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Research Article

Application of Topological Degree Method for Solutions of Coupled Systems of Multipoints Boundary Value Problems of Fractional Order Hybrid Differential Equations

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We established the theory to coupled systems of multipoints boundary value problems of fractional order hybrid differential equations with nonlinear perturbations of second type involving Caputo fractional derivative. The proposed problem is as follows: ${}^cD^{\alpha}[x(t)-f(t,x(t))]=g(t,y(t),I^{\alpha}y(t)), t\in J=[0,1]$, ${}^cD^{\alpha}[y(t)-f(t,y(t))]=g(t,x(t),I^{\alpha}x(t)), t\in J=[0,1]$, ${}^cD^{p}x(0)=\psi(x(\eta_1))$, $x'(0)=0,\dots,x^{n-2}(0)=0$, ${}^cD^{p}x(1)=\psi(x(\eta_2))$, ${}^cD^{p}y(0)=\psi(y(\eta_1))$, $y'(0)=0,\dots,y^{n-2}(0)=0$, ${}^cD^{p}y(1)=\psi(y(\eta_2))$, where $p,\eta_1,\eta_2\in(0,1)$, ψ is linear, ${}^cD^{\alpha}$ is Caputo fractional derivative of order α, with $n-1<\alpha\le n$, $n\in\mathbb{N}$, and I^{α} is fractional integral of order α. The nonlinear functions f, g are continuous. For obtaining sufficient conditions on existence and uniqueness of positive solutions to the above system, we used the technique of topological degree theory. Finally, we illustrated the main results by a concrete example.

1. Introduction

Due to a wide range of applications of fractional calculus in various scientific disciplines such as optimization theory, electric networks, signal processing, nonlinear control theory, nonlinear biological systems, controlled thermonuclear fusion, viscoelasticity, chemistry, turbulence, mechanics, oscillation, diffusion, fluid dynamics, stochastic dynamical system, polymer physics, plasma physics, astrophysics, chemical physics, and economics [1–4], the subject area has received much attention among the scientific community. Recently, the theory on existence and uniqueness of solutions to boundary value problems (BVPs) of fractional differential equations (DEs) are well studied and many results are available in literature (see, e.g., [5–10] and the references herein). The perturbed DEs are categorized into various types. Dhage [11] classified different types of perturbations for nonlinear integral and DEs. An important class of DEs which captured great attention in last few decades is the quadratic perturbations of nonlinear differential equations known as hybrid differential equations (HDEs). This class is well studied for BVPs with ordinary DEs. However, existence theory for BVPs with fractional hybrid differential equations (FHDEs) are not well explored and few results are available in the literature (see [12, 13]). This class of DEs includes perturbations of dynamical systems in different ways and hence includes several dynamical systems as special cases. Modern control-command systems often include controllers that perform nonlinear computations to control a physical system, which can typically be described by hybrid automaton containing high dimensional systems of nonlinear DEs. The hybrid systems are dynamical systems that involve the interaction of continuous (real valued) states and discrete (finite valued) states.

Recently, existence of solutions to some classes of HDEs is studied with the use of hybrid fixed point theory (see [14–18]). Dhage and Jadhav [17] developed sufficient conditions for existence of maximal and minimal solutions to the following first-order HDEs:

$$\begin{aligned} &\frac{d}{dt}\left[r(t)-\Theta(t,r(t))\right]=\varphi(t,r(t)), \quad \text{a.e } t\in I, \\ &r(t_0)=r_0\in\mathbb{R}, \end{aligned} \tag{1}$$

where $I = [t_0, t_0 + a) \in \mathbb{R}$ for some t_0, $a \in \mathbb{R}$ with $a > 0$ and $\Theta, \varphi \in C(I \times \mathbb{R}, \mathbb{R})$. The results of [17] were generalized by Lu et al. [18] to the case of fractional order and developed conditions for existence and uniqueness to the following FHDEs:

$$\begin{aligned} &D^q\left[r(t)-\Theta(t,r(t))\right]=\varphi(t,r(t)), \\ &\qquad \text{a.e } t\in I,\ 0<q<1, \\ &r(t_0)=r_0\in\mathbb{R}. \end{aligned} \tag{2}$$

Bashiri et al. in [13] used coupled fixed point theorem, a Krasnoselskii type generalization of fixed point theorem of Burton [19] in Banach spaces, to the case of coupled systems and developed sufficient conditions for existence of solutions to the following coupled systems of two-point BVPs for FHDEs:

$$\begin{aligned} &D^q\left[r(t)-\Theta(t,r(t))\right]=\psi\left(t,s(t),I^{\alpha}(s(t))\right), \\ &\qquad \text{a.e } t\in I, \\ &D^q\left[s(t)-\Theta(t,s(t))\right]=\psi\left(t,r(t),I^{\alpha}(r(t))\right), \\ &\qquad \text{a.e } t\in I,\ 0<q<1,\ \alpha>0, \\ &r(0)=0, \\ &s(0)=0. \end{aligned} \tag{3}$$

Leray Schauder theory is powerful tool in solving operator equations of the form $(I - T)u = w$, where T is compact. But in many situations, T is not compact. Therefore, it is natural to ask whether the solutions of the above operator equation are possible if T is not compact. Schauder constructed an example and showed that it is impossible. But later on Browder, Sadovski and Vath, and so on proved that it is possible to define a complete analogue of the Leray Schauder theory for condensing type mapping with compactness. They called this method the "topological degree method"; see for detail [20]. In 1970, Mawhin introduced the mentioned degree theory for nonlinear Volterra integral equations and differential equations with boundary conditions. On the other hand, using the classical fixed point theory such as "Schauder fixed point theorem" and "Banach contraction principle" required stronger conditions on the nonlinear functions and thus restrict the applicability of these results to limited classes of applied problems and to some specialized systems of BVPs. Finding the fixed points of the respective operator equations corresponding to fractional integral equations needs strong conditions for the compactness of the operator. To relax the criteria and establish weaker conditions in order to extend tools to more classes of BVPs, researchers need to look for some other refined tools of functional analysis. One of such tools is "topological degree theory." The topological degree method is a powerful tool for existence of solutions to BVPs of many mathematical models that arise in applied nonlinear analysis. The concerned method is also called the "prior estimate method." By coincidence degree theory approach, Mawhin [21] studied existence of solutions to the following BVPs:

$$\begin{aligned} &\frac{d}{dt}s(t)=\psi(t,s(t)), \quad t\in[0,1], \\ &s(0)=s(1), \end{aligned} \tag{4}$$

$$\begin{aligned} &-\frac{d^2}{dt^2}s(t)=\psi\left(t,s(t),\frac{d}{dt}s(t)\right), \quad t\in[0,\pi], \\ &s(0)=s(\pi)=0, \end{aligned} \tag{5}$$

under appropriate assumptions. Dinca et al. [22] used this method together with Leray Schauder degree to prove the existence of solutions of the Dirichlet problems with p-Laplacian:

$$\begin{aligned} &-\Delta_p w=\psi(t,w), \quad \text{in } \Omega \\ &w \mid \partial\Omega=0. \end{aligned} \tag{6}$$

Isaia [23] used this method along with "the degree for condensing maps" and proved the existence of solutions for the following integral equation by using appropriate assumption on the functions φ and Θ, where $\varphi : [a_1, a_2] \times \mathbb{R} \to \mathbb{R}$ and $\Theta : [a_1, a_2] \times [a_1, a_2] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $a_1, a_2 \in \mathbb{R}$,

$$s(t)=\varphi(t,s(t))+\int_{a_1}^{a_2}\Theta(t,\xi,s(\xi))\,d\xi, \quad t\in[a_1,a_2]. \tag{7}$$

Wang et al. [24] used "topological degree method" to a class of "nonlocal Cauchy problems" of the following form to study the existence and uniqueness of solutions:

$$\begin{aligned} &{}^cD^p s(t)=\psi(t,s(t)), \quad t\in[0,T], \\ &s(0)+g(s)=s_0, \end{aligned} \tag{8}$$

where ${}^cD^p$ is the Caputo fractional derivative of order $p \in (0, 1]$, $s_0 \in \mathbb{R}$, and $\psi : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous. For more study of fractional DEs via topological degree method, we refer to [6, 24–26]. Recently, Shah et al. [27] applied the "topological degree method" and established sufficient conditions for the existence of at least one solution to the following coupled system of nonlinear ordinary fractional equations with four-point boundary conditions:

$$\begin{aligned} &{}^cD^p x(t)=\phi(t,x(t),y(t)), \quad t\in[0,1], \\ &{}^cD^q y(t)=\psi(t,x(t),y(t)), \quad t\in[0,1], \\ &x(0)=f(x), \\ &x(1)=\lambda x(\eta), \quad \lambda,\eta\in(0,1), \\ &y(0)=g(y), \\ &y(1)=\delta y(\xi), \quad \delta,\xi\in(0,1), \end{aligned} \tag{9}$$

where $p, q \in (1,2]$ and $\phi, \psi : [0,1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous, also $f, g \in ([0,1], \mathbb{R})$. The above system (9) contains two nonlinear functions, namely, ϕ, ψ. While the proposed coupled system contains four nonlinearities as follows:

$$\begin{aligned} {}^cD^{\alpha}\left[x(t) - f(t, x(t))\right] &= g\left(t, y(t), I^{\alpha} y(t)\right), \\ & \quad t \in J = [0,1], \\ {}^cD^{\alpha}\left[y(t) - f(t, y(t))\right] &= g\left(t, x(t), I^{\alpha} x(t)\right), \\ & \quad t \in J = [0,1], \\ {}^cD^{p} x(0) &= \psi\left(x(\eta_1)\right), \\ x'(0) &= 0, \ldots, x^{n-2}(0) = 0, \\ {}^cD^{p} x(1) &= \psi\left(x(\eta_2)\right), \\ {}^cD^{p} y(0) &= \psi\left(y(\eta_1)\right), \\ y'(0) &= 0, \ldots, y^{n-2}(0) = 0, \\ {}^cD^{p} y(1) &= \psi\left(y(\eta_2)\right), \end{aligned} \tag{10}$$

which made the considered problem more general and complicated. Furthermore, in the proposed coupled system (10) of FHDEs, the order of fractional differential operator lies in $(n-1, n]$. Moreover, the boundary conditions of the proposed problem involve Caputo fractional order derivative as well as ordinary derivative of higher order. Moreover, to the best of our knowledge, the topological degree method has not been applied properly for the systems of nonlinear hybrid fractional differential equations.

Motivated by this consideration, our main focus in the present article is to use topological degree approach for condensing mapping to investigate existence of solutions of coupled system (10). We would like to solve the dynamics of the system to determine how the state will evolve in the future, that is, to find a function $x(t)$ called trajectory or solution of the system.

2. Preliminaries

In the following, we provide some basic definitions and results of fractional calculus and topological degree theory. For detailed study, we refer to [1–4, 10, 28–30].

Definition 1 (see [1]). The fractional integral operator of order $r \in \mathbb{R}^+$ of a function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t (t-\xi)^{r-1} f(\xi)\, d\xi, \tag{11}$$

provided that integral on the right is pointwise defined on $(0, \infty)$.

Definition 2 (see [1]). The Caputo fractional order derivative of order $p \in \mathbb{R}^+$ of a continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined as

$${}^cD^p f(t) = \frac{1}{\Gamma(m-p)} \int_0^t (t-\xi)^{m-p-1} f^{(m)}(\xi)\, d\xi, \tag{12}$$

where $m = [p] + 1$, provided that integral on the right is pointwise defined on $(0, \infty)$.

Lemma 3 (see [1, 28]). *The following results hold for fractional integral and Caputo derivative.*

(i) ${}^cD^p[\lambda_1 f(t) + \lambda_2 g(t)] = \lambda_1 {}^cD^p f(t) + \lambda_2 {}^cD^p g(t)$, $\lambda_1, \lambda_2 \in \mathbb{R}$.

(ii) ${}^cD^p I^q f(t) = I^{q-p} f(t)$, ${}^cD^p I^p f(t) = f(t)$.

(iii) ${}^cD^p t^q = (\Gamma(q+1)/\Gamma(q+1-p)) t^{q-p}$, $D^p C = 0$, *where C is a constant.*

(iv) $I^q\, {}^cD^q f(t) = f(t) - \sum_{k=0}^{n-1} (D^k f(0)/\Gamma(k-q+1)) t^k = f(t) - d_0 - d_1 t - d_2 t^2 - d_3 t^3 - \cdots - d_{n-1} t^{n-1}$, $d_i \in \mathbb{R}$, *for* $0 \le i \le n-1$.

Let $J = [0,1]$; the spaces of all continuous functions $Y = C(J, \mathbb{R})$ and $Z = C(J, \mathbb{R})$ are Banach spaces under the usual norms $\|y\| = \sup\{|y(t)| : t \in J\}$ and $\|z\| = \sup\{|z(t)| : t \in J\}$, respectively. Moreover, the product space $Y \times Z$ is a Banach space under the norm $\|(y,z)\| = \|y\| + \|z\|$ and norm $\|(y,z)\| = \max\{\|y\| + \|z\|\}$. In the following, Y is a Banach space and $\mathscr{M} \subset P(Y)$ is the family of all its bounded sets.

We recall the following notions, which can be found in [29].

Definition 4. "The function $\Upsilon : \mathscr{M} \rightarrow \mathbb{R}^+$ defined as $\mu(M_k) = \inf\{d > 0 : M_k$ admits a finite cover by sets of diameter $\le d\}$, where $M_k \in \mathscr{M}$ is called the (Kuratowski-) measure of noncompactness."

Some of the properties of this measure are listed below (without proof).

Proposition 5. *The following assertions hold for Kuratowski measure Υ:*

(i) $\Upsilon(M_k) = 0$ *iff M_k is relatively compact.*

(ii) Υ *is a seminorm; that is,* $\Upsilon(\sigma M_k) = |\sigma| \Upsilon(M_k)$ *and* $\Upsilon(M_{k1} + M_{k2}) \le \Upsilon(M_{k1}) + \Upsilon(M_{k2})$, *where* $M_{k1}, M_{k2} \in \mathscr{M}$, *and* $\sigma \in \mathbb{R}$.

(iii) $M_{k1} \subset M_{k2}$ *implies* $\Upsilon(M_{k1}) \le \Upsilon(M_{k2})$ *and* $\Upsilon(M_{k1} \cup M_{k2}) = \max\{\Upsilon(M_{k1}), \Upsilon(M_{k2})\}$.

(iv) $\Upsilon(\operatorname{conv} M_k) = \Upsilon(M_k)$.

(v) $\Upsilon(\overline{M_k}) = \Upsilon(M_k)$.

Definition 6. Let $\mathscr{L} \subset Y$ and $\Theta : \mathscr{L} \rightarrow Y$ be a continuous bounded mapping. Then Θ is Υ-Lipschitz if $\exists\, k \ge 0 \ni$

$$\Upsilon(\Theta(L)) \le k \Upsilon(L) \quad (\forall)\, L \subset \mathscr{L} \text{ bounded}. \tag{13}$$

Furthermore, if $k < 1$, then Θ is a strict Υ-contraction.

Definition 7. The function Θ is Υ-condensing if

$$\Upsilon(\Theta(L)) < \Upsilon(L) \quad (\forall) L \subset \mathscr{L} \text{ bounded with } \Upsilon(L) > 0. \tag{14}$$

If $\Upsilon(\Theta(L)) \geq \Upsilon(L)$, then $\Upsilon(L) = 0$.

It may be noted that the class of strict Υ-contractions contains the class of Υ-condensing maps. Also every member of Υ-condensing map is Υ-Lipschitz with constant $k = 1$.

Definition 8. $\Theta : \mathscr{L} \to Y$ is called Lipschitz if $\exists\ k > 0 \ni$

$$\|\Theta y_1 - \Theta y_2\| \leq k \|y_1 - y_2\| \quad (\forall) y_1, y_2 \in \mathscr{L}. \tag{15}$$

If $k < 1$, then Θ is a strict contraction.

Proposition 9 (see [23]). *If $\Theta : \mathscr{L} \to Y$ is compact, then Θ is Υ-Lipschitz with zero constant.*

Proposition 10 (see [23]). *If $\Theta : \mathscr{L} \to Y$ is Lipschitz with constant k_1, then Θ is Υ-Lipschitz with the same constant k_1.*

Proposition 11 (see [23]). *If $\Theta_1, \Theta_2 : \mathscr{L} \to Y$ are Υ-Lipschitz with constants k_1 and k_2, respectively, then $\Theta_1 + \Theta_2 : \mathscr{L} \to Y$ is Υ-Lipschitz with constant $k_1 + k_2$.*

The following theorem from [23] plays a vital role for our main result.

Theorem 12. *Let $\Theta : Y \to Y$ be Υ-condensing and*

$$S = \{y \in Y : (\exists) \sigma \in J \ni y = \sigma \Theta y\}. \tag{16}$$

If S is a bounded set in Y, so there exist $r > 0 \ni\ S \subset B_r(0)$; then the degree

$$D(I - \sigma\Theta, B_r(0), 0) = 1 \quad (\forall) \sigma \in J. \tag{17}$$

Consequently, Θ has at least one fixed point and the set of fixed points of Θ lies in $B_r(0)$.

Definition 13 (see [31]). An element $(y_1, y_2) \in Y \times Y$ is called "a coupled fixed point of a mapping" $T : Y \times Y \to Y$ if $T(y_1, y_2) = y_1$ and $T(y_2, y_1) = y_2$.

3. Main Result

Assume that $f : J \times \mathbb{R} \to \mathbb{R}$ and $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Consider the following hypotheses:

($\mathscr{A}1$) $(\partial^k/\partial t^k) f(t, x(t))|_{t=0} = 0$.

($\mathscr{A}2$) There exist $k_1 \in [0, 1) \ni\ \forall (t, x), (t, y) \in J \times \mathbb{R}$,

$$|f(t, x) - f(t, y)| \leq k_1 |x - y|. \tag{18}$$

($\mathscr{A}3$) There exist $c_1, N_1 \geq 0, \ni\ \forall (t, x) \in J \times \mathbb{R}$,

$$|f(t, x)| \leq c_1 |x|^{p_1} + N_1, \quad \text{where } p_1 \in (0, 1). \tag{19}$$

($\mathscr{A}4$) There exist $c_2, N_2 \geq 0, \ni\ \forall (t, x, I^\alpha x) \in J \times \mathbb{R} \times \mathbb{R}$,

$$|g(t, x, I^\alpha x)| \leq c_2 |x|^{p_2} + N_2, \quad \text{where } p_2 \in (0, 1). \tag{20}$$

($\mathscr{A}5$) There exist a continuous function $h \in Y \ni\ |g(t, x(t), y(t))| \leq h(t)$, for $x, y \in \mathbb{R}$ and $t \in J$.

The following lemma is useful in the existence results.

Lemma 14. *If $\mathscr{H} : J \to \mathbb{R}$ is α-time integrable and assuming that the hypothesis ($\mathscr{A}1$) holds, then the solutions of the multipoints BVPs:*

$$\begin{gathered} {}^cD^\alpha [x(t) - f(t, x(t))] = \mathscr{H}(t), \quad t \in J, \\ {}^cD^p x(0) = \psi(x(\eta_1)), \\ x'(0) = 0, \dots, x^{n-2}(0) = 0, \\ {}^cD^p x(1) = \psi(x(\eta_2)) \\ \text{where } \psi \text{ is linear}, \ 0 < \eta_1, \eta_2 < 1, \end{gathered} \tag{21}$$

are the following integral equation:

$$x(t) = f(t, x(t)) + I^\alpha \mathscr{H}(t) + a_1 t^{n-1} + a_2, \quad \text{where } a_1, a_2 \in \mathbb{R}. \tag{22}$$

Proof. Applying I^α on ${}^cD^\alpha[x(t) - f(t, x(t))] = \mathscr{H}(t)$ and using Lemma 3, we obtain

$$\begin{aligned} x(t) &= f(t, x(t)) + I^\alpha \mathscr{H}(t) + d_0 + d_1 t + d_2 t^2 + d_3 t^3 \\ &\quad + d_4 t^4 + \cdots + d_{n-2} t^{n-2} + d_{n-1} t^{n-1}, \\ &\quad \text{where } d_i \in \mathbb{R}, \ \text{for } i = 0, 1, 2, \dots, n-1. \end{aligned} \tag{23}$$

Now by conditions $x'(0) = 0, \dots, x^{n-2}(0) = 0$ and hypothesis ($\mathscr{A}1$), (23) implies $d_1 = d_2 = \cdots = d_{n-2} = 0$. Hence,

$$x(t) = f(t, x(t)) + I^\alpha \mathscr{H}(t) + d_0 + d_{n-1} t^{n-1}. \tag{24}$$

Applying ${}^cD^p$ on (24) and using Lemma 3

$$\begin{aligned} {}^cD^p x(t) &= {}^cD^p f(t, x(t)) + I^{\alpha-p} \mathscr{H}(t) \\ &\quad + d_{n-1} \frac{\Gamma(n)}{\Gamma(n-p)} t^{n-1-p}. \end{aligned} \tag{25}$$

By conditions ${}^cD^p x(0) = \psi(x(\eta_1))$ and ${}^cD^p x(1) = \psi(x(\eta_2))$ and linearity of ψ,

$$\begin{aligned} 0 &= \psi(f(\eta_1, x(\eta_1))) + \psi(I^\alpha \mathscr{H}(\eta_1)) + \psi(d_0) \\ &\quad + d_{n-1} \psi(\eta_1^{n-1}), \end{aligned} \tag{26}$$

$$\begin{aligned} {}^cD^p f(1, x(1)) &+ I^{\alpha-p} \mathscr{H}(1) + d_{n-1} \frac{\Gamma(n)}{\Gamma(n-p)} \\ &= \psi(f(\eta_2, x(\eta_2))) + \psi(I^\alpha \mathscr{H}(\eta_2)) + \psi(d_0) \\ &\quad + d_{n-1} \psi(\eta_2^{n-1}). \end{aligned} \tag{27}$$

Subtracting (26) from (27) and rearranging, we get

$$d_{n-1} = \frac{1}{\{\psi(\eta_2^{n-1}) - \psi(\eta_1^{n-1}) - \Gamma(n)/\delta_2\Gamma(n-p)\}}\left[{}^cD^p f(1, x(1)) + I^{\alpha-p}\mathscr{H}(1) - \{\psi(f(\eta_2, x(\eta_2))) - \psi(f(\eta_1, x(\eta_1))) + \psi(I^\alpha\mathscr{H}(\eta_2)) - \psi(I^\alpha\mathscr{H}(\eta_1))\}\right]. \quad (28)$$

Using (28) in (26), we get

$$d_0 = \frac{-\psi(f(\eta_1, x(\eta_1))) - \psi(I^\alpha\mathscr{H}(\eta_1))}{\psi(1)} - \frac{\psi(\eta_1^{n-1})}{\psi(1)\{\psi(\eta_2^{n-1}) - \psi(\eta_1^{n-1}) - \Gamma(n)/\delta_2\Gamma(n-p)\}}\left[{}^cD^p f(1, x(1)) + I^{\alpha-p}\mathscr{H}(1) - \{\psi(f(\eta_2, x(\eta_2))) - \psi(f(\eta_1, x(\eta_1))) + \psi(I^\alpha\mathscr{H}(\eta_2)) - \psi(I^\alpha\mathscr{H}(\eta_1))\}\right]. \quad (29)$$

From (24), (28), and (29), it follows that

$$x(t) = f(t, x(t)) + I^\alpha\mathscr{H}(t) + a_1 t^{n-1} + a_2, \quad (30)$$

where $a_1 = \kappa_3\psi(1)/\kappa_2$, $a_2 = -\kappa_o - (\kappa_1/\kappa_2)\kappa_3$, $\kappa_o = (-\psi(f(\eta_1, x(\eta_1))) - \psi(I^\alpha\mathscr{H}(\eta_1)))/\psi(1)$, $\kappa_1 = \psi(\eta_1^{n-1})$, $\kappa_2 = \psi(1)\{\psi(\eta_2^{n-1}) - \psi(\eta_1^{n-1}) - \Gamma(n)/\delta_2\Gamma(n-p)\}$ and $\kappa_3 = {}^cD^p f(1, x(1)) + I^{\alpha-p}\mathscr{H}(1) + \psi(f(\eta_1, x(\eta_1))) - \psi(f(\eta_2, x(\eta_2))) + \psi(I^\alpha\mathscr{H}(\eta_1)) - \psi(I^\alpha\mathscr{H}(\eta_2))$. □

In view of Lemma 14, system (10) is equivalent to the following coupled systems of integral equations:

$$x(t) = f(t, x(t)) + I^\alpha g(t, y(t), I^\alpha y(t)) + a_1 t^{n-1} + a_2,$$
$$y(t) = f(t, y(t)) + I^\alpha g(t, x(t), I^\alpha x(t)) + b_1 t^{n-1} + b_2, \quad (31)$$
$$\text{where } a_1, b_1, a_2, b_2 \in \mathbb{R}.$$

Define operators $F_1, F_2, G : Y \to Y$ by

$$(F_1 z)t = f(t, z(t)) + a_1 t^{n-1} + a_2,$$
$$(F_2 z)t = f(t, z(t)) + b_1 t^{n-1} + b_2, \quad (32)$$
$$(Gz)t = I^\alpha g(t, z(t), I^\alpha z(t)).$$

By virtue of these operators, system (31) can be written as

$$x(t) = (F_1 x)t + (Gy)t = T_1(x, y),$$
$$y(t) = (F_2 y)t + (Gx)t = T_2(x, y).$$

Which implies $(x, y) = (T_1, T_2)(x, y)$.

$$\text{If } \widetilde{F} = (F_1, F_2), \quad \widetilde{G} = (G, G), \quad \widetilde{T} = (T_1, T_2), \quad \text{then } (x, y) = \widetilde{T}(x, y) \Longrightarrow u = \widetilde{T}u, \quad \text{where } u = (x, y); \quad (33)$$

and solutions of system (10) are fixed points of $\widetilde{T}$.

Lemma 15. *Assume that hypotheses ($\mathscr{A}2$) and ($\mathscr{A}3$) hold, then the operator $\widetilde{F} : Y \times Y \to Y$ is Υ-Lipschitz with constant k_1.*

Proof. For $= (x_1, y_1)$, $v = (x_2, y_2) \in Y \times Y$, using ($\mathscr{A}2$), it follows that

$$\begin{aligned}\left\|(\widetilde{F}u)t - (\widetilde{F}v)t\right\| &= \|(F_1, F_2)(x_1, y_1)t - (F_1, F_2)(x_2, y_2)t\| = \sup_{t\in J}|((F_1x_1)t, (F_2y_1)t) - ((F_1x_2)t, (F_2y_2)t)| = \sup_{t\in J}|((F_1x_1)t - (F_1x_2)t, (F_2y_1)t - (F_2y_2)t)| \\ &= \sup_{t\in J}[|(F_1x_1)t - (F_1x_2)t| + |(F_2y_1)t - (F_2y_2)t|] \\ &= \sup_{t\in J}\left[\left|f(t, x_1(t)) + a_1t^{n-1} + a_2 - \{f(t, x_2(t)) + a_1t^{n-1} + a_2\}\right| + \left|f(t, y_1(t)) + b_1t^{n-1} + b_2 - \{f(t, y_2(t)) + b_1t^{n-1} + b_2\}\right|\right] \\ &= \sup_{t\in J}[|f(t, x_1(t)) - f(t, x_2(t))| + |f(t, y_1(t)) - f(t, y_2(t))|] \le k_1|x_1(t) - x_2(t)| + k_1|y_1(t) - y_2(t)| \le k_1(\|x_1 - x_2\| + \|y_1 - y_2\|) = k_1\|(x_1 - x_2, y_1 - y_2)\| = k_1\|u - v\|.\end{aligned} \quad (34)$$

Thus, $\widetilde{F}$ is Lipschitz with constant k_1. Hence, by Proposition 10, $\widetilde{F}$ is Υ-Lipschitz with constant k_1. Moreover, by using ($\mathscr{A}3$), we get the following condition for $\widetilde{F}$:

$$\left\|\widetilde{F}u\right\| \le c_1\|u\|^{p_1} + N_3, \quad \text{where } N_3 = N_1 + a_1 + a_2. \quad (35)$$

□

Lemma 16. *Assume that the hypotheses ($\mathscr{A}4$) and ($\mathscr{A}5$) hold; then the operator $\widetilde{G} : Y \times Y \to Y$ is Υ-Lipschitz with zero constant.*

Proof. To show that $\widetilde{G}$ is compact it is enough to show that $\widetilde{G}$ is uniformly bounded and equicontinuous. Let $\widetilde{S} = \{u_n = (x_n, y_n) : \|(x_n, y_n)\| \le R\} \subset Y \times Y \ni (x_n, y_n) \to (x, y)$ as $n \to \infty$ in $\widetilde{S}$. We have to show that $\|\widetilde{G}u_n - \widetilde{G}u\| \to 0$.

$$\begin{aligned}
\lim_{n\to\infty} \left\{\left(\widetilde{G}u_n\right) t\right\} &= \lim_{n\to\infty} \left\{(G,G)\left(x_n, y_n\right) t\right\} \\
&= \left(\lim_{n\to\infty} \left(Gx_n\right) t, \lim_{n\to\infty} \left(Gy_n\right) t\right) \\
&= \left(\lim_{n\to\infty} I^{\alpha} g\left(t, x_n(t), I^{\alpha} x_n(t)\right),\right. \\
&\quad \left.\lim_{n\to\infty} I^{\alpha} g\left(t, y_n(t), I^{\alpha} y_n(t)\right)\right) = \left(\frac{1}{\Gamma(\alpha)}\right. \\
&\quad \cdot \lim_{n\to\infty} \int_0^t (t-\xi)^{\alpha-1} g\left(t, x_n(t), I^{\alpha} x_n(t)\right) d\xi, \frac{1}{\Gamma(\alpha)} \\
&\quad \left.\cdot \lim_{n\to\infty} \int_0^t (t-\xi)^{\alpha-1} g\left(t, y_n(t), I^{\alpha} y_n(t)\right) d\xi\right)
\end{aligned} \tag{36}$$

which by Lebesgue dominated convergence theorem gives

$$\begin{aligned}
\lim_{n\to\infty} \left\{\left(\widetilde{G}u_n\right) t\right\} &= \left(\frac{1}{\Gamma(\alpha)}\right. \\
&\quad \cdot \int_0^t (t-\xi)^{\alpha-1} g\left(t, x(t), I^{\alpha} x(t)\right) d\xi, \frac{1}{\Gamma(\alpha)} \\
&\quad \left.\cdot \int_0^t (t-\xi)^{\alpha-1} g\left(t, y(t), I^{\alpha} y(t)\right) d\xi\right) \\
&= \left(I^{\alpha} g\left(t, x(t), I^{\alpha} x(t)\right), I^{\alpha} g\left(t, y(t), I^{\alpha} y(t)\right)\right) \\
&= \left((Gx)t, (Gy)t\right) = \left(\widetilde{G}u\right) t.
\end{aligned} \tag{37}$$

Thus, the image of a convergent sequence is convergent, so $\widetilde{G}$ is a continuous on $\widetilde{S}$. Moreover, in view of ($\mathscr{A}4$), $\widetilde{G}$ satisfies the following condition:

$$\left\|\widetilde{G}u\right\| \le c_2 \|u\|^{p_2} + N_2. \tag{38}$$

Now, using ($\mathscr{A}5$), we obtain

$$\begin{aligned}
\left|\left(\widetilde{G}u\right) t\right| &= \left|\left((Gx)t, (Gy)t\right)\right| = |(Gx)t| + |(Gy)t| \\
&= \left|I^{\alpha} g\left(t, x(t), I^{\alpha} x(t)\right)\right| + \left|I^{\alpha} g\left(t, y(t), I^{\alpha} y(t)\right)\right| \\
&= \frac{1}{\Gamma(\alpha)} \left|\int_0^t (t-\xi)^{\alpha-1} g\left(t, x(t), I^{\alpha} x(t)\right) d\xi\right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left|\int_0^t (t-\xi)^{\alpha-1} g\left(t, y(t), I^{\alpha} y(t)\right) d\xi\right| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \left|g\left(t, x(t), I^{\alpha} x(t)\right)\right| d\xi \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \left|g\left(t, y(t), I^{\alpha} y(t)\right)\right| d\xi \\
&\le \frac{2\|h\|}{\Gamma(\alpha)} \int (t-\xi)^{\alpha-1} d\xi = \frac{2\|h\|}{\Gamma(\alpha+1)} t^{\alpha}.
\end{aligned} \tag{39}$$

Upon using $t \le 1$, (39) implies that

$$\left\|\widetilde{G}u\right\| \le \frac{2\|h\|}{\Gamma(\alpha+1)}. \tag{40}$$

Hence, $\widetilde{G}$ is uniformly bounded. Now, for $t_1, t_2 \in J$, and any $u \in \widetilde{S}$, consider

$$\begin{aligned}
\left|\left(\widetilde{G}u\right) t_1 - \left(\widetilde{G}u\right) t_2\right| &= \left|G(x,y) t_1 - G(x,y) t_2\right| \\
&= \left|(Gx) t_1 - (Gx) t_2, (Gy) t_1 - (Gy) t_2\right| = \left|(Gx) t_1\right. \\
&\quad \left.- (Gx) t_2\right| + \left|(Gy) t_1 - (Gy) t_2\right| \\
&= \left|I^{\alpha} g\left(t_1, x(t_1), I^{\alpha} x(t_1)\right)\right. \\
&\quad \left.- I^{\alpha} g\left(t_2, x(t_2), I^{\alpha} x(t_2)\right)\right| \\
&\quad + \left|I^{\alpha} g\left(t_1, y(t_1), I^{\alpha} y(t_1)\right)\right. \\
&\quad \left.- I^{\alpha} g\left(t_2, y(t_2), I^{\alpha} y(t_2)\right)\right| = \frac{1}{\Gamma(\alpha)} \left|\int_0^{t_1} (t_1\right. \\
&\quad - \xi)^{\alpha-1} g\left(\xi, x(\xi), I^{\alpha} x(\xi)\right) d\xi - \int_0^{t_2} (t_2-\xi)^{\alpha-1} \\
&\quad \left.\cdot g\left(\xi, x(\xi), I^{\alpha} x(\xi)\right) d\xi\right| + \frac{1}{\Gamma(\alpha)} \left|\int_0^{t_1} (t_1\right. \\
&\quad - \xi)^{\alpha-1} g\left(\xi, y(\xi), I^{\alpha} y(\xi)\right) d\xi - \int_0^{t_2} (t_2-\xi)^{\alpha-1} \\
&\quad \left.\cdot g\left(\xi, y(\xi), I^{\alpha} y(\xi)\right) d\xi\right| \\
&= \frac{1}{\Gamma(\alpha)} \left|\left[\int_0^{t_1} (t_1-\xi)^{\alpha-1} g\left(\xi, x(\xi), I^{\alpha} x(\xi)\right) d\xi\right.\right. \\
&\quad - \int_0^{t_1} (t_2-\xi)^{\alpha-1} g\left(\xi, x(\xi), I^{\alpha} x(\xi)\right) d\xi \\
&\quad + \int_0^{t_1} (t_2-\xi)^{\alpha-1} g\left(\xi, x(\xi), I^{\alpha} x(\xi)\right) d\xi \\
&\quad \left.\left.- \int_0^{t_2} (t_2-\xi)^{\alpha-1} g\left(\xi, x(\xi), I^{\alpha} x(\xi)\right) d\xi\right]\right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left|\left[\int_0^{t_1} (t_1-\xi)^{\alpha-1} g\left(\xi, y(\xi), I^{\alpha} y(\xi)\right) d\xi\right.\right. \\
&\quad - \int_0^{t_1} (t_2-\xi)^{\alpha-1} g\left(\xi, y(\xi), I^{\alpha} y(\xi)\right) d\xi \\
&\quad + \int_0^{t_1} (t_2-\xi)^{\alpha-1} g\left(\xi, y(\xi), I^{\alpha} y(\xi)\right) d\xi \\
&\quad \left.\left.- \int_0^{t_2} (t_2-\xi)^{\alpha-1} g\left(\xi, y(\xi), I^{\alpha} y(\xi)\right) d\xi\right]\right|
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \left| \left[\int_0^{t_1} \left[(t_1 - \xi)^{\alpha-1} - (t_2 - \xi)^{\alpha-1} \right] \cdot g(\xi, x(\xi), I^\alpha x(\xi))\, d\xi + \int_0^{t_1} (t_2 - \xi)^{\alpha-1} \cdot g(\xi, x(\xi), I^\alpha x(\xi))\, d\xi + \int_{t_2}^{0} (t_2 - \xi)^{\alpha-1} \cdot g(\xi, x(\xi), I^\alpha x(\xi))\, d\xi \right] \right| + \frac{1}{\Gamma(\alpha)} \left| \left[\int_0^{t_1} \left[(t_1 - \xi)^{\alpha-1} - (t_2 - \xi)^{\alpha-1} \right] \cdot g(\xi, y(\xi), I^\alpha y(\xi))\, d\xi + \int_0^{t_1} (t_2 - \xi)^{\alpha-1} \cdot g(\xi, y(\xi), I^\alpha y(\xi))\, d\xi + \int_{t_2}^{0} (t_2 - \xi)^{\alpha-1} \cdot g(\xi, y(\xi), I^\alpha y(\xi))\, d\xi \right] \right| \le \frac{\|h\|}{\Gamma(\alpha)} \left[\left| \int_0^{t_1} \left\{ (t_1 - \xi)^{\alpha-1} - (t_2 - \xi)^{\alpha-1} \right\} d\xi \right| + \left| \int_{t_2}^{t_1} (t_2 - \xi)^{\alpha-1} d\xi \right| \right] + \frac{\|h\|}{\Gamma(\alpha)} \left[\left| \int_0^{t_1} \left\{ (t_1 - \xi)^{\alpha-1} - (t_2 - \xi)^{\alpha-1} \right\} d\xi \right| + \left| \int_{t_2}^{t_1} (t_2 - \xi)^{\alpha-1} d\xi \right| \right] \le \frac{2\|h\|}{\Gamma(\alpha+1)} \left[\left| t_1^\alpha - t_2^\alpha \right| + \left| t_2 - t_1 \right|^\alpha - \left| t_2 - t_1 \right|^\alpha \right] = \frac{2\|h\|}{\Gamma(\alpha+1)} \left| t_1^\alpha - t_2^\alpha \right|. \tag{41}$$

Since t^α is uniformly continuous on J for $n-1 < \alpha \le n$, so for $\epsilon > 0$ $\exists \delta > 0$ $\ni$ if $|t_1 - t_2| < \delta$, then $|t_1^\alpha - t_2^\alpha| < (\Gamma(\alpha + 1)/2\|h\|)\,\epsilon$. Thus, (41) becomes

$$\left| \left(\widetilde{G}u\right) t_1 - \left(\widetilde{G}u\right) t_2 \right| < \frac{2\|h\|}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)}{2\|h\|} \epsilon. \tag{42}$$

Thus, $|(\widetilde{G}u)t_1 - (\widetilde{G}u)t_2| < \epsilon$ if $|t_1 - t_2| < \delta$; thus, G is equicontinuous. Hence, by Arzela-Ascoli Theorem $\widetilde{G}$ is compact and by virtue of Proposition 9, $\widetilde{G}$ is Υ-Lipschitz with zero constant. □

Theorem 17. *If $f : Y \times Y \to Y$ and $g : Y \times Y \to Y$ satisfying conditions ($\mathscr{A}1$)–($\mathscr{A}5$), then the integral equation*

$$u(t) = (x(t), y(t)) = \big(f(t, x(t)) + a_1 t^{n-1} + a_2 + I^\alpha g(t, y(t), I^\alpha y(t)), f(t, y(t)) + b_1 t^{n-1} + b_2 + I^\alpha g(t, x(t), I^\alpha x(t)) \big), \quad t \in J, \tag{43}$$

has at least one solution $u \in J$ and the set of solutions of system (10) is bounded in Y.

Proof. Let $\widetilde{F}, \widetilde{G}, \widetilde{T} : Y \times Y \to Y$ be the operators defined in (32). These are continuous and bounded. By Lemma 15, it follows that $\widetilde{F}$ is Υ-Lipschitz with constant k_1. Also by Lemma 16, $\widetilde{G}$ is Υ-Lipschitz with zero constant. Thus, by using Proposition 11, $\widetilde{T}$ is Υ-Lipschitz with constant k_1. Set

$$\widetilde{S} = \left\{ u \in Y \times Y : (\exists)\, \sigma \in J \ni u = \sigma \widetilde{T} u \right\}. \tag{44}$$

We will prove that $\widetilde{S}$ is bounded in $Y \times Y$. For this let $u \in \widetilde{S}$; then $u = \sigma \widetilde{T} u$, where $\sigma \in J$. Now by (35) and (38),

$$\begin{aligned} \|u\| &= \|(x, y)\| = \left\| \sigma \widetilde{T}(x, y) \right\| \le \left\| \widetilde{T}(x, y) \right\| \\ &= \left\| \widetilde{F}(x, y) \right\| + \left\| \widetilde{G}(x, y) \right\| \\ &\le c_1 \|u\|^{p_1} + N_3 + c_2 \|u\|^{p_2} + N_2 \\ 1 &\le \frac{c_1}{\|u\|^{1-p_1}} + \frac{N_3}{\|u\|} + \frac{c_2}{\|u\|^{1-p_2}} + \frac{N_2}{\|u\|}. \end{aligned} \tag{45}$$

If $\|u\| \to \infty$, then $1 \le 0$, a contradiction. Thus, $\|u\|$ is bounded in $Y \times Y$. Consequently, we deduced by Theorem 12 that $\widetilde{T}$ has at least one fixed point which is bounded in $Y \times Y$. □

Remark 18. Here we remark that the conditions ($\mathscr{A}3$), ($\mathscr{A}4$) hold for $p_1 = p_2 = 1$. Therefore, in view of this remark, Theorem 17 is also valid for $p_1 = p_2 = 1$.

4. Illustrative Example

Example 1. Consider the coupled system of FHDEs given by

$$D^{5/2} \left[x(t) - \frac{\cos t\, |x(t)|}{4(10 + |x(t)|)} \right] = \frac{t \left[y(t) + I^{5/2} y(t) \right]}{10 + |y(t)|}, \quad t \in J = [0, 1],$$

$${}^cD^{5/2} \left[y(t) - \frac{\cos t\, |y(t)|}{4(10 + |y(t)|)} \right] = \frac{t \left[x(t) + I^{5/2} x(t) \right]}{10 + |x(t)|}, \quad t \in J = [0, 1],$$

$${}^cD^{1/2} x(0) = \sum_{k=0}^{10} \frac{1}{20} x\left(\frac{1}{2}\right),$$

$$x'(0) = 0, \ldots, x^{n-2}(0) = 0,$$

$$
\begin{aligned}
{}^{c}D^{1/2}x(1) &= \sum_{k=0}^{10} \frac{1}{20} x\left(\frac{1}{2}\right), \\
{}^{c}D^{1/2}y(0) &= \sum_{k=0}^{10} \frac{1}{20} y\left(\frac{1}{2}\right), \\
& \qquad y'(0) = 0, \dots, y^{n-2}(0) = 0, \\
{}^{c}D^{1/2}y(1) &= \sum_{k=0}^{10} \frac{1}{20} y\left(\frac{1}{2}\right).
\end{aligned} \tag{46}
$$

From system (46) we see that $f(t, x(t)) = \cos t|x(t)|/4(10 + |x(t)|)$, $g(t, x(t), I^{\alpha}x(t)) = t[x(t) + I^{5/2}x(t)]/(10 + |x(t)|)$, $\eta_1 = \eta_2 = 1/2$, $\alpha = 5/2$, $p = 1/2$, $\psi(x(\eta_1)) = \psi(x(\eta_2)) = \sum_{k=0}^{10}(1/20)x(1/2)$. Upon computation, we have $c_1 = 1/4$, $p_1 = 1$, $N_1 = 0$, $p_2 = 1$, $c_2 = 1/4$, $N_2 = 0$, $a_1 = 0.0055$, $a_2 = 1.0852$, and $N_3 = 1.0907$. In view of Theorem 17, $\widetilde{S} = \{u = (x, y) \in Y \times Y : u = (1/2)\widetilde{T}u\}$ is the solution set; then

$$
\begin{aligned}
\|u\| &\le \left\|\widetilde{T}u\right\| = \left\|\widetilde{F}u\right\| + \left\|\widetilde{G}u\right\| \\
&\le c_1 \|u\|^{p_1} + N_3 + c_2 \|u\|^{p_2} + N_2.
\end{aligned} \tag{47}
$$

From which, we have $\|u\| \le 2N_3 = 2.1814$. Thus, system (46) has at least one solution and the set of solutions of $\widetilde{S}$ is bounded in $C(J, \mathbb{R}) \times C(J, \mathbb{R})$.

Conflicts of Interest

There are no conflicts of interest with regard to this paper.

Authors' Contributions

All authors equally contributed to this paper.

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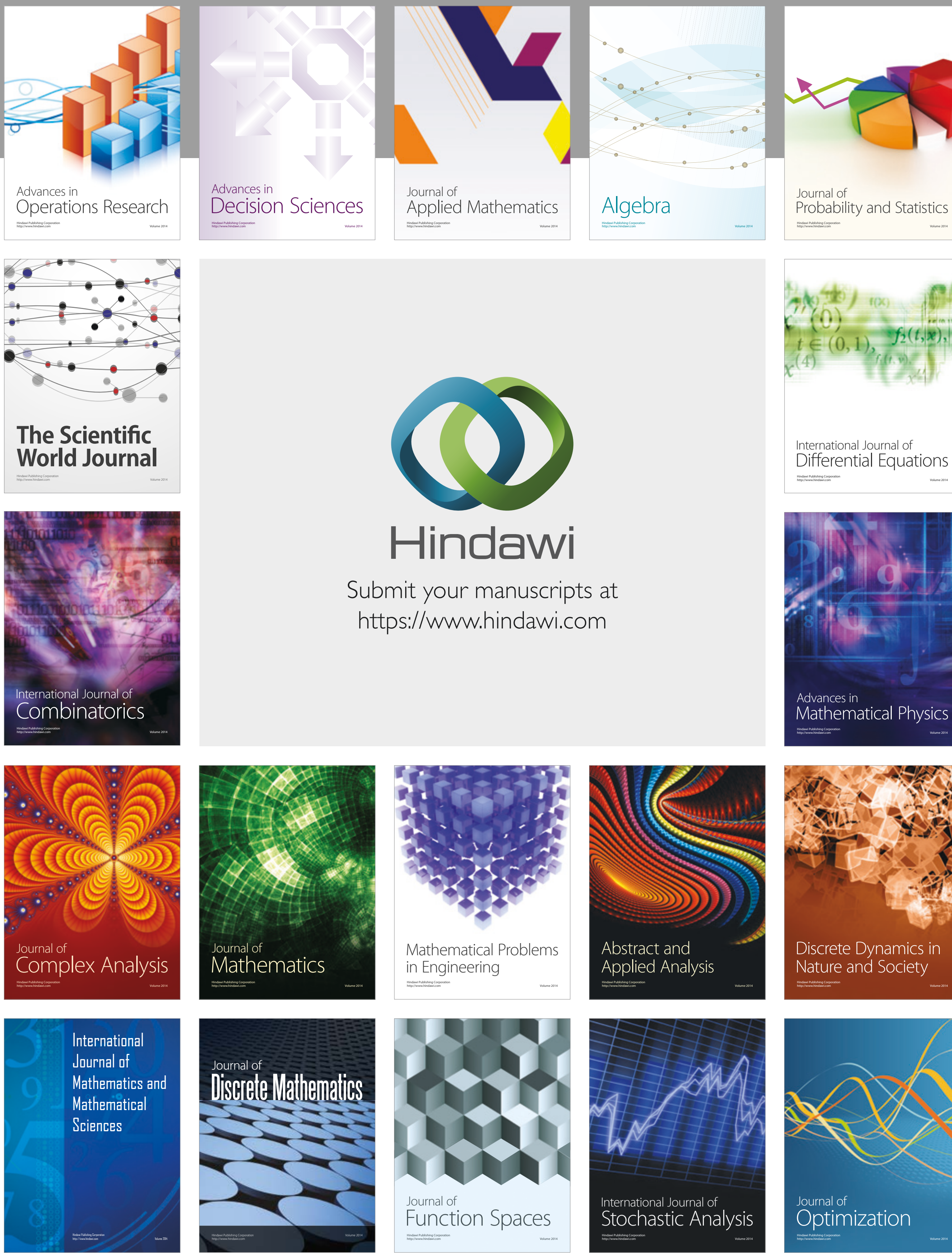

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