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Research Article

Multiplicity of Periodic Solutions for Third-Order Nonlinear Differential Equations

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We study the existence of periodic solutions for third-order nonlinear differential equations. The method of proof relies on Schauder's fixed point theorem applied in a novel way, where the original equation is transformed into second-order integrodifferential equation through a linear integral operator. Finally, examples are presented to illustrate applications of the main results.

1. Introduction

Questions on the existence and the multiplicity of periodic solutions are important topics in qualitative analysis of differential equations. Much work related to periodic solutions for second-order differential equations has been done by using various theorems and methods of nonlinear functional analysis; see [1–10] and references therein. In this paper, we investigate existence of periodic solutions of the following differential equation:

$$u'''(t) = g(u(t)) - f(t, u(t)), \qquad (1)$$

where $g(u) : R \to R$, $f(t,u)$ is ω-periodic in t, and $\omega > 0$. Third-order differential equations of the above type arise, for example, in various fields of agriculture, biology, economics, and physics [11–14]. Questions related to this class of differential equations have recently attracted considerable attention from the researcher community; see, for example, [15–19].

A naive idea in study of higher-order (in particular third-order) differential equations is to translate the equation into a first-order system of differential equations by defining $x_1 = x$, $x_2 = x'$, $x_3 = x''$, ... This method works well, if the task is to show the existence of periodic solutions. However, it does not obviously lead to existence proofs for positive periodic solutions, since the condition $x = x_1 \geq 0$ of positivity for the higher-order equation is different from the natural positivity condition ($x_1, x_2, \ldots \geq 0$) for the corresponding system.

Another frequently used approach is to transform the third-order equation into a corresponding integral equation and to establish the existence of positive periodic solutions based on a fixed point theorem in cones. In order to follow this path, one needs an explicit representation of Green's function for corresponding ordinary equation; see [20, 21]. In [20], Agarwal gave the explicit Green function for the nth-order and $2m$th-order differential equations. Futhermore, Anderson has studied Green's function for a third-order boundary value problem in [21].

It should be noted that (1) includes many important models. For example, it arises in many fields of science and technology, such as physics, mechanics, and engineering. We refer the reader to [22–27] for recent results on such models.

The main purpose of this paper is to show the existence of periodic solutions of (1) by means of Schauder's fixed point theorem. The method of proof used in this paper is based on a simple but novel idea, and it consists of two steps as follows.

(1) The first step is to transform the original equation into a second-order integrodifferential equation through a linear integral operator.

(2) The second step is to apply Schauder's fixed point theorem.

After the above steps, the existence of a single periodic solution for (1) has been established under suitable behavior of functions g and f on some closed set. In addition, some information on the location of a periodic solution is obtained, leading to results on the multiplicity of solutions. To our belief, neither this method nor similar results can be found in the literature.

The paper is organized as follows. In Section 2 we introduce a lemma, which is crucial in proving the main results. Section 3 is devoted to the existence results on solutions of (1). In Section 4 we give some examples to illustrate potential applications of the main results.

2. Preliminaries

Let $X = \{u \in C(R,R) : u(t+\omega) = u(t) \text{ for all } t \in R\}$ with the norm $\|u\| = \max_{t\in[0,\omega]}|u(t)|$; then X is a Banach space.

Let $p > 0$, $h_1, h_2 \in X$, and consider the following two differential equations:

$$u'' - pu' + p^2u = h_1(t), \tag{2}$$

$$u'' + pu' + p^2u = h_2(t). \tag{3}$$

Lemma 1. *Assume that $0 < p\omega < 2\pi/\sqrt{3}$. Then (2) has a unique ω-periodic solution $\overline{u}$ satisfying*

$$\overline{u}(t) = \int_0^\omega G(t,s)\,h_1(s)\,ds, \quad t \in [0,\omega], \tag{4}$$

and (3) has a unique ω-periodic solution $\widehat{u}$ satisfying

$$\widehat{u}(t) = \int_0^\omega H(t,s)\,h_2(s)\,ds, \quad t \in [0,\omega], \tag{5}$$

where

$$G(t,s) = \frac{1}{A} \cdot \begin{cases} e^{(p/2)(\omega+t-s)} \cdot \left[\sin\frac{\sqrt{3}}{2}p(\omega+t-s) + e^{(p/2)\omega}\sin\frac{\sqrt{3}}{2}p(s-t)\right], & 0 \le t \le s, \\ e^{(p/2)(\omega+t-s)} \cdot \left[\sin\frac{\sqrt{3}}{2}p(\omega+s-t) + e^{-(p/2)\omega}\sin\frac{\sqrt{3}}{2}p(t-s)\right], & t \ge s, \end{cases}$$

$$H(t,s) = \frac{1}{B} \cdot \begin{cases} e^{-(p/2)(\omega+t-s)} \cdot \left[\sin\frac{\sqrt{3}}{2}p(\omega+t-s) + e^{-(p/2)\omega}\sin\frac{\sqrt{3}}{2}p(s-t)\right], & 0 \le t \le s, \\ e^{-(p/2)(\omega+t-s)} \cdot \left[\sin\frac{\sqrt{3}}{2}p(\omega+s-t) + e^{(p/2)\omega}\sin\frac{\sqrt{3}}{2}p(t-s)\right], & t \ge s, \end{cases}$$

$$A = \sqrt{3}\left(\frac{1+e^{p\omega}}{2} - e^{(p/2)\omega}\cos\frac{\sqrt{3}}{2}p\omega\right)p > 0,$$

$$B = \sqrt{3}\left(\frac{1+e^{-p\omega}}{2} - e^{-(p/2)\omega}\cos\frac{\sqrt{3}}{2}p\omega\right)p > 0. \tag{6}$$

Proof. We only consider (2). Let u_1, u_2 be two ω-periodic solutions of (2). Then, $u^* = u_1 - u_2$ is ω-periodic solution of the differential equation

$$u'' - pu' + p^2u = 0. \tag{7}$$

Note that u^* can be written as

$$u^* = c_1e^{(p/2)t}\cos\frac{\sqrt{3}}{2}pt + c_2e^{(p/2)t}\sin\frac{\sqrt{3}}{2}pt. \tag{8}$$

From $u^{*(i)}(0) = u^{*(i)}(\omega)$, $i = 0, 1$, we obtain $c_1 = c_2 = 0$. Thus $u_1 = u_2$.

Next, we show that

$$\begin{gathered} \overline{u}'' - p\overline{u}' + p^2\overline{u} = h_1(t), \quad t \in [0,\omega], \\ \overline{u}^{(i)}(0) = \overline{u}^{(i)}(\omega), \quad i = 0, 1. \end{gathered} \tag{9}$$

By direct computation, we get

$$\begin{aligned} \overline{u}'(t) &= \frac{p}{2A} \cdot \int_0^t e^{(p/2)(\omega+t-s)}h_1(s) \cdot \left(\sin\frac{\sqrt{3}}{2}p(\omega+s-t) + e^{-p\omega/2}\sin\frac{\sqrt{3}p}{2}(t-s)\right)ds \\ &+ \frac{\sqrt{3}p}{2A} \cdot \int_0^t e^{(p/2)(\omega+t-s)}h_1(s) \cdot \left(-\cos\frac{\sqrt{3}}{2}p(\omega+s-t) + e^{-p\omega/2}\cos\frac{\sqrt{3}p}{2}(t-s)\right)ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{p}{2A}\\
&\cdot\int_t^{\omega} e^{(p/2)(\omega+t-s)}h_1(s)\\
&\quad\cdot\left(\sin\frac{\sqrt{3}}{2}p(\omega+t-s)+e^{p\omega/2}\sin\frac{\sqrt{3}p}{2}(s-t)\right)ds\\
&+\frac{\sqrt{3}p}{2A}\\
&\cdot\int_t^{\omega} e^{(p/2)(\omega+t-s)}h_1(s)\\
&\quad\cdot\left(\cos\frac{\sqrt{3}}{2}p(\omega+t-s)-e^{p\omega/2}\cos\frac{\sqrt{3}p}{2}(s-t)\right)ds,\\
&\overline{u}''(t)\\
&=-\frac{p^2}{2A}\\
&\cdot\int_0^{t} e^{(p/2)(\omega+t-s)}h_1(s)\\
&\quad\cdot\left(\sin\frac{\sqrt{3}}{2}p(\omega+s-t)+e^{-p\omega/2}\sin\frac{\sqrt{3}p}{2}(t-s)\right)ds\\
&+\frac{\sqrt{3}p^2}{2A}\\
&\cdot\int_0^{t} e^{(p/2)(\omega+t-s)}h_1(s)\\
&\quad\cdot\left(-\cos\frac{\sqrt{3}}{2}p(\omega+s-t)+e^{-p\omega/2}\cos\frac{\sqrt{3}p}{2}(t-s)\right)ds\\
&-\frac{p^2}{2A}\\
&\cdot\int_t^{\omega} e^{(p/2)(\omega+t-s)}h_1(s)\\
&\quad\cdot\left(\sin\frac{\sqrt{3}}{2}p(\omega+t-s)+e^{p\omega/2}\sin\frac{\sqrt{3}p}{2}(s-t)\right)ds\\
&+\frac{\sqrt{3}p^2}{2A}\\
&\cdot\int_t^{\omega} e^{(p/2)(\omega+t-s)}h_1(s)\\
&\quad\cdot\left(\cos\frac{\sqrt{3}}{2}p(\omega+t-s)-e^{p\omega/2}\cos\frac{\sqrt{3}p}{2}(s-t)\right)ds.
\end{aligned}
\tag{10}
$$

Hence, we have

$$\overline{u}''-p\overline{u}'+p^2\overline{u}=h_1(t). \tag{11}$$

Moreover, it is easy to check that $\overline{u}^{(i)}(0)=\overline{u}^{(i)}(\omega)$, $i=0,1$. □

Remark 2. If $0<p\omega<2\pi/\sqrt{3}$, then

$$G(t,s)>0,\quad H(t,s)>0,\quad t,s\in[0,\omega]. \tag{12}$$

Remark 3. Consider

$$\int_0^{\omega}G(t,s)\,ds=\frac{1}{p^2},\qquad \int_0^{\omega}H(t,s)\,ds=\frac{1}{p^2}. \tag{13}$$

Define two operators $T,S:C^0(X)\to C^2(X)$ by

$$
\begin{aligned}
T&:h_1\longrightarrow\overline{u}(t)=Th_1,\\
S&:h_2\longrightarrow\hat{u}(t)=Sh_2.
\end{aligned}
\tag{14}
$$

From Lemma 1, one can easily check that T, S are compact, increasing operators for $0<p\omega<2\pi/\sqrt{3}$.

Remark 4. If $h\equiv c$ is a constant function, then

$$Th=\frac{c}{p^2},\qquad Sh=\frac{c}{p^2}, \tag{15}$$

for $0<p\omega<2\pi/\sqrt{3}$.

Next, we define two operators J and K on X by

$$
\begin{aligned}
(Ju)(t)&=\int_t^{t+\omega}\frac{e^{(s-t)p}}{e^{p\omega}-1}u(s)\,ds,\quad u\in X,\\
(Ku)(t)&=\int_t^{t+\omega}\frac{e^{(t+\omega-s)p}}{e^{p\omega}-1}u(s)\,ds,\quad u\in X,
\end{aligned}
\tag{16}
$$

where $p>0$ is a constant. For any $u\in X$, $Ju\in X\cap C^1(R)$ and $Ku\in X\cap C^1(R)$.

Lemma 5. *If $u\in X\cap C^2(R)$ satisfies the differential equation*

$$u''-pu'+p^2u=g(Ju)-f(t,Ju)+p^3(Ju), \tag{17}$$

then Ju is a ω-periodic solution of (1).

If $u\in X\cap C^2(R)$ satisfies the differential equation

$$-u''-pu'-p^2u=g(Ku)-f(t,Ku)-p^3(Ku), \tag{18}$$

then Ku is a ω-periodic solution of (1).

Proof. Note that

$$
\begin{aligned}
&(Ju)(t+\omega)\\
&\quad=\int_{t+\omega}^{t+2\omega}\frac{e^{(s-t-\omega)p}}{e^{p\omega}-1}u(s)\,ds\\
&\quad=\int_t^{t+\omega}\frac{e^{(r-t)p}}{e^{p\omega}-1}u(r+\omega)\,dr\\
&\quad=(Ju)(t),\quad u\in X,\\
&(Ju)'(t)=-p(Ju)(t)+u(t).
\end{aligned}
\tag{19}
$$

If $u\in X\cap C^1(R)$, $Ju\in X\cap C^2(R)$ and

$$
\begin{aligned}
(Ju)''(t)&=-p(Ju)'(t)+u'(t)\\
&=p^2(Ju)(t)-pu(t)+u'(t).
\end{aligned}
\tag{20}
$$

If $u \in X \cap C^2(R)$, $Ju \in X \cap C^3(R)$ and

$$(Ju)'''(t) = p^2(-p(Ju)(t) + u(t)) - pu'(t) + u''(t) = -p^3(Ju)(t) + p^2u(t) - pu'(t) + u''(t). \quad (21)$$

Hence

$$(Ju)'''(t) + p^3(Ju)(t) = u''(t) - pu'(t) + p^2u(t). \quad (22)$$

If

$$u''(t) - pu'(t) + p^2u(t) = h((Ju)(t)) := g((Ju)(t)) - f(t,(Ju)(t)) + p^3(Ju)(t), \quad (23)$$

then

$$(Ju)'''(t) + p^3(Ju)(t) = g((Ju)(t)) - f(t,(Ju)(t)) + p^3(Ju)(t). \quad (24)$$

If $u \in X \cap C^2(R)$ satisfies (17), we have

$$(Ju)'''(t) = g((Ju)(t)) - f(t,(Ju)(t)). \quad (25)$$

Hence, Ju is a ω-periodic solution of (1).

On the other hand, we have

$$(Ku)'(t) = p(Ku)(t) - u(t). \quad (26)$$

If $u \in X \cap C^1(R)$, then $Ku \in X \cap C^2(R)$ and

$$(Ku)''(t) = p(Ku)'(t) - u'(t) = p^2(Ku)(t) - pu(t) - u'(t). \quad (27)$$

From the above, we see that if $u \in X \cap C^2(R)$, then $Ku \in X \cap C^3(R)$ and

$$(Ku)'''(t) = p^2(p(Ku)(t) - u(t)) - pu'(t) - u''(t) = p^3(Ku)(t) - p^2u(t) - pu'(t) - u''(t). \quad (28)$$

We have

$$(Ku)'''(t) - p^3(Ku)(t) = -p^2u(t) - pu'(t) - u''(t). \quad (29)$$

If

$$-\left(u''(t) + pu'(t) + p^2u(t)\right) = \overline{h}((Ku)(t)) := g((Ku)(t)) - f(t,(Ku)(t)) - p^3(Ku)(t), \quad (30)$$

then

$$(Ku)'''(t) - p^3(Ku)(t) = g((Ku)(t)) - f(t,(Ku)(t)) - p^3(Ku)(t). \quad (31)$$

If $u \in X \cap C^2(R)$ satisfies (18), we have

$$(Ku)'''(t) = g((Ku)(t)) - f(t,(Ku)(t)). \quad (32)$$

Hence, Ku is a ω-periodic solution of (1). □

Lemma 6 (see [28]). *Let X be a Banach space so that $D \subset X$ is closed and convex. Assume that $T : D \to D$ is a completely continuous operator. Then T has a fixed point in D.*

3. Main Results

The following theorems are the main results of this paper.

Theorem 7. *Assume that there exist constants $m < M$ such that*

(H_1) $g \in C^1[m, M]$ and $g'(u) < (2\pi/\sqrt{3}\omega)^3$ in $u \in [m, M]$;

(H_2) $f(t,u) \in C(R \times [m, M])$ and

$$g(m) \le f(t,u) \le g(M) \quad (33)$$

for any $(t,u) \in R \times [m, M]$.

Then (1) has at least one ω-periodic solution u with $m \le u \le M$.

Proof. From (H_1), we obtain that there exists a constant $p \in (0, 2\pi/\sqrt{3}\omega)$ with $g'(u) < p^3$ in $u \in [m, M]$. Consider the differential equation

$$u'' + pu' + p^2u = f(t, Ku) + p^3(Ku) - g(Ku). \quad (34)$$

From Lemma 1 we see that if u is a solution of (34), then u satisfies

$$u = (S \circ W)u, \quad (35)$$

where $S \circ W$ is composition of S defined by $(S \circ W)u = S(Wu)$ and the operator W is defined by

$$(Wu)(t) = f(t, Ku) + p^3(Ku) - g(Ku). \quad (36)$$

Set $\Omega = \{u \in X : pm \le u(t) \le pM,\ t \in R\}$. For all $u \in \Omega$, we have $m \le Ku \le M$. Put $G(u) = p^3u - g(u)$; then $G'(u) = p^3 - g'(u) > 0$ for $u \in [m, M]$. Thus

$$p^3m - g(m) \le G(u) \le p^3M - g(M) \quad (37)$$

for any $u \in [m, M]$. Hence, for all $u \in \Omega$,

$$p^3m - g(m) \le G(Ku) = p^3(Ku)(t) - g((Ku)(t)) \le p^3M - g(M). \quad (38)$$

Using (H_2), we obtain that, for all $u \in \Omega$,

$$\begin{aligned}(Wu)(t) &= p^3(Ku) - g(Ku) + f(t, Ku) \\ &\le p^3M - g(M) + f(t, Ku) \le p^3M, \\ (Wu)(t) &= p^3Ku(t) - g(Ku) + f(t, Ku) \\ &\ge p^3m - g(m) + f(t, Ku) \ge p^3m.\end{aligned} \quad (39)$$

Since S is an increasing operator, we obtain that, for $u \in \Omega$,

$$S\left(p^3m\right) \le (S \circ W)u \le S\left(p^3M\right). \quad (40)$$

By Remark 4, we have

$$pm \le (S \circ W)\, u \le pM \tag{41}$$

for $u \in \Omega$; that is, $(S \circ W)(\Omega) \subset \Omega$.

Also, from the facts that S is completely continuous and W is continuous it follows that $S \circ W : \Omega \to \Omega$ is a continuous and compact map. By Lemma 6, $S \circ W$ has at least one fixed point u in Ω and $m \le Ku \le M$ is periodic solution of (1). The proof is complete. □

Theorem 8. *Assume that there exist constants $m < M$ such that*

- (H_3) *$g \in C^1[m, M]$ and $g'(u) \ge -(2\pi/\sqrt{3}\omega)^3$ in $u \in [m, M]$;*
- (H_4) *$f(t, u) \in C(R \times [m, M])$ and*

$$g(M) \le f(t, u) \le g(m) \tag{42}$$

for any $(t, u) \in R \times [m, M]$.

Then (1) has at least one ω-periodic solution u with $m \le u \le M$.

Proof. From (H_3), we obtain that there exists a constant $p \in (0, 2\pi/\sqrt{3}\omega)$ with $g'(u) \ge -p^3$ in $u \in [m, M]$. Consider the differential equation

$$u'' - pu' + p^2 u = g(Ju) - f(t, Ju) + p^3 (Ju). \tag{43}$$

From Lemma 1, if u is a solution of (43), u satisfies

$$u = (T \circ V)\, u, \tag{44}$$

where $T \circ V$ is composition of T and V defined as $(T \circ V)u = T(Vu)$, and the operator V is defined as

$$(Vu)(t) = g(Ju) + p^3 (Ju) - f(t, Ju). \tag{45}$$

Set $\Omega = \{u \in X : pm \le u(t) \le pM, t \in R\}$, where $p \in (0, 2\pi/\sqrt{3}\omega)$ with $g'(u) \ge -p^3$ in $[m, M]$. We set $I(u) = u + (1/p^3)g(u)$; then $I'(u) = 1 + (1/p^3)g'(u) \ge 0$; we have

$$m + \frac{1}{p^3} g(m) \le I(u) \le M + \frac{1}{p^3} g(M), \tag{46}$$

for any $u \in [m, M]$. Hence, for all $u \in \Omega$, we have

$$m + \frac{1}{p^3} g(m) \le I(Ju) \le M + \frac{1}{p^3} g(M). \tag{47}$$

Using (H_4), we obtain that, for all $u \in \Omega$,

$$\begin{aligned} (Vu)(t) &= p^3 (Ju) + g(Ju) - f(t, Ju) \\ &\le p^3 M + g(M) - f(t, Ju) \le p^3 M, \\ (Vu)(t) &= p^3 (Ju) + g(Ju) - f(t, Ju) \\ &\ge p^3 m + g(m) - f(t, Ju) \ge p^3 m. \end{aligned} \tag{48}$$

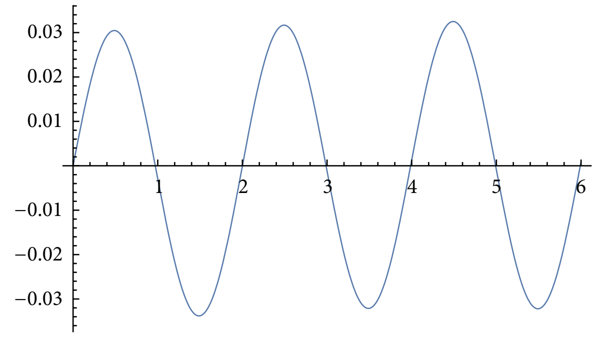

Figure 1: A numerical approximation to a 2-periodic solution to (51). The approximation is obtained by using the built-in `NDsolve` function of Mathematica 10.

Since T is an increasing operator, we obtain that, for $u \in \Omega$,

$$T\left(p^3 m\right) \le (T \circ V)\, u \le T\left(p^3 M\right). \tag{49}$$

By Remark 4, we have

$$pm \le (T \circ V)\, u \le pM \tag{50}$$

for $u \in \Omega$; that is, $(T \circ V)(\Omega) \subset \Omega$.

Also, from the facts that T is completely continuous and V is continuous, it follows that $T \circ V : \Omega \to \Omega$ is a continuous and compact map. By Lemma 6, $T \circ V$ has at least one fixed point in Ω. The proof is complete. □

4. Some Examples

In this section, two examples are provided to highlight potential applications of the results obtained in the previous section.

Example 1. Consider the differential equation

$$x''' = \sin x - \cos\left(\pi t + x^2\right). \tag{51}$$

Note that here

$$\begin{gathered} g(x) = \sin x, \qquad \omega = 2, \qquad f(t, x) = \cos\left(\pi t + x^2\right), \\ g'(x) \le 1 < \left(\frac{2\pi}{\sqrt{3}\omega}\right)^3, \quad \forall x \in R. \end{gathered} \tag{52}$$

Letting $M = 2n\pi + (1/2)\pi$, $m = 2n\pi - \pi/2$, $n \in Z$,

$$g(m) = -1 \le f(t, x) \le 1 = g(M). \tag{53}$$

By Theorem 7, (51) has infinitely many 2-periodic solutions. One such solution is illustrated in Figure 1.

Example 2. Consider the differential equation

$$x''' = \frac{1}{2} x (x - 1) - \frac{1}{20} e^{x^2/2 + \sin 2t/2}. \tag{54}$$

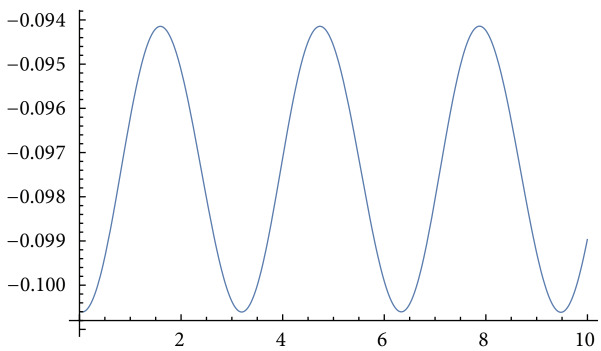

FIGURE 2: A numerical approximation of a π-periodic solution to (54). The approximation is obtained by using the built-in `NDsolve` function of Mathematica 10.

We claim that (54) has at least two π-periodic solutions. Note that here

$$g(x) = \frac{1}{2}x(x-1), \qquad f(t,x) = \frac{1}{20}e^{x^2/2+\sin 2t/2}, \qquad \omega = \pi. \tag{55}$$

Let $m_1 = -1/2$, $M_1 = 1/2$, $m_2 = 1$, and $M_2 = 2$. Then

$$\begin{gathered} g'(x) = x - \frac{1}{2} \geq -1 > -\left(\frac{2\pi}{\sqrt{3}\omega}\right)^3, \quad \forall x \in [m_1, M_1], \\ g'(x) = x - \frac{1}{2} \leq \frac{3}{2} < \left(\frac{2\pi}{\sqrt{3}\omega}\right)^3, \quad \forall x \in [m_2, M_2], \\ g(M_1) = -\frac{1}{8}, \qquad g(m_1) = \frac{1}{8}, \\ g(M_2) = 1, \qquad g(m_2) = 0, \\ g(M_1) < \frac{1}{20} < f(t,x) \leq \frac{1}{20}e^{5/8} < \frac{1}{8} = g(m_1), \\ \forall x \in [m_1, M_1], \\ g(m_2) < f(t,x) \leq \frac{1}{20}e^{5/2} < 1 = g(M_2), \\ \forall x \in [m_2, M_2]. \end{gathered} \tag{56}$$

By Theorems 7 and 8, (54) has at least two π-periodic solutions $x_i \in [m_i, M_i]$, $i = 1, 2$. One such solution is illustrated in Figure 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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