Internat. J. Math. & Math. Sci.
Vol. 1 (1978) 479-496

CHARACTERISTIC APPROXIMATION PROPERTIES OF QUADRATIC IRRATIONALS

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(Received April 19, 1978)

ABSTRACT. Some characteristic approximation properties of quadratic irrationals are studied in this paper. It is shown that the limit points of the sequence δ_n form a subset C(x), and D(x) can be generated from C(x) in a relatively simple way. Another proof of Lekkerkerker's theorem is given using relations between δ_{n-1}, δ_n, δ_{n+1} which are independent of x and n.

KEY WORDS AND PHRASES. Quadratic Irrationals, Approximation of numbers, Badly Approximable Numbers.

AMS(MOS) SUBJECT CLASSIFICATION (1970) CODES. 10F05, 10F35.

0. Throughout this paper x will denote a real irrational number. We introduce

$$||x|| = \min_{k\in\mathbb{Z}} |x-k| \quad , \quad r(x) = x - \left[x + \frac{1}{2}\right]$$

which implies $r(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, $|r(x)| = ||x||$.

Given x , the sequence $n||nx||$, $n \in \mathbb{N}$, contains bounded subsequences (e.g. $n||nx||<1/\sqrt{5}$ for infinitely many n by Hurwitz's theorem), and it seems natural to investigate the set $D(x)$ of all its limit points which describes the various qualities of approximation of x by rationals which occur again and again [1]. A number x is "well approximable" if $0 \in D(x)$ (e.g. if $x=e=2.71...$ or if x is a Liouville number) and "badly approximable" if $0 \notin D(x)$. If $0 \in D(x)$ then [2] $D(x) = [0,\infty)$, hence interesting numbers in this context are the badly approximable numbers.

Let x be represented by the continued fraction $[b_o, b_1, ...]$, let A_n/B_n denote its convergents and let

$$\delta_n = \delta_n(x) = B_n|B_n x - A_n| \quad , \quad n \geq -2 \quad (\delta_n = B_n||B_n x|| \text{ for } n \geq 1). \quad (1)$$

The limit points of the sequence δ_n form a subset $C(x)$ (which is in a sense constructive) and we shall show that $D(x)$ can be generated from $C(x)$ in a relatively simple way (Theorem 1), so the structure of $C(x)$ is basic in our context.

A theorem of Lekkerkerker [5] shows that for a badly approximable number x the set $C(x)$ is finite if and only if x is a quadratic irrational, and the connection between $C(x)$ and $D(x)$ shows that $D(x)$ is discrete if and only if

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1) For results on $\inf D(x)$, which is the inverse of Perron's modular function [5], see [1] and the bibliography of this paper.

2) Let $\eta_i = n_i||n_i x|| \to 0$, choose $0< \alpha \in \mathbb{R}$, and let $n_i^* = n_i\left[\sqrt{\frac{\alpha}{\eta_i}}\right]$. Then $\eta_i\left[\sqrt{\frac{\alpha}{\eta_i}}\right]^2 = n_i^*||n_i^* x||$ for i large and $\eta_i\left[\sqrt{\frac{\alpha}{\eta_i}}\right]^2 \to \alpha$. Hence $\alpha \in D(x)$.

x is (badly approximable and) a quadratic irrational. We will also give another proof of Lekkerkerker's theorem using relations between $\delta_{n-1}, \delta_n, \delta_{n+1}$ which are independent of x and n and seem to tell the whole structure of the δ_n's (Lemma 3, Theorem 3).

1. THE BASIC FORMULAS.

Writing $x = [b_o, b_1, \ldots] = [b_o, b_1, \ldots, b_{n-1} + \frac{1}{\xi_n}]$, $\xi_n = [b_n, b_{n+1}, \ldots]$ and $\rho_n = \frac{B_n}{B_{n-1}}$, $n \geq 1$, $1/\rho_o = 0$ we have for $n \geq 0$ the following well known formulas

$$\xi_n = b_n + \frac{1}{\xi_{n+1}} \tag{2}$$

$$B_n(B_n x - A_n) = \frac{(-1)^n}{\xi_{n+1} + \frac{1}{\rho_n}} \tag{3}$$

$$b_{n+1} = \rho_{n+1} - \frac{1}{\rho_n} \tag{4}$$

(cf. [7] , 13; (4) is a consequence of $B_{n+1} = b_{n+1} B_n + B_{n-1}$, $n \geq -1$).

LEMMA 1. For $n \geq 1$

$$\delta_n + \delta_{n-1} < 1 \quad \text{unless}^{3)} \quad n = 1 \ , \ b_1 = 1 \ , \tag{5}$$

$$\rho_n = \frac{1+\sqrt{1-4\delta_n \delta_{n-1}}}{2\,\delta_{n-1}} \ , \ \frac{1}{\rho_n} = \frac{1-\sqrt{1-4\,\delta_n\delta_{n-1}}}{2\delta_n} \ . \tag{6}$$

PROOF. It follows from (2) and (4) that

$\xi_n + \frac{1}{\rho_{n-1}} = b_n + \frac{1}{\xi_{n+1}} + \frac{1}{\rho_{n-1}} = \frac{1}{\xi_{n+1}} + \rho_n$ ($n \geq 1$) . This and (1), (3)

3) If $b_1 = 1$ then $\delta_o + \delta_1 = (x-[x]) - (x-[x]-1) = 1$.

show that

$$\delta_n + \delta_{n-1} = \frac{\xi_{n+1} + \rho_n}{1 + \rho_n \xi_{n+1}} \qquad \text{for} \quad n \geq 1 \quad , \tag{7}$$

which implies (5) (note that $\xi_{n+1} > 1$) . In order to prove (6) we note that the foregoing calculations also show that

$$1 - 4\delta_n \delta_{n-1} = 1 - 4 \frac{\rho_n \xi_{n+1}}{(1 + \rho_n \xi_{n+1})^2} = \left(\frac{\rho_n \xi_{n+1} - 1}{1 + \rho_n \xi_{n+1}} \right)^2$$

and this leads immediately to (6).

Formulas (4) and (6) suggest the introduction of the function

$$\phi(x,y;z) = \frac{\sqrt{1-4xz} + \sqrt{1-4yz}}{2z} \quad , \quad z > 0 \quad , \quad 4xz < 1 \quad , \quad 4yz < 1 \quad .$$

using this notation, we have

$$b_{n+1} = \phi(\delta_{n-1} \, , \, \delta_{n+1} \, ; \delta_n) \quad , \quad n \geq 0 \qquad (\delta_{-1} = 0) \; . \tag{8}$$

The following properties of ϕ will be used in later sections of this paper :

$$\phi(x,y;z) = \phi(y,x;z) \quad , \tag{9}$$

$$\phi(x,y;z) \downarrow \text{ (strictly) if } \quad x\uparrow \, , \, y\uparrow \quad \text{or} \quad z\uparrow \, , \tag{10}$$

$$\phi(x,1-z;z) = \frac{|2z-1| + \sqrt{1-4xz}}{2z} \quad , \tag{11}$$

$$\phi(x,0;z) - \phi(x,1-z;z) = \frac{1-|2z-1|}{2z} = \begin{cases} 1 & \text{if} \quad z \leq 1/2 \\ \frac{1-z}{z} < 1 & \text{if} \quad z > 1/2 \end{cases} \tag{12}$$

In conclusion we mention that (5) contains Vahlen's result (see e.g. [7] , §14) that at least one of δ_n, δ_{n-1} is $< 1/2$, and Borel's result (see [7],§14) that at least one of δ_{n-1}, δ_n, δ_{n+1} is $< 1/\sqrt{5}$ follows from (6), (8) and (10). Indeed, if this were not true then one of the δ's would be $> 1/\sqrt{5}$ (since $\delta_n = \delta_{n+1} = 1/\sqrt{5}$ and (6) would imply $\rho_{n+1} = \frac{\sqrt{5}+1}{2}$, but ρ_n is rational) and this and (8) and (10) imply

$$b_{n+1} = \phi(\delta_{n-1}, \delta_{n+1}; \delta_n) < \phi(1/\sqrt{5}, 1/\sqrt{5}; 1/\sqrt{5}) = 1 ,$$

but $b_{n+1} \geq 1$.

2. THE RELATION BETWEEN C(x) AND D(x).

In addition to $d(x)$ and $C(x)$ we introduce the sets

$D_s(x)$: the limit points of the sequence $n\, r(nx)$,

$C_s(x)$: the limit points of the sequence $B_n r(B_n x)$.

These sets contain information on the sign of the approximations of x by rationals, and $D(x)$ or $C(x)$ is known if $D_s(x)$ or $C_s(x)$ is known. Let $||nx|| = |nx-m|$, sign $(nx - m) = \varepsilon$. Then it follows from

$$\begin{aligned} n &= \lambda B_k + \mu B_{k-1} \\ m &= \lambda A_k + \mu A_{k-1} \quad , \quad k \geq -1 \end{aligned} \tag{13}$$

by Cramer's rule that $\lambda, \mu \in \mathbb{Z}$ and that

$$\begin{aligned} \lambda &= n|xB_{k-1} - A_{k-1}| + (-1)^k \varepsilon\, B_{k-1}\, ||nx|| \quad , \\ \mu &= n|xB_k - A_k| - (-1)^k \varepsilon\, B_k\, ||nx|| \quad . \end{aligned} \tag{14}$$

THEOREM 1. Let $0 \notin D(x)$. Then $\alpha \in D_s(x)$ if and only if

$$\alpha = \lambda^2\gamma - \lambda\mu\sqrt{1+4\beta\gamma}\ \operatorname{sign}\gamma + \mu^2\beta \quad , \tag{15}$$

where $\lambda, \mu \in \mathbb{N}_o$, $(\lambda,\mu) \neq (0,0)$ and $\beta = \lim B_{k_i-1} r(B_{k_i-1}x)$, $\gamma = \lim B_{k_i} r(B_{k_i}x)$ for some sequence $k_i \to \infty$.

<u>COROLLARY</u>. Formula (15) and $\beta\gamma < 0$ show that $D(x)$ and $C(x)$ are connected by

$$\alpha = \left| \lambda^2|\gamma| - \lambda\mu\sqrt{1-4|\beta|\gamma|} - \mu^2 |\beta| \right| . \tag{16}$$

<u>PROOF of Theorem 1.</u>

Let $n_i r(n_i x) = n_i(n_i x - m_i) \to \alpha \in D_s(x)$, and select $k_i \in \mathbb{N}$ (for all large i) such that

$$B_{k_i} ||n_i x|| \leq n_i ||B_{k_i} x|| \quad , \tag{17}$$

$$B_{k_i+1}||n_i x|| > n_i ||B_{k_i+1}x|| \quad . \tag{18}$$

Define numbers λ_i, μ_i by (13) (with n_i, m_i, k_i instead of n, m, k). It follows from (17) and (14) that $\lambda_i, \mu_i \in \mathbb{N}_o$. Condition (17) implies $B_{k_i} \leq n_i$, since otherwise $||n_i x|| > ||B_{k_i} x||$ by Lagrange's Theorem ([7], §15) which leads to a contradiction to (17). On the other hand, it follows from $||B_{k_i+1}x|| > (B_{k_i+1} + B_{k_i+2})^{-1}$ ([7], §13) and (18) that

$$\frac{n_i^2}{B_{k_i+1} + B_{k_i+2}} \leq n_i^2||B_{k_i+1}x|| < B_{k_i+1} n_i ||n_i x|| = B_{k_i+1}(|\alpha| + o(1))$$

which implies $n_i \leq 2|\alpha|^{1/2} B_{k_i+2}$ for all large i.

It follows from $0 \notin D(x)$ and $B_k||B_k x|| < \frac{1}{b_{k+1}}$ ([7], 13) that $b_{k+1} = O(1)$. Hence, there is a constant $C = C(\alpha, x)$ such that

$$B_{k_i} \le n_i \le C(\alpha, x) B_{k_i - 1} \qquad \text{for all large } i, \tag{19}$$

From (19) and (14) we infer that

$$0 \le \lambda_i \le K_1(\alpha, x) \qquad , \qquad 0 \le \mu_i \le K_2(\alpha, x)$$

for constants K_1, K_2 and all large i.

By taking subsequences, the foregoing shows that sequences $n_i \to \infty$, $k_i \to \infty$ exist such that

$$(20) \quad \begin{cases} n_i r(n_i x) \to \alpha \\ n_i = \lambda B_{k_i} + \mu B_{k_i - 1},\ m_i = \lambda A_{k_i} + \mu A_{k_i - 1},\ \lambda, \mu \in \mathbb{N}_o,\ (\lambda, \mu) \neq (0, 0) \\ B_{k_i - 1}\, r(B_{k_i - 1} x) \to \beta \quad , \quad B_{k_i}\, r(B_{k_i} x) \to \gamma \quad . \end{cases}$$

Let n_i, k_i satisfy (20). Then (note that $r(B_n x) = B_n x - A_n$ for $n \ge 1$)

$$n_i r(n_i x) = \lambda^2 B_{k_i} r(B_{k_i} x) + \lambda\mu (\rho_{k_i} B_{k_i - 1} r(B_{k_i - 1} x) + \frac{1}{\rho_{k_i}} B_{k_i} r(B_{k_i} x) + \mu^2 B_{k_i - 1} r(B_{k_i - 1} x).$$

This and (6) show that every $\alpha \in D_s$ has a representation (15) and that every number (15) belongs to D_s.

REMARKS. 1. Let $K > 0$. Then the proof of Theorem 1 shows that for every $\alpha \in D_s(x)$, $|\alpha| \le K$, a representation (15) holds for some λ and μ which are bounded by a constant which depends on K and x only. Hence, if $C(x)$ is discrete (i.e. $C(x)$ is finite since $B_n||B_n x|| \le 1$), then $D(x)$ is discrete and vice versa.

2. A slight modification of the proof of Theorem 1 also shows that $n||nx|| = n\,|nx-m| < 1/2$ $(n \in \mathbb{N})$ implies $n/m = A_\nu / B_\nu$ for some ν ([7], §13;[2]

Theorem 184; for a more general result compare [4], Proposition 4). In fact, choose $k \geq 1$ such that $B_{k-1} < n \leq B_k$ (n=1 is a trivial case). If $\varepsilon = (-1)^k$ and $n < B_k$, then (14) leads to the contradiction $0 < \lambda < 2n||nx|| < 1$, hence $n = B_k$. If $\varepsilon = (-1)^{k-1}$, then (14) implies $\mu > 0$, $\lambda > -n||nx|| > -1/2$, hence $\lambda \geq 0$. But $\lambda < 1$ since $n \leq B_k$, hence $n = \mu B_{k-1}$, $m = \mu A_{k-1}$.

3. THE STRUCTURE OF C(x) WHEN x IS A QUADRATIC IRRATIONALITY.

We show first that $C(x)$ is finite when x is a quadratic irrationality.

LEMMA 2. If x belongs to a quadratic number field, then $0 \notin C(x)$ and $C_s(x)$ and $C(x)$ are finite.

This Lemma is essentially due to Lekkerkerker [5], see also Perron [6], p.6. The following proof contains an explicit representation of the elements of $C_s(x)$.

PROOF. $x = [b_o, b_1, \ldots]$ is represented in this case by a periodic continued fraction, i.e. $x = [b_o, \ldots, b_{r-1}, \overline{p_o, p_1, \ldots, p_{k-1}}]$, $r \geq 1$, $k \geq 1$. It follows that $b_{r+nk+\nu} = p_\nu$ for $\nu = 0,1,\ldots,k-1$, $n \in \mathbb{N}_o$, and if $x_\nu = [\overline{p_\nu, p_{\nu+1}, \ldots, p_{k-1}, p_o, \ldots p_{\nu-1}}]$, then $\xi_{r+nk+\nu} = x_\nu$.

It follows from (4) that $\rho_n = [b_n, b_{n-1}, \ldots, b_1]$, hence $\rho_{r+nk+\nu-1} \to [\overline{p_{\nu-1}, p_{\nu-2}, \ldots, p_o, p_{k-1}, \ldots, p_\nu}] = c_\nu$ $(n \to \infty)$, and the statement of Lemma 2 follows from (3).

REMARK. It follows from a theorem of Galois ([7], §23) that $c_\nu = -\dfrac{1}{\bar{x}_\nu}$, where $\bar{x}_\nu$ is the conjugate of x_ν. Hence, the elements of C_s are

$$\frac{(-1)^{r+\nu-1}}{x_\nu - \bar{x}_\nu} \quad \text{if } k \text{ is even} \quad , \quad \frac{\pm 1}{x_\nu - \bar{x}_\nu} \quad \text{if } k \text{ is odd.} \qquad (21)$$

This formula leads to an even more explicit representation of the elements of $C_s(x)$.

This representation uses the notation $A_{n,j}/B_{n,j}$ for the convergents of $[b_j, b_{j+1}, \dots]$ ([7], §5). Let A_n/B_n denote the convergents of $[\overline{p_o, \dots, p_{k-1}}]$. Then the elements of $C_s(x)$ are

$$\begin{cases} (-1)^{r+\nu-1} \dfrac{B_{k-1,\nu}}{\sqrt{D}} \text{ if k is even }, \pm \dfrac{B_{k-1,\nu}}{\sqrt{D}} \text{ if k is odd }, \\ \nu = 0, 1, \dots, k-1 \quad , \quad D = (A_{k-1} + B_{k-2})^2 + 4(-1)^{k-1} \end{cases} \tag{22}$$

In fact, we have $x_\nu = \dfrac{A_{k-1,\nu} - B_{k-2,\nu} + \sqrt{D_\nu}}{2B_{k-1,\nu}}$, $D_\nu = (A_{k-1,\nu} + B_{k-2,\nu})^2 + 4(-1)^{k-1}$ ([7], § 19). But $B_{i,j} = A_{i-1,j+1}$, $A_{i,j} = b_j A_{i-1,j+1} + B_{i-1,j+1}$ ([7], §5), and it follows that

$$A_{k-1,\nu-1} + B_{k-2,\nu-1} = b_{\nu-1} A_{k-2,\nu} + B_{k-2,\nu} + A_{k-3,\nu} = b_{k-1+\nu} A_{k-2,\nu} + A_{k-3,\nu} + B_{k-2,\nu}$$

$= A_{k-1,\nu} + B_{k-2,\nu}$. Hence $D_\nu = D_o$, and (22) follows.

4. THE RELATION BETWEEN THREE CONSECUTIVE δ's.

Formula (8) shows that b_{n+1} is a function of $\delta_{n-1}, \delta_n, \delta_{n+1}$. The following Lemma shows that b_{n+1} is also a function of δ_{n-1}, δ_n alone. This fact is the key to the following considerations, which will show that the converse of Lemma 2 is also true.

LEMMA 3. For $n \geq 0$

$$b_{n+1} = \phi(\delta_{n-1}, 0; \delta_n) \quad , \text{ and } \phi(\delta_{n-1}, 0; \delta_n) \notin \mathbb{N} . \tag{23}$$

PROOF. Formulas (3), (6) and (8) imply

$$\xi_{n+1} = \frac{1}{\delta_n} - \frac{1}{\rho_n} = \frac{1 + \sqrt{1 - 4\delta_n \delta_{n-1}}}{2\delta_n} = \phi(\delta_{n-1}, 0; \delta_n) \quad (n \geq 0)$$

and (23) follows from $\xi_{n+1} = [b_{n+1}, b_{n+2}, \dots]$, $b_{n+1} = [\xi_{n+1}]$

(note that ξ_{n+1} is irrational).

REMARK. Formulas (6) and (4) show that

$$\phi(\delta_{n+1}, 0; \delta_n) = \frac{1 + \sqrt{1-4\delta_n\delta_{n+1}}}{2\delta_n} = \rho_{n+1} = [b_{n+1}, b_n, \dots, b_1] \quad (n \geq 0)$$

and it follows

$$b_{n+1} = \phi(\delta_{n+1}, 0; \delta_n) \quad , \quad \phi(\delta_{n+1}, 0; \delta_n) \notin \mathbb{N} \quad , \tag{24}$$

if $n \geq 2$ or if $n = 1$, $b_1 > 1$.

The first formula (24) remains true for $n = 0$.

Lemma 3 shows that a (universal) function Ψ exists such that

$$b_{n+1} = \Psi(\delta_n, \delta_{n-1}) \quad , \quad n \geq 0 \quad , \tag{25}$$

and the remark shows that also $b_{n+1} = \Psi(\delta_n, \delta_{n+1})$ unless $n = 1$, $b_1 = 1$, i.e. unless $n = 1$, $\delta_o > 1/2$.

It follows from (8) that $\Psi(\delta_n, \delta_{n-1}) = \phi(\delta_{n-1}, \delta_{n+1}, \delta_n)$, hence there exists by (10) a function χ such that

$$\delta_{n+1} = \chi(\delta_n, \delta_{n-1}) \quad , \quad n \geq 0 \quad , \tag{26}$$

and similarly $\delta_{n-1} = \chi(\delta_n, \delta_{n+1})$ unless $n = 1$, $b_1 = 1$.

Using the function Ψ , we find explicitely

$$\delta_{n+1} = \chi(\delta_n, \delta_{n-1}) = \frac{1}{4\delta_n}\left[1 - \left(2\delta_n \Psi(\delta_n, \delta_{n-1}) - \sqrt{1-4\delta_{n-1}\delta_n}\right)^2\right] \tag{27}$$

The following theorem gives Ψ in a more convenient form than Lemma 3.

THEOREM 2. Let $n \geq 0$, $k_n = \left[\frac{1}{\delta_n}\right]$. Then $\delta_{n-1} \neq k_n(1-k_n\delta_n)$ and

$$b_{n+1} = \Psi(\delta_n, \delta_{n-1}) = \begin{cases} k_n & \text{if } \delta_{n-1} \in [0, k_n(1-k_n\delta_n)) \\ k_n - 1 & \text{if } \delta_{n-1} \in (k_n(1-k_n\delta_n), (1-\delta_n))\ ^{4)} \end{cases}.$$

PROOF. Assume that $\delta_{n-1} = k_n(1-k_n\delta_n)$. Then

$$\phi(\delta_{n-1},0;\delta_n) = \frac{1+\sqrt{(2\delta_n k_n - 1)^2}}{2\delta_n} = k_n \tag{28}$$

(note that $2\delta_n k_n > 1$) which contradicts (23).

Let $\delta_{n-1} \in [0, k_n(1-k_n\delta_n))$. Then by (10) and (28)

$$k_n + 1 > \frac{1}{\delta_n} = \phi(0,0;\delta_n) \geq \phi(\delta_{n-1},0;\delta_n) > \phi(k_n(1-k_n\delta_n), 0;\delta_n) = k_n \quad \text{and}$$

$k_n = b_{n+1}$ follows from Lemma 3.

Let $\delta_{n-1} \in (k_n(1-k_n\delta_n), 1-\delta_n)$ which implies $n \geq 1$ since $\delta_{-1} = 0$. Then, by (28), (10), (5) and (12)

$$k_n = \phi(k_n(1-k_n\delta_n), 0; \delta_n) > \phi(\delta_{n-1},0;\delta_n) \geq \phi(1-\delta_n,0;\delta_n)$$

$$\geq \phi(0,0;\delta_n) - 1 = \frac{1}{\delta_n} - 1 > k_n - 1 \quad ,$$

and $k_n - 1 = b_{n+1}$ follows from Lemma 3 .

Figure 1 shows the areas of constancy for the function Ψ.

5. THE INFLUENCE OF $0 \notin C(x)$.

Our next step is to introduce the assumption $0 \notin C(x)$, i.e. $\delta_n \geq \lambda > 0$, $n \in \mathbb{N}$, for some λ into our considerations.

LEMMA 4. Let $0 \leq \lambda \leq 1/\sqrt{2}$.

If $n \geq 1$ and if δ_{n-1} and δ_{n+2} are $> \lambda$, then

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4) This interval is empty if $k_n = 1$.

$$\delta_n + \delta_{n+1} < \sqrt{1 - \lambda^2} \quad . \tag{29}$$

PROOF. Our proof depends on the inequality

$$\phi(\lambda, \sqrt{1 - \lambda^2} - z; z) \le 1 \quad \text{if} \quad \frac{1}{2}\sqrt{1 - \lambda^2} \le z < 1 \ , \ 4\lambda z < 1 \ , \tag{30}$$

In order to prove (30) we observe that

$$\sqrt{1-4\lambda z} \le \sqrt{1 - 2\lambda\sqrt{1-\lambda^2}} = \sqrt{(\sqrt{1 - \lambda^2} - \lambda)^2} = \sqrt{1 - \lambda^2} - \lambda,$$

$$\sqrt{1 - 4z(\sqrt{1 - \lambda^2} - z)} = \sqrt{(2z - \sqrt{1 - \lambda^2})^2 + \lambda^2} \le (2z - \sqrt{1-\lambda^2}) + \lambda \ ,$$

and (30) follows from

$$\phi(\lambda, \sqrt{1- \lambda^2} - z; z) \le \frac{1}{2z}\left(\sqrt{1- \lambda^2} - \lambda + 2z - \sqrt{1- \lambda^2} + \lambda\right) = 1 \ .$$

Assume that the assumptions of Lemma 4 hold and that $\delta_n + \delta_{n+1} \ge \sqrt{1- \lambda^2}$.

If $\delta_n \ge \frac{1}{2}\sqrt{1-\lambda^2}$, then by (8), (10) and (30)

$$b_{n+1} = \phi(\delta_{n-1}, \delta_{n+1}; \delta_n) < \phi(\lambda, \delta_{n+1}; \delta_n) \le \phi(\lambda, \sqrt{1-\lambda^2} - \delta_n; \delta_n) \le 1 \quad ,$$

but $b_{n+1} \ge 1$.

Similarily, if $\delta_{n+1} \ge \frac{1}{2}\sqrt{1-\lambda^2}$,

$$b_{n+2} = \phi(\delta_n, \delta_{n+2}; \delta_{n+1}) < \phi(\lambda, \delta_n; \delta_{n+1}) \le \phi(\lambda, \sqrt{1-\lambda^2} - \delta_{n+1}; \delta_{n+1}) \le 1 \ ,$$

but $b_{n+2} \ge 1$.

REMARK. Formula (5) is for $n \ge 2$ a special case of (29) .
If $\delta_n > \lambda$ for all n , then it follows form (29) that $2\lambda < \sqrt{1-\lambda^2}$,
hence $\lambda < 1/\sqrt{5}$.

Lemma 4 will be used now to show that the points (δ_n, δ_{n-1}) keep a certain distance from the discontinuities of Ψ if $0 \notin C(x)$. We introduce the notation

$$\eta_n = k_n(1-k_n\delta_n) \ ,$$

and we assume that $\delta_n > \lambda > 0$ for some $\lambda > 0$ and all $n \in \mathbb{N}$. Let δ_n 1/2 for some fixed $n \geq 2$. Formula (8) and Theorem 2 imply

$$\sqrt{1-4\delta_n\delta_{n-1}} + \sqrt{1-4\delta_n\delta_{n+1}} = \left\{ \begin{array}{lll} 2\,\delta_n k_n & \text{if} & \delta_{n-1} < \eta_n \\ 2\,\delta_n k_n - 2\delta_n & \text{if} & \delta_{n-1} > \eta_n \end{array} \right\} \qquad (31)$$

In what follows we need the inequality $2\delta_n k_n > \frac{2k_n}{k_n+1} \geq \frac{4}{3}$ (note that $k_n \geq 2$) and the formulas $1-4\delta_n\eta_n = (2\,\delta_n k_n - 1)^2$, $1-4\,\delta_n(1-\delta_n) = (1-2\delta_n)^2$.

Let $\delta_{n-1} > \eta_n$, Then it follows from (31) that

$$\sqrt{1} - \sqrt{1-4\delta_n\delta_{n+1}} = \sqrt{1-4\delta_n\delta_{n-1}} - \sqrt{1-4\delta_n\eta_n} \ ,$$

hence (use $\sqrt{a} - \sqrt{b} = (a-b) \,/\, (\sqrt{a} + \sqrt{b})$)

$$\frac{\lambda}{2} \leq \frac{\delta_{n+1}}{2} \leq \frac{\delta_{n+1}}{1+\sqrt{1-4\delta_n\delta_{n+1}}} = \frac{\eta_n - \delta_{n-1}}{\sqrt{1-4\delta_n\delta_{n-1}} + (2\delta_n k_n - 1)} \leq \frac{\eta_n - \delta_{n-1}}{1/3} \ .$$

It follows that

$$\delta_{n-1} \leq \eta_n - \frac{\lambda}{6} \ . \qquad (32)$$

Let $\delta_{n-1} > \eta_n$. Then it follows from (31) that

$$\sqrt{1-4\delta_n\eta_n} - \sqrt{1-4\delta_n\delta_{n-1}} = \sqrt{1-4\delta_n\delta_{n+1}} - \sqrt{1-4\delta_n(1-\delta_n)} \ ,$$

hence, by Lemma 4

$$\frac{\delta_{n-1}-\eta_n}{1/3} \geq \frac{\delta_{n-1}-\eta_n}{2\delta_n k_n - 1 + \sqrt{1-4\delta_n\delta_{n-1}}} = \frac{1-(\delta_n+\delta_{n+1})}{1-4\delta_n\delta_{n+1}+(1-2\delta_n)} \geq \frac{1-\sqrt{1-\lambda^2}}{2} \ .$$

It follows that

$$\delta_{n-1} \geq \eta_n + \frac{1-\sqrt{1-\lambda^2}}{6} \tag{33}$$

Formula (4) implies that all points (δ_n, δ_{n-1}) , $n \geq 2$, are in a certain open triangle, and some straight lines inside of this triangle are excluded by Theorem 2 (cf. figure 1) .

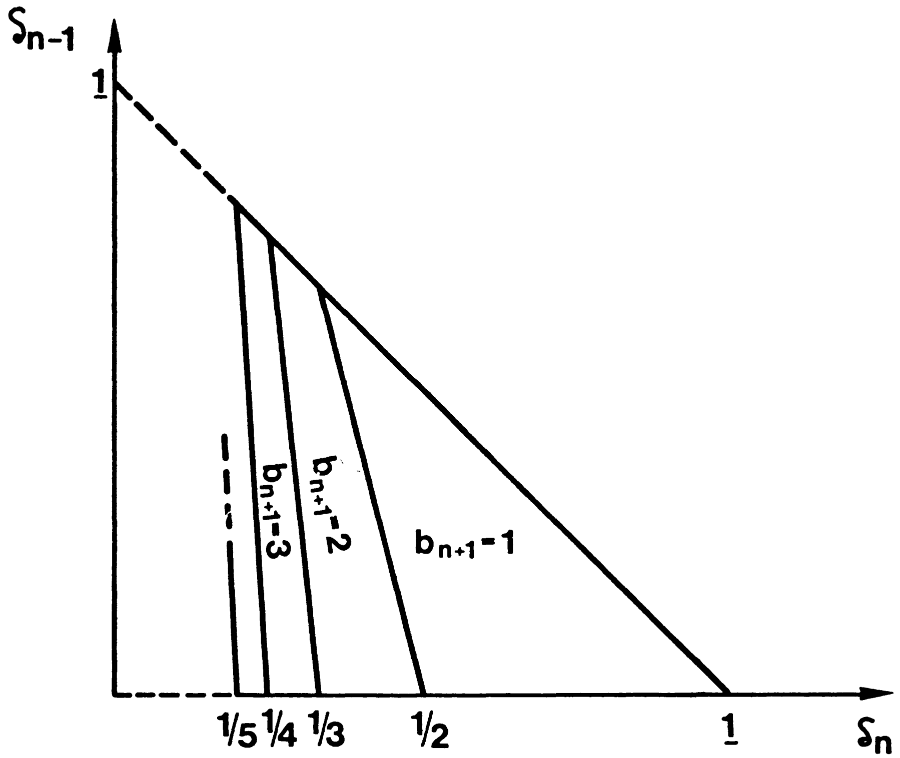

Fig. 1

Moreover, if $\delta_n > \lambda > 0$, then (29), (32) and (33) introduce some additional restriction for (δ_n, δ_{n-1}) . To describe the remaining region we introduce the following set.

Let $M(\lambda)$, $0 \leq \lambda < 1/\sqrt{5}$, denote the (open) set of points (x, y) with the properties

$$x > \lambda\ ,\ y > \lambda\ ,\quad x + y < \sqrt{1-\lambda^2}$$

and for $x < 1/2$

$$y < \left[\frac{1}{x}\right]\left(1 - x\left[\frac{1}{x}\right]\right) - \frac{\lambda}{6} \qquad \text{or} \qquad y > \left[\frac{1}{x}\right]\left(1 - x\left[\frac{1}{x}\right]\right) + \frac{1-\sqrt{1-\lambda^2}}{6}$$

(Figure 2 illustrates $M(\lambda)$ for $\lambda = 1/5$.)

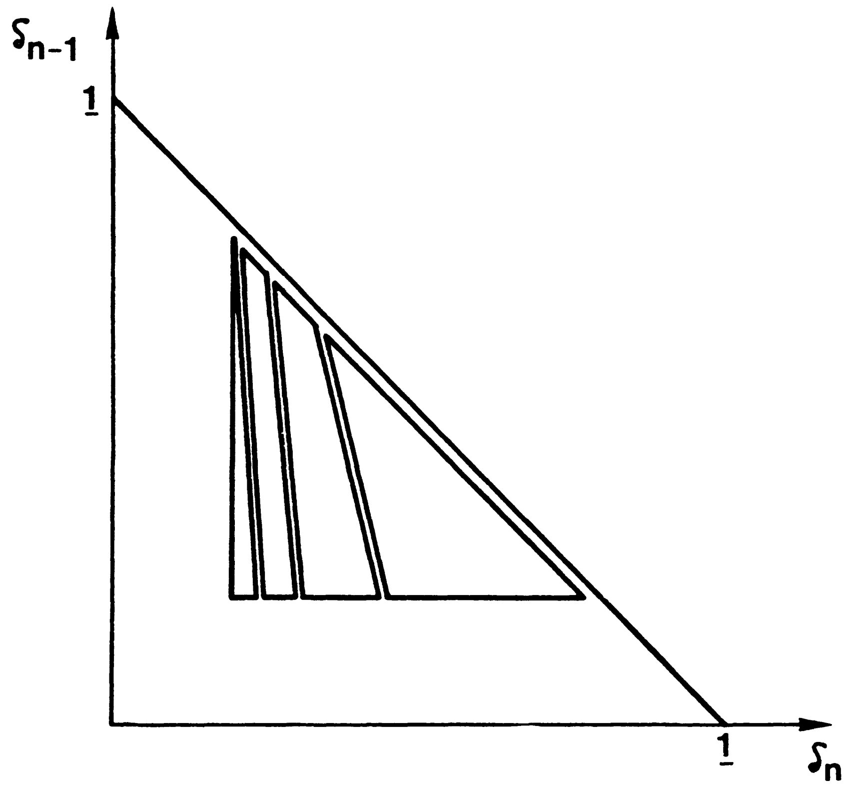

Fig. 2

If $\delta_n > \lambda \geq 0$ for all $n \in \mathbb{N}$, then $(\delta_n, \delta_{n-1}) \in M(\lambda)$ for $n \geq 3$ by (29), (32) and (33). The combination of this result with the results of section 4 leads immediately to

THEOREM 3. There are (universal) functions Ψ and χ , defined on $M(0)$, such that $b_{n+1} = \Psi(\delta_n, \delta_{n-1})$, $\delta_{n+1} = \chi(\delta_n, \delta_{n-1})$, $n \geq 0$.

The functions Ψ and χ are continuous on every $M(\lambda)$. $\lambda > 0$. If $\delta_n > \lambda > 0$ $(\lambda < 1/\sqrt{5})$ for all $n \in \mathbb{N}$, then $(\delta_n, \delta_{n-1}) \in M(\lambda)$ for $n \geq 3$.

6. THE CONVERSE OF LEMMA 2.

We use Theorem 3 to prove the following result of Lekkerkerker [5].

THEOREM 4. If $C_s(x)$ is finite and $0 \notin C_s(x)$, then x belongs to a quadratic number field.

PROOF. Let α_i denote the elements of $C(x)$, and let A be the set to all pairs (α_i, α_j) with $(\delta_n, \delta_{n-1}) \to (\alpha_i, \alpha_j)$ on a subsequence. Since $0 \notin C(s)$, there is some $\lambda > 0$ such that $(\delta_n, \delta_{n-1}) \in M(\lambda)$ for all large n , and $a \in M(\lambda)$ for every $a \in A$.
If $\mathbf{a} = (\alpha_i, \alpha_j) \in A$ then $a' = (\chi(\alpha_i, \alpha_j), \alpha_i) \in A$ since $\delta_{n_k} \to \alpha_i$, $\delta_{n_k-1} \to \alpha_j$ implies $\delta_{n_k+1} = \chi(\delta_{n_k}, \delta_{n_k-1}) \to (\alpha_i, \alpha_j)$ by Theorem 3.

We call a' the successor of a. The set A is finite, hence if $a \in A$ then one of its later successors is again a .
Let $U(a,\varepsilon) = \{(x,y) \,\big|\, |(x,y) - a| < \varepsilon\}$, $a \in A$. Choose $\varepsilon > 0$ such that $u(a,\varepsilon) \subsetneqq M(\lambda)$ for every $a \in A$, $U(a,\varepsilon) \cap U(b,\varepsilon) = \emptyset$ if $a \neq b$.
It follows that Ψ is constant on every $U(a,\varepsilon)$.
Choose $\varepsilon^* \in (0, \varepsilon)$ such that for every $a \in A$

$$\left\{ (\chi(x,y)\ ,\ x)\ \middle|\ (x,y) \in U(a,\varepsilon^*) \right\} \subseteqq U(a',\varepsilon) . \tag{34}$$

Let $N \in \mathbb{N}$ be so large that $(\delta_n, \delta_{n-1}) \subsetneqq U(\mathbf{a},\varepsilon^*)$ for exactly one $a \in A$ depending on $n \geq N$. This establishes a mapping $a = F(\delta_n, \delta_{n-1})$ for every $n \geq N$ which is "successor preserving" , i.e. if $F(\delta_n, \delta_{n-1}) = a$ then $F(\delta_{n+1}, \delta_n) = a'$. Indeed, if $F(\delta_n, \delta_{n-1}) = a$, i.e. $(\delta_n, \delta_{n-1}) \in U(a,\varepsilon^*)$, then $(\delta_{n+1}, \delta_n) = (\chi(\delta_n, \delta_{n-1}), \delta_n) \subseteqq U(a',\varepsilon)$ by (34), hence $(\delta_{n+1}, \delta_n) \in U(a',\varepsilon^*)$ since $n \geq N$.

Take a fixed $n \geq N$, and let $a = F(\delta_n, \delta_{n-1})$. Consider a sequence of successors $a = a^{(o)}$, a' , $a'', \ldots, a^{(\ell)}$, $\ell \in \mathbb{N}$, with $a^{(\ell)} = a$. It follows that

$$F(\delta_{n+\nu+k\ell}, \delta_{n-1+\nu+k\ell}) = a^{(\nu)} \quad , \quad \nu = 0, 1, \ldots, \ell-1, \; k = 0, 1, 2, \ldots \qquad (35)$$

Since Ψ is constant on every $U(a,\varepsilon^*)$, it follows from (35) that $b_{n+\nu+k\ell+1} = \Psi\,(\delta_{n+\nu+K\ell}, \delta_{n+\nu+k\ell-1})$ is independent of k, i.e. the continued fraction for x is periodic. This proves Theorem 4.

REMARK. As conclusion we explain our results in the simplest case $x = (1 + \sqrt{5})/2 = [1,1,\ldots]$. Here $C(x)$ consists of the single point $1/\sqrt{5}$ by (22), and $D(x)$ consists of the points $|\lambda^2 - \lambda\mu - \mu^2|/\sqrt{5}$ with integral $(\lambda,\mu) \neq (0,0)$ by (16). It is well-known (see [3], p. 554) that

$$\lambda^2 - \lambda\mu - \mu^2 = (\lambda - \mu \frac{1+\sqrt{5}}{2})\;(\lambda - \mu \frac{1-\sqrt{5}}{2})$$

represents exactly the integers for which the exponents in the prime factorization must be even for all primes $\equiv$ 2 or 3 mod 5 . So

$$D(x) = \{ \frac{1}{\sqrt{5}}, \frac{4}{\sqrt{5}}, \frac{5}{\sqrt{5}}, \frac{9}{\sqrt{5}}, \frac{11}{\sqrt{5}}, \frac{16}{\sqrt{5}}, \frac{19}{\sqrt{5}}, \frac{20}{\sqrt{5}}, \ldots \}$$

Since this set contains only one element $\in (0,1)$ it determines $C(x)$ uniquely. Furthermore, given $C(y) = \{1/\sqrt{5}\}$, all possible y which produce this set are given by integral transformations $y = \frac{ax+b}{cx+d}$, $ad - bc = \pm 1$.
This follows because the proof of Theorem 4 works with $\ell = 1$, so the continued fraction for y has period 1 (the terms before the period being of no influence with quotients 1 by (22).

ACKNOWLEDGEMENT. This research was supported in part by the National Science Foundation.

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