Internat. J. Math. & Math. Sci.
VOL. 17 NO. 4 (1994) 799-806

ROTATORY VIBRATION OF SPHERE OF HIGHER ORDER VISCOELASTIC SOLID

P. R. SENGUPTA

Department of Mathematics
University of Kalyani
Kalyani, West Bengal, India

NIBEDITA DE (DAS) and MANIDIPA KAR

201 Manicktola Main Road, Suite #42
Calcutta - 700 054, India

and

LOKENATH DEBNATH

Department of Mathematics
University of Central Florida
Orlando, Florida 32816-1364, U.S.A.

(Received January 8, 1992 and in revised form August 8, 1993)

ABSTRACT. An attempt is made to investigate the rotatory vibration of a sphere of higher order viscoelastic solid considering higher order strain rate and stress rate. The general frequency equation is obtained for this type of vibration of a sphere. As a special case of this analysis, the frequency equations for the first order and the second order viscoelastic solids are derived. It is shown that the classical frequency equation for an isotropic elastic solid follows from this analysis.

KEYWORDS AND PHRASES. Vibration, viscoelastic solid and frequency equation.
1991 AMS SUBJECT CLASSIFICATION CODES. 73D30.

1. INTRODUCTION.

Hopkins [1] has made an extensive study on the problem of dynamic expansion of cavities in elastic solids. Considerable work has been done on the theory of wave propagation in elastic solids and this work is available in many books including Nowacki [2], Eringen and Suhubi [3], Ewing et al. [4]. In addition, several authors including Sengupta and Roy [5-6], Sharpe [7], Jeffreys [8], and Kawasomi and Yosiyama [9] have considered various problems of propagation of waves in an infinite elastic solid medium due to pressure applied on a spherical cavity within the medium. On the other hand, Chakrabarty [10] has discussed the rotational waves in a visco-elastic medium due to a twist on a spherical cavity. Bhattacharyya and Sengupta [11] have investigate the disturbances produced in a visco-elastic medium of higher order due to impulsive forces acting on the surface of a spherical cavity within the medium. Further, Sengupta and his associates [12-14] have studied the problem of rotatory vibration of a sphere of viscoelastic solid. In spite of a vast literature on the subject, relatively less attention has been given to the effects of viscosity, nonhomogeneity and gravity on the wave motions in a viscoelastic solid medium.

So this paper deals with the rotatory vibration of a sphere of higher order viscoelastic solids. The general frequency equation is derived for this type of vibration of a sphere. As a special case of this analysis, the frequency equations for the first order and the second order viscoelastic solids are also found. It is shown that the classical frequency equation for an isotropic elastic solid follows from this study.

2. BASIC EQUATIONS, BOUNDARY CONDITIONS AND SOLUTIONS.

The stress-strain relations in a viscoelastic medium, considering strain rate as well as stress rate of general nature are given by

$$D_\eta \sigma_{ij} = D_\lambda \Delta \delta_{ij} + 2D_\mu e_{ij} \qquad (i,j=1,2,3), \tag{2.1}$$

where σ_{ij} and e_{ij} are the stress tensor and strain tensor respectively. The u_i's $(i=1,2,3)$ are the components of displacement, δ_{ij} is the KRONECKER delta and also the differential operators in t are

$$\begin{aligned} D_\lambda &= \lambda_0 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2} + \cdots\cdots\cdots + \lambda_n \frac{\partial^n}{\partial t^n}, \\ D_\mu &= \mu_0 + \mu_1 \frac{\partial}{\partial t} + \mu_2 \frac{\partial^2}{\partial t^2} + \cdots\cdots\cdots + \mu_n \frac{\partial^n}{\partial t^n}, \\ D_\eta &= \eta_0 + \eta_1 \frac{\partial}{\partial t} + \eta_2 \frac{\partial^2}{\partial t^2} + \cdots\cdots\cdots + \eta_n \frac{\partial^n}{\partial t^n}, \end{aligned} \tag{2.2abc}$$

λ_0, μ_0, η_0 are the Lame elastic constants, $\lambda_k, \mu_k, \eta_k (k=1,2......n)$ are the parameters representing the effect of viscoelasticity of the kth order.

When there are no body forces, the equation of motion of a general viscoelastic solid is

$$\begin{aligned} \rho D_\eta \frac{\partial^2 u_i}{\partial t^2} = &\left[(\lambda_0+\mu_0)+(\lambda_1+\mu_1)\frac{\partial}{\partial t}+(\lambda_2+\mu_2)\frac{\partial^2}{\partial t^2}+\cdots\cdots(\lambda_n+\mu_n)\frac{\partial^n}{\partial t^n}\right]\mathrm{grad}\Delta \\ &+\left(\mu_0+\mu_1\frac{\partial}{\partial t}+\mu_2\frac{\partial^2}{\partial t^2}+\cdots\cdots+\mu_n\frac{\partial^n}{\partial t^n}\right)\nabla^2 u_i \end{aligned} \tag{2.3}$$

where ρ is the density of the material medium and ∇^2 is the Laplacian.

Equation (2.3) can be written in the following form

$$\rho D_\eta \frac{\partial^2 u_i}{\partial t^2} = \sum_{k=0}^{n}\left[(\lambda_k+\mu_k)D_k\right]\mathrm{grad}\Delta + \sum_{k=0}^{n}(\mu_k D_k)\nabla^2 u_i \tag{2.4}$$

where the differential operator D_k is defined as $D_k = \frac{\partial^k}{\partial t^k}$, $(k=1,2,...n)$ and $D_o = 1$.

If every spherical surface concentric with the boundary of the sphere turns round that the z-axis through a small angle proportional to $\phi_1(r)$, where $r=\sqrt{x^2+y^2+z^2}$, and if the components of displacement are u, v, w parallel to the axes Ox, Oy, Oz, then u, v, w are given by

$$u = Ay\phi_1(r)e^{ipt}, \quad \mathrm{v} = -Ax\phi_1(r)e^{ipt}, \quad w=0, \tag{2.5abc}$$

where A is an arbitrary small constant representing the amplitude of the vibrating motion.

We now consider a function $\phi(r)$ such that $\phi'(r) = \frac{\partial \phi}{\partial r} = r\phi_1(r)$. Putting the function $\phi(r)$ in equation (2.5), we get

$$u = Ae^{ipt}\frac{\partial\phi}{\partial y}, \quad v = -Ae^{ipt}\frac{\partial\phi}{\partial x}, \quad w = 0 \tag{2.6abc}$$

These components of displacement make the dilatation $\Delta = 0$, and the equations of motion become

$$\sum_{k=0}^{n}(\mu_k D_k)\nabla^2 u = \rho D_\eta \frac{\partial^2 u}{\partial t^2} \quad \text{and} \quad \sum_{k=0}^{n}(\mu_k D_k)\nabla^2 v = \rho D_\eta \frac{\partial^2 v}{\partial t^2} \tag{2.7ab}$$

Thus, the problem is to solve the equation (2.7abc) subject to the value of u, v from (2.6ab) and to find the value of $\phi_1(r)$.

Now, from the first equation of (2.7ab) and from the value of u from (2.6a) we obtain

$$\nabla^2\left(\frac{\partial\phi}{\partial y}\right) = -\rho p^2\left(\frac{m_2 + in_2}{m_1 + in_1}\right)\frac{\partial\phi}{\partial y}, \tag{2.8}$$

where

$$m_1 = \mu_0 - \mu_2 p^2 + \mu_4 p^4 - \mu_6 p^6 + \cdots\cdots, \quad n_1 = \mu_1 p - \mu_3 p^3 + \mu_5 p^5 - \mu_7 p^7 + \cdots\cdots, \tag{2.9ab}$$

$$m_2 = \eta_0 - \eta_2 p^2 + \eta_4 p^4 - \eta_6 p^6 + \cdots\cdots, \quad n_2 = \eta_1 p - \eta_3 p^3 + \eta_5 p^5 - \eta_7 p^7 + \cdots\cdots. \tag{2.10ab}$$

Equation (2.8) can be put in the form

$$\nabla^2\left(\frac{\partial\phi}{\partial y}\right) + K_v^2\frac{\partial\phi}{\partial y} = 0 \tag{2.11}$$

where
$$K_v^2 = p^2\rho\frac{(m_1 m_2 + n_1 n_2) + i(m_1 n_2 - m_2 n_1)}{m_1^2 + n_1^2} \tag{2.12}$$

Finally, we obtain the following equations from (2.7ab)

$$\frac{\partial}{\partial y}\left[\left(\nabla^2 + K_v^2\right)\phi\right] = 0 \quad \text{and} \quad \frac{\partial}{\partial x}\left[\left(\nabla^2 + K_v^2\right)\phi\right] = 0. \tag{2.13ab}$$

These two equations are satisfied simultaneously if we take

$$\left(\nabla^2 + K_v^2\right)\phi = 0, \tag{2.14}$$

and the spherical symmetry leads to

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right), \tag{2.15}$$

and finally we get

$$\frac{\partial^2}{\partial r^2}(r\phi) + K_v^2(r\phi) = 0. \tag{2.16}$$

Clearly, the solution of (2.16) can be put in the form

$$r\phi = \left(C_1 e^{-\beta r} + C_2 e^{\beta r}\right)\cos\alpha r + i\left(C_1 e^{-\beta r} - C_2 e^{\beta r}\right)\sin\alpha r, \tag{2.17}$$

where C_1, C_2 are arbitrary constants and

$$\alpha = p\left[\rho\left\{\left(m_1^2+n_1^2\right)^{1/2}\left(m_2^2+n_2^2\right)^{1/2} + \left(m_1 m_2 + n_1 n_2\right)\right\}\Big/2\left(m_1^2+n_1^2\right)\right]^{1/2} \tag{2.18a}$$

$$\beta = p\left[\rho\left\{\left(m_1^2+n_1^2\right)^{1/2}\left(m_2^2+n_2^2\right)^{1/2} - \left(m_1 m_2 + n_1 n_2\right)\right\}\Big/2\left(m_1^2+n_1^2\right)\right]^{1/2} \tag{2.18b}$$

For a sphere ϕ is finite everywhere including the origin and as such when r tends to zero the left hand side of the above equation tends to zero, leading to the conclusion that the right hand side should also tend to zero and this implies that $C_1 = -C_2$. Imposing this condition and writing $2C_2 = C$, we have

$$\phi = C[\sinh\beta r\cos\alpha r - i\cosh\beta r\sin\alpha r]r^{-1} \tag{2.19}$$

Using the relation $r\phi_1 = \dfrac{\partial\phi}{\partial r}$, we have

$$\phi_1 = C\Big[r\left\{(\beta\cosh\beta r\cos\alpha r - \alpha\sinh\beta r\sin\alpha r) - i(\beta\sinh\beta r\sin\alpha r + \alpha\cosh\beta r\cos\alpha r)\right\} - \sinh\beta r\cos\alpha r + i\cosh\beta r\sin\alpha r\Big]r^{-3} \tag{2.20}$$

Putting this value of ϕ_1 from equation (2.20) into equation (2.5abc), we have the following components of stresses:

$$\left.\begin{aligned}
X_x &= 2A\left(\frac{m_1+in_1}{m_2+in_2}\right)\phi_1'(r)xyr^{-1}e^{ipt},\\
Y_y &= -2A\left(\frac{m_1+in_1}{m_2+in_2}\right)\phi_1'(r)xyr^{-1}e^{ipt},\\
Z_z &= 0\\
X_y &= A\left(\frac{m_1+in_1}{m_2+in_2}\right)\phi_1'(r)\left(y^2-x^2\right)r^{-1}e^{ipt},\\
Y_z &= -A\left(\frac{m_1+in_1}{m_2+in_2}\right)\phi_1'(r)xzr^{-1}e^{ipt},\\
X_z &= A\left(\frac{m_1+in_1}{m_2+in_2}\right)\phi_1'(r)yzr^{-1}e^{ipt}
\end{aligned}\right\} \tag{2.21}$$

Now if l,m,n are the direction cosines of the normal to the surface of the sphere, then

$$X_r = lX_x + mX_y + nX_z,\quad Y_r = lY_x + mY_y + nY_z,\quad Z_r = lZ_x + mZ_y + nZ_z. \tag{2.22abc}$$

where $(l,m,n) = \left(\dfrac{x}{r},\dfrac{y}{r},\dfrac{z}{r}\right)$. In view of these relations we obtain

$$X_r = A\left(\frac{m_1 + in_1}{m_2 + in_2}\right)y\phi_1'(r)e^{ipt}, \quad Y_r = -A\left(\frac{m_1 + in_1}{m_2 + in_2}\right)x\phi_1'(r)e^{ipt}, \quad Z_r = 0 \tag{2.23abc}$$

For free vibration, the surface of the sphere is free from traction. This implies that

$$\frac{\partial}{\partial r}[\phi_1(r)] = 0 \text{ when } r = a \tag{2.24}$$

Putting the value of $\phi_1(r)$ from equation (2.20) into (2.24) and simplifying, we obtain the frequency equation

$$\tan(\alpha a) = \left[(P_1P_2 + Q_1Q_2) + i(Q_1P_2 - P_1Q_2)\right]/\left(P_2^2 + Q_2^2\right), \tag{2.25}$$

where

$$P_1 = \left(-\beta^2a^2 + \alpha^2a^2 - 3\right)\sinh\beta a + 3\beta a\cosh\beta a, \quad Q_1 = -3\alpha a\cosh\beta a + 2\alpha\beta a^2\sinh\beta a,$$

$$P_2 = 3\alpha a\sinh\beta a - 2\alpha\beta a^2\cosh\beta a, \quad Q_2 = 3\beta a\sinh\beta a + \left(-\beta^2a^2 + \alpha^2a^2 - 3\right)\cosh\beta a.$$

This frequency equation (2.25) determines the frequency of vibration of a sphere of a general viscoelastic solid.

In order to obtain the frequency equation for similar vibration of an isotropic elastic solid sphere we simply put

$$\mu_1 = \mu_2 = \mu_3 = \cdots\cdots = \mu_n = 0, \quad \eta_1 = \eta_2 = \eta_3 = \cdots\cdots = \eta_n = 0, \quad \eta_0 = 1$$

in (2.25), and in that case the value of m_1, n_1, m_2, n_2 are reduced to $m_1 = \mu_0$, $n_1 = 0$, $m_2 = \eta_0 = 1$, $n_2 = 0$ and as such the values of α and β then become

$$\alpha = p\left(\frac{\rho}{\mu_0}\right)^{\frac{1}{2}} \text{ and } \beta = 0. \tag{2.26ab}$$

Under these circumstances we get

$$P_1 = 0,\ P_2 = 0,\ Q_1 = -3pa\left(\frac{\rho}{\mu_0}\right)^{\frac{1}{2}},\ Q_2 = p^2\frac{\rho a^2}{\mu_0} - 3 \tag{2.27}$$

and equation (2.25) reduces to the form

$$\tan(ka) = \frac{3ka}{3 - k^2a^2}, \tag{2.28}$$

where $k^2 = p^2\rho/\mu_0$. This is in agreement with the corresponding classical result of isotropic elastic solid.

We next consider some particular cases of viscoelastic problems.

3. THE VISCOELASTIC SOLID OF VOIGT TYPE.

We now consider the particular case when the material is a viscoelastic solid of VOIGT type. Then the corresponding values of m_1n_1 and m_2n_2 are given by

$$m_1 = \mu_0,\ n_1 = \mu_1 p,\ m_2 = \eta_0,\ n_2 = \eta_1 p \tag{3.1}$$

and then

$$\alpha = p(\rho\eta_0/\mu_0)^{1/2} \text{ and } \beta = p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)/2\mu_0^{3/2}\eta_0^{1/2} \tag{3.2}$$

where higher powers of the small quantities μ_1, η_1 have been neglected. Making use of these values of α and β, the frequency equation (2.25) takes the form

$$\tan\left(\sqrt{\frac{\rho\eta_0}{\mu_0}} \cdot pa\right) = \left[(L_1L_2 + M_1M_2) + i(L_2M_1 - L_1M_2)\right] / \left(L_2^2 + M_2^2\right) \tag{3.3}$$

where

$$L_1 = \left[-\frac{p^4\rho}{4\mu_0^3\eta_0}(\mu_1\eta_0 - \mu_0\eta_1)^2 a^2 + \frac{p^2\rho\eta_0}{\mu_0}a^2 - 3\right]\sinh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}} \cdot a\right] + \frac{3p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}} a \cosh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}} \cdot a\right] \tag{3.4}$$

$$M_1 = -3p\rho^{1/2}\left(\frac{\eta_0}{\mu_0}\right)^{1/2} a \cosh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}} \cdot a\right] + \frac{p^3\rho a^2}{\mu_0^2}(\mu_1\eta_0 - \mu_0\eta_1) \times \sinh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}} \cdot a\right], \tag{3.5}$$

$$L_2 = 3p\rho^{1/2}\left(\frac{\eta_0}{\mu_0}\right)^{1/2} a \sinh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}} a\right] - \frac{p^3\rho a^2}{\mu_0^2} \cosh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}} a\right], \tag{3.6}$$

$$M_2 = \frac{3p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}} a \sinh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}} a\right] + \left[p^2\rho\left(\frac{\eta_0}{\mu_0}\right)a^2 - \frac{p^4\rho(\mu_1\eta_0 - \mu_0\eta_1)^2}{4\mu_0^3\eta_0}a^2 - 3\right]\cosh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0 - \mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}} a\right]. \tag{3.7}$$

In the particular case of an isotropic elastic solid medium the corresponding frequency equation is obtained by putting $\mu_1 = 0, \eta_1 = 0, \eta_0 = 1$ and $L_1 = 0, M_1 = -3p\left(\frac{\rho}{\mu_0}\right)^{1/2} a, L_2 = 0, M_2 = p^2\left(\frac{\rho}{\mu_0}\right)a^2 - 3$ in equation (3.3). Clearly the classical frequency equation (2.28) can be recovered.

4. THE SECOND ORDER VISCOELASTIC SOLIDS.

We now consider the case when the effect of viscoelasticity upto the second order is included. In this case the values of m_1, n_1 and m_2, n_2 become

$$m_1 = \mu_0 - \mu_2 p^2, n_1 = \mu_1 p; \quad m_2 = \eta_0 - \eta_2 p^2, n_2 = \eta_1 p, \tag{4.1}$$

and consequently, the approximate value of α is given by

$$\alpha = p\left(\frac{\rho\eta_0}{\mu_0}\right)^{1/2}\left[1 + \frac{1}{2}\left(\frac{\mu_2\eta_0 - \mu_0\eta_2}{\mu_0\eta_0}\right)p^2\right], \tag{4.2}$$

where the above expression for α is obtained from the general expression by neglecting the higher powers of μ_1, η_1 and μ_2, η_2. Similarly, we derive the value of β as

$$\beta=\frac{p^2\rho^{1/2}}{2\mu_0^{3/2}\eta_0^{1/2}}(\mu_1\eta_0-\mu_0\eta_1)\left[1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right]. \tag{4.3}$$

Now, substituting these values of α and β the frequency equation (2.25) reduces to

$$\tan\left[p\left(\frac{\rho\eta_0}{\mu_0}\right)^{1/2}\left\{1+\frac{1}{2}\left(\frac{\mu_2\eta_0-\mu_0\eta_2}{\mu_0\eta_0}\right)p^2\right\}\right]=\{(R_1R_2+S_1S_2)+i(S_1R_2+S_2R_1)\}/(R_2^2+S_2^2) \tag{4.4}$$

where,

$$\begin{aligned}R_1=&\left[-\frac{p^4\rho}{4\mu_0^3\eta_0}(\mu_1\eta_0-\mu_0\eta_1)^2\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}^2a^2\right.\\&\left.+\frac{p^2\rho\eta_0}{\mu_0}\left\{1+\frac{1}{2}\left(\frac{\mu_2\eta_0-\mu_0\eta_2}{\mu_0\eta_0}\right)p^2\right\}^2a^2-3\right]\sinh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0-\mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}}\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}a\right]\\&+\left[\frac{3}{2}\cdot\frac{p^2\rho^{1/2}a(\mu_1\eta_0-\mu_0\eta_1)}{\mu_0^{3/2}\eta_0^{1/2}}\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}\right]\\&\times\cosh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0-\mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}}\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}a\right],\end{aligned} \tag{4.5}$$

$$\begin{aligned}S_1=&\left[-\frac{3p\rho^{1/2}\eta_0^{1/2}}{\mu_0^{1/2}}\left\{1+\frac{1}{2}\left(\frac{\mu_2\eta_0-\mu_0\eta_2}{\mu_0\eta_0}\right)p^2\right\}a\right]\cosh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0-\mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}}\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}a\right]\\&+\left[\frac{p^3\rho a^2}{\mu_0^2}(\mu_1\eta_0-\mu_0\eta_1)\left(1+\frac{2\mu_2}{\mu_0}p^2\right)\right]\sinh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0-\mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}}\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}a\right],\end{aligned} \tag{4.6}$$

$$\begin{aligned}R_2=&\left[\frac{3p\rho^{1/2}\eta_0^{1/2}}{\mu_0^{1/2}}\left\{1+\frac{1}{2}\left(\frac{\mu_2\eta_0-\mu_0\eta_2}{\mu_0\eta_0}\right)p^2\right\}a\right]\sinh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0-\mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}}\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}a\right]\\&-\left[\frac{p^3\rho a^2}{\mu_0^2}(\mu_1\eta_0-\mu_0\eta_1)\left(1+\frac{2\mu_2}{\mu_0}p^2\right)\right]\cosh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0-\mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}}\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}a\right],\end{aligned} \tag{4.7}$$

$$\begin{aligned}S_2=&\left[\frac{3p^2\rho^{1/2}a(\mu_1\eta_0-\mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}}\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}\right]\\&\times\sinh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0-\mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}}\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}a\right]\\&+\left[-\frac{p^4\rho a^2}{4\mu_0^3\eta_0}(\mu_1\eta_0-\mu_0\eta_1)^2\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}^2\right.\\&\left.+\frac{p^2\rho\eta_0}{\mu_0}\left\{1+\frac{1}{2}\left(\frac{\mu_2\eta_0-\mu_0\eta_2}{\mu_0\eta_0}\right)p^2\right\}^2a^2-3\right]\cosh\left[\frac{p^2\rho^{1/2}(\mu_1\eta_0-\mu_0\eta_1)}{2\mu_0^{3/2}\eta_0^{1/2}}\left\{1+\left(\frac{3}{2}\cdot\frac{\mu_2}{\mu_0}+\frac{1}{2}\cdot\frac{\eta_2}{\eta_0}\right)p^2\right\}a\right],\end{aligned} \tag{4.8}$$

In particular, for an isotropic elastic solid medium, the corresponding frequency equation is obtained by putting

$$\mu_1 = 0, \mu_2 = 0, \eta_1 = 0, \eta_2 = 0, \eta_0 = 1, \tag{4.9}$$

and

$$R_1 = 0,\ S_1 = \frac{-3p\rho^{1/2}}{\mu_0^{1/2}}a,\ R_2 = 0,\ S_2 = \frac{p^2\rho a^2}{\mu_0} - 3 \tag{4.10}$$

in equation (3.3). Consequently, the classical result for elastic solids follows from the above analysis.

REFERENCE

[1] HOPKINS, H. G. Progress in Solid Mechanics, Vol. 1, edited by I. N. Sneddon and R. Hill, North Holland: Amsterdam, 1961, p. 83.

[2] NOWACKI, W. Dynamics of Elastic System, Chapman and Hall: London, 1963, pp. 298-302.

[3] ERINGEN, A. C. and SUHUBI, E. S. Elastodynamics, Vol. II. (Linear Theory), Academic Press: New York, 1975.

[4] EWING, W. M., JARDETZKY, W. S., and PRESS, F. Elastic Waves in Layered Media, McGraw-Hill: New York, 1957, pp. 257-259, 311.

[5] SENGUPTA, P. R. and ROY, L. K. *Indian J. Pure and Appl. Math.* **4**, (1973), 9/10, 723.

[6] SENGUPTA, P. R. and ROY, R. *Pure and Appl. Geophys.* **86**, (1971/III), 18-27.

[7] SHARPE, J. A. *Geophysics* **7** (1962), 144, 311.

[8] JEFFREYS, H. *Monthly Not. Roy. Astron. Soc., Geophys. Suppl.* **2**, (1931), 407.

[9] KAWASOMI, H. and YOSIYAMA, R. *Bull. Earthque. Res. Inst. Tokyo* **13**, (1935), 496.

[10] CHAKRABARTY, S. K. *Geofis. Pure and Appl.* **48**, (1961), 23-26.

[11] BHATTACHARYYA, S. and SENGUPTA, P. R. *Gerlands Beiter, Geophysik* **87**, (1978), 57-62.

[12] SENGUPTA, P. R. and ROY, S. K. *Gerlands Beiter Geophysik*, Leipzig **92**, (1988) 1, S-70-76.

[13] ROY BHOWMIK, M. and SENGUPTA, P. R. *Ind. J. Theo. Phys.* **35**(4), (1984), 295-302.

[14] DATTA, B. K. and SENGUPTA, P. R. *Acta Ciencia Indica*, **10** (3), (1984), 233.

[15] LOVE, A. E. H. The Mathematical Theory of Elasticity, Fourth edition, Dover Publications: New York, 1944.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization