^h x_0\big)^2,$$

which implies

$$\begin{aligned} d\big(T^m x_0, T^h x_0\big) \leq{}& f_0^2\big(2\alpha_m^2(x_0) + 2\alpha_h^2(x_0) - 4\big) + 2r_n^2(x_0) + 2r_h^2(x_0) \\ &+ 4f_0\big(\alpha_m(x_0) r_m(x_0) + \alpha_h(x_0) r_h(x_0)\big). \end{aligned}$$

Since T is of weak asymptotic pointwise nonexpansive type, we get

$$\limsup_{m,h\to\infty} d\big(T^m x_0, T^h x_0\big) \leq 0,$$

which implies $\{T^n x_0\}$ is a Cauchy sequence. Let $v = \lim_{n\to\infty} T^n x_0$. By the proof of Theorem 3 $Tv = v$ and $\mathrm{Fix}(T)$ is closed. In order to prove that $\mathrm{Fix}(T)$ is convex, it is enough to prove $\frac{x \oplus y}{2} \in \mathrm{Fix}(T)$, whenever $x, y \in \mathrm{Fix}(T)$. Let $z = \frac{x\oplus y}{2}$. The $CAT(0)$ inequality implies

$$d\big(T^n z, z\big)^2 \leq \frac{1}{2} d\big(x, T^n z\big)^2 + \frac{1}{2} d\big(y, T^n z\big)^2 - \frac{1}{4} d(x, y)^2$$

for any $n \geq 1$. Since

$$d\big(x, T^n z\big)^2 = d\big(T^n x, T^n z\big)^2 \leq \big(\alpha_n(z) d(z, x) + r_n(z)\big)^2 = \left(\alpha_n(z) \frac{d(x, y)}{2} + r_n(z)\right)^2$$

and

$$d\big(y, T^n z\big)^2 = d\big(T^n y, T^n z\big)^2 \leq \big(\alpha_n(z) d(z, y) + r_n(z)\big)^2 = \left(\alpha_n(z) \frac{d(x, y)}{2} + r_n(z)\right)^2$$

we get

$$d\big(T^n z, z\big)^2 \leq \frac{1}{4}\big(\alpha_n^2(z) - 1\big) d(x, y)^2 + r_n^2(z) + 2\alpha_n(z) r_n(z) d(x, y)$$

for any $n \geq 1$. Since T is of weak asymptotic pointwise nonexpansive type, we get $\lim_{n\to\infty} T^n z = z$, which implies that $T(z) = z$, *i.e.*, $z \in \mathrm{Fix}(T)$. □

Before we state the next and final result of this work, we need the following notation:

$$\{x_n\} \rightharpoonup z \quad \text{if and only if} \quad f(z) = \inf_{x\in C} f(x),$$

where C is a closed convex subset which contains the bounded sequence $\{x_n\}$ and $f(x) = \limsup_{n\to\infty} d(x_n, x)$.

Theorem 6 *Let M be a complete $CAT(0)$ metric space. Let C be a bounded closed nonempty convex subset of M. Let $T : C \to C$ be a weak asymptotic pointwise nonexpansive type. Let $\{x_n\} \in C$ be an approximate fixed point sequence, i.e., $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{x_n\} \rightharpoonup z$. Then we have $Tz = z$.*

Proof Since $\{x_n\}$ is an approximate fixed point sequence, we have

$$f(x) = \limsup_{n\to\infty} d\bigl(T^m x_n, x\bigr),$$

for any $m \geq 1$. Hence $f(T^m x) \leq \alpha_m(x) f(x) + r_m(x)$ (see the proof of Theorem 3). In particular, $\lim_{n\to\infty} f(T^m z) = f(z)$. The $CAT(0)$ inequality implies

$$d\left(x_n, \frac{z \oplus T^n z}{2}\right)^2 \leq \frac{1}{2} d(x_n, z)^2 + \frac{1}{2} d\bigl(x_n, T^m z\bigr)^2 - \frac{1}{4} d\bigl(z, T^m z\bigr)^2,$$

for any $m, n \geq 1$. If $n \to \infty$, we will get

$$f\left(\frac{z \oplus T^n z}{2}\right)^2 \leq \frac{1}{2} f(z)^2 + \frac{1}{2} f\bigl(T^m z\bigr)^2 - \frac{1}{4} d\bigl(z, T^m z\bigr)^2,$$

for any $m \geq 1$. The definition of z implies

$$f(z)^2 \leq \frac{1}{2} f(z)^2 + \frac{1}{2} f\bigl(T^m z\bigr)^2 - \frac{1}{4} d\bigl(z, T^m z\bigr)^2$$

for any $m \geq 1$, or

$$d\bigl(z, T^m z\bigr)^2 \leq 2f\bigl(T^m z\bigr)^2 - 2f(z)^2 \leq 2\bigl(\alpha_m(z) f(z) + r_m(z)\bigr)^2 - 2f(z)^2.$$

Letting $m \to \infty$, we will get $\lim_{n\to\infty} d(z, T^m z) = 0$. The rest of the proof is similar to the one used for Theorem 3. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details

[1] Department of Mathematics, Sanandaj Branch, Islamic Azad University, Sanandaj, Iran. [2] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [3] Young Researchers and Elite Club, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran.

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