Hindawi Publishing Corporation
Journal of Discrete Mathematics
Volume 2014, Article ID 538423, 5 pages
http://dx.doi.org/10.1155/2014/538423

Research Article

Chaotification for Partial Difference Equations via Controllers

Wei Liang,[1] Yuming Shi,[2] and Zongcheng Li[3]

[1] *School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454000, China*
[2] *Department of Mathematics, Shandong University, Jinan, Shandong 250100, China*
[3] *Department of Mathematics, Shandong Jianzhu University, Jinan, Shandong 250100, China*

Correspondence should be addressed to Wei Liang; wliang@hpu.edu.cn

Received 1 December 2013; Accepted 22 January 2014; Published 13 March 2014

Academic Editor: Zhan Zhou

Chaotification problems of partial difference equations are studied. Two chaotification schemes are established by utilizing the snap-back repeller theory of general discrete dynamical systems, and all the systems are proved to be chaotic in the sense of both Li-Yorke and Devaney. An example is provided to illustrate the theoretical results with computer simulations.

1. Introduction

Consider the following first-order partial difference equation:

$$x(n+1,m) = f(x(n,m), x(n,m+1)), \qquad (1)$$

where $n \geq 0$ is time step, m is the lattice point with $0 \leq m \leq k < +\infty$, and $f : D \subset \mathbf{R}^2 \to \mathbf{R}$ is a map.

Equation (1) is a discretization of the partial differential equation

$$w_t(t,s) = \tilde{f}(w(t,s), w_s(t,s)), \qquad (2)$$

where $t \geq 0$ is time variable, s is spatial variable, and $\tilde{f} : \tilde{D} \subset \mathbf{R}^2 \to \mathbf{R}$ is a map. Equation (1) often appears in imaging and spatial dynamical systems and so forth [1, 2]. Chen and Liu studied the chaos for (1) in $\mathbf{R}^3$ by constructing spatial periodic orbits in 2003 [3]. Chen et al. [4] reformulated (1) to a discrete system:

$$x_{n+1} = h(x_n), \quad n \geq 0. \qquad (3)$$

Applying this approach, the second author of the present paper gave several criteria of chaos for (1) [5]. She with her coauthors established some chaotification schemes for (1) and proved all the systems are chaotic [6, 7]. Recently, Li studied the chaotification for delay difference equations [8]. However, only a few papers study the chaotification problems of (1) except for [6–8]. In this paper, the chaotification of (1) is studied.

This paper is organized as follows. First, (1) is reformulated to a discrete system, and several concepts and lemmas are listed. Then, we give two chaotification schemes for (1) via controllers and prove that all the systems are chaotic in the sense of both Li-Yorke and Devaney. Finally, we give one example with computer simulation result to verify the theoretical predictions.

2. Preliminaries

Consider the following boundary condition for (1):

$$x(n,k+1) = \varphi(x(n,0)), \quad n \geq 0, \qquad (4)$$

where $\varphi : I \subset \mathbf{R} \to \mathbf{R}$ is a map. For the initial condition

$$x(0,m) = \phi(m), \quad 0 \leq m \leq k+1, \qquad (5)$$

where ϕ satisfies (4), (1) has a unique solution $\{x(n,m) : n \geq 0,\ 0 \leq m \leq k\}$, and it can be easily proved by iterations.

Let

$$x_n = (x(n,0), x(n,1), \ldots, x(n,k))^T \in \mathbf{R}^{k+1}, \quad n \geq 0; \qquad (6)$$

then (1) with (4) can be rewritten in the following form:

$$x_{n+1} = F(x_n), \quad n \geq 0, \qquad (7)$$

where

$$F(x_n) = (f(x(n,0), x(n,1)), f(x(n,1), x(n,2)), \ldots, f(x(n,k), \varphi(x(n,0))))^T. \quad (8)$$

F is said to be the induced map by f and φ, and (7) is called the induced system by (1) with (4).

Definition 1 (see [9]). Let (X,d) be a metric space and let $F: X \to X$ be a map. A subset S of X is called a scrambled set of F if for any two different points $x, y \in S$,

$$\liminf_{n\to\infty} d(F^n(x), F^n(y)) = 0,$$
$$\limsup_{n\to\infty} d(F^n(x), F^n(y)) > 0. \quad (9)$$

The map F is said to be chaotic in the sense of Li-Yorke if there exists an uncountable scrambled set S of F.

Definition 2 (see [10]). A map $F: V \subset X \to V$ is said to be chaotic on V in the sense of Devaney if

(i) F is topologically transitive in V;

(ii) the periodic points of F in V are dense in V;

(iii) F has sensitive dependence on initial conditions in V.

Chaos of Devaney is stronger than that of Li-Yorke in some conditions [11].

Definition 3 (see [6]). A point $x \in \mathbf{R}^{k+1}$ is called a fixed point of (1) with (4) if $F(x) = x$; that is, it is a fixed point of its induced system (7).

It follows from Definition 3 that $x = \{x(m)\}_{m=0}^{k}$ is a fixed point of (1) with (4) if and only if it satisfies

$$x(m) = f(x(m), x(m+1)), \quad 0 \le m \le k-1,$$
$$x(k) = f(x(k), \varphi(x(0))). \quad (10)$$

Definition 4 (see [6]). Equation (1) with (4) is said to be chaotic in the sense of Li-Yorke (or Devaney) on $V \subset \mathbf{R}^{k+1}$ if its induced system (7) is chaotic in the sense of Li-Yorke (or Devaney) on V.

Recently, some chaotification schemes of the discrete system (3) were established in [7]; we list them as follows. For convenience, let $C^k(U, R^n)$ be the set of all the maps $f: U \subset R^n \to R^n$ that are k times continuously differentiable in U.

Lemma 5 (see [7]). *Consider the controlled system*

$$x_{n+1} = f(x_n) + g(\mu x_n), \quad n \ge 0, \quad (11)$$

in Y_k ($k \le \infty$). Assume that

(i) *$x^* = 0$ is a fixed point of f and there exist positive constants r and L such that $f \in C^0([-r,r]^k, Y_k)$, $f \in C^1((-r,r)^k, Y_k)$, and $\|Df(x)\| \le L$ for any $x \in (-r,r)^k$;*

(ii) *g satisfies the following conditions:*

(iia) *$g \in C^0([-r,r]^k \cup [a,b]^k, Y_k)$ and $g \in C^1((-r,r)^k \cup (a,b)^k, Y_k)$ with $r < a < b$;*

(iib) *$x^* = 0$ is a fixed point of g and there exists a point $\xi \in (a,b)^k$ such that $g(\xi) = 0$;*

(iic) *$Dg(x)$ is an invertible linear operator for each $x \in (-r,r)^k \cup (a,b)^k$ and there exists a positive constant N such that for any $x, y \in [-r,r]^k \cup [a,b]^k$,*

$$\|g(x) - g(y)\| \ge N\|x - y\|. \quad (12)$$

Then, for any constant μ satisfying

$$\mu > \mu_0 := \max\left\{\frac{b}{r}, \frac{Lr+b}{Nr}, \frac{Lb}{N(\|\xi\|_0 - a)}, \frac{Lb}{N(b - \|\xi\|)}\right\}, \quad (13)$$

where $\|\xi\|_0 = \min\{|\xi_i| : 0 \le i \le k\}$, and for any neighborhood U of $x^* = 0$, there exist a positive integer $n > 2$ and a Cantor set $\Lambda \subset U$ such that $F_\mu^n : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma: \Sigma_2^+ \to \Sigma_2^+$, where $F_\mu(x) = f(x) + g(\mu x)$. Consequently, there exists a compact and perfect invariant set $D \subset X$ containing a Cantor set such that the controlled system is chaotic on D in the sense of both Devaney and Li-Yorke.

A map is said to be an invertible linear map if it is a bounded linear map and bijective and if it has a bounded linear inverse map [6].

Lemma 6 (see [7]). *Consider the controlled system*

$$x_{n+1} = f(x_n) + \mu g(x_n), \quad n \ge 0, \ x_n \in Y_k, \quad (14)$$

where $k \le \infty$. Assume that

(i) *assumption (i) in Lemma 5 holds;*

(ii) *g satisfies the following conditions:*

(iia) *$g \in C^0([-a,a]^k \cup [b,r]^k, Y_k)$ and $g \in C^1((-a,a)^k \cup (b,r)^k, Y_k)$ with $0 < a < b < r$;*

(iib) *$x^* = 0$ is a fixed point of g and there exists a point $\xi \in (b,r)^k$ such that $g(\xi) = 0$;*

(iic) *$Dg(x)$ is an invertible linear operator for each $x \in (-a,a)^k \cup (b,r)^k$ and there exists a positive constant N such that (12) holds for any $x, y \in [-a,a]^k \cup [b,r]^k$.*

Then, for each constant μ satisfying

$$\mu > \mu_0 := \max\left\{\frac{La+r}{Na}, \frac{Lr}{N(\|\xi\|_0 - b)}, \frac{Lr}{N(r - \|\xi\|)}\right\}, \quad (15)$$

all the results in Lemma 5 hold for $F_\mu(x) = f(x) + \mu g(x)$ therein.

3. Chaotification Problems for (1)

Assume that $f \in C^1([-r,r]^2, \mathbf{R})$ for $r > 0$. Let $f_x(x,y)$ and $f_y(x,y)$ be the first-order partial derivatives of f for the 1st and the 2nd variables at (x,y). Let

$$L := \max\left\{\left|f_x(x,y)\right| + \left|f_y(x,y)\right| : x, y \in [-r,r]\right\}. \quad (16)$$

Theorem 7. *Consider the following system:*

$$x(n+1,m) = f(x(n,m), x(n,m+1)) + g(\mu x(n,m)), \quad n \geq 0,\ 0 \leq m \leq k, \quad (17)$$

with (4)*, where $g : \mathbf{R} \to \mathbf{R}$ is a map and $\mu > 0$ is a constant. Assume that*

(i) *$f \in C^1([-r,r]^2, R)$ and $f(0,0) = 0$;*

(ii) *$g \in C^1([-r,r] \cup [a,b], R)$, and $g'(x) \neq 0$ for any $x \in [-r,r] \cup [a,b]$, where $r < a < b$;*

(iii) *$g(0) = 0$ and there is a point $\xi \in (a,b)$ satisfying $g(\xi) = 0$;*

(iv) *$\varphi \in C^1([-r,r,-r,r])$ and $\varphi(0) = 0$.*

Then, for

$$\mu > \mu_0 := \max\left\{\frac{b}{r}, \frac{Mr+b}{Nr}, \frac{Mb}{N(\xi - a)}, \frac{Mb}{N(b-\xi)}\right\} \quad (18)$$

and for any neighborhood U of $x = 0$, there exist a Cantor set $\Lambda \subset U^{k+1}$ and a perfect as well as compact invariant set $E \subset \mathbf{R}^{k+1}$ containing Λ such that system (17) *with* (4) *is chaotic on E in the sense of both Li-Yorke and Devaney, where $M = \max\{L, |f_x(x,y)| + |f_y(x,y)\varphi'(x(0))| : x, y, x(0) \in [-r,r]\}$, L is given in* (16)*, and $N = \min\{|g'(x)| : x \in [-r,r] \cup [a,b]\}$.*

Proof. Assume that $\mu > \mu_0$ in the proof. System (17) with (4) can be rewritten as

$$x_{n+1} = F(x_n) + G(\mu x_n), \quad n \geq 0, \quad (19)$$

where F is defined by (8), and

$$G(x_n) = \left(g(x(n,0)), g(x(n,1)), \ldots, g(x(n,k))\right)^T. \quad (20)$$

By assumptions (i), (iv), and Definition 3, $\{x^*(m) = 0 : 0 \leq m \leq k\}$ is a fixed point of (1) with (4), and then $F(x^*) = x^*$, for $x^* := 0 \in \mathbf{R}^{k+1}$, and $F \in C^1([-r,r]^{k+1}, R^{k+1})$. Further, for any $x = \{x(j)\}_{i=0}^{k} \in [-r,r]^{k+1}$,

$$DF(x) = \begin{pmatrix} f_x(\alpha(0)) & f_y(\alpha(0)) & 0 & \cdots & 0 \\ 0 & f_x(\alpha(1)) & f_y(\alpha(1)) & \cdots & 0 \\ 0 & 0 & f_x(\alpha(2)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_y(\alpha(k))\varphi'(x(0)) & 0 & 0 & \cdots & f_x(\alpha(k)) \end{pmatrix}_{(k+1)\times(k+1)}, \quad (21)$$

where $\alpha(i) = (x(i), x(i+1))$ for $0 \leq i \leq k$ with $x(k+1) = \varphi(x(0))$. So, for $\omega = \{\omega(i)\}_{i=0}^{k} \in \mathbf{R}^{k+1}$,

$$\begin{aligned} DF(x)z = \big(&f_x(\alpha(0))\omega(0) + f_y(\alpha(0))\omega(1), \\ &f_x(\alpha(1))\omega(1) + f_y(\alpha(1))\omega(2), \ldots, \\ &f_x(\alpha(k))\omega(k) + f_y(\alpha(k))\varphi'(x(0))\omega(0)\big)^T. \end{aligned} \quad (22)$$

Therefore,

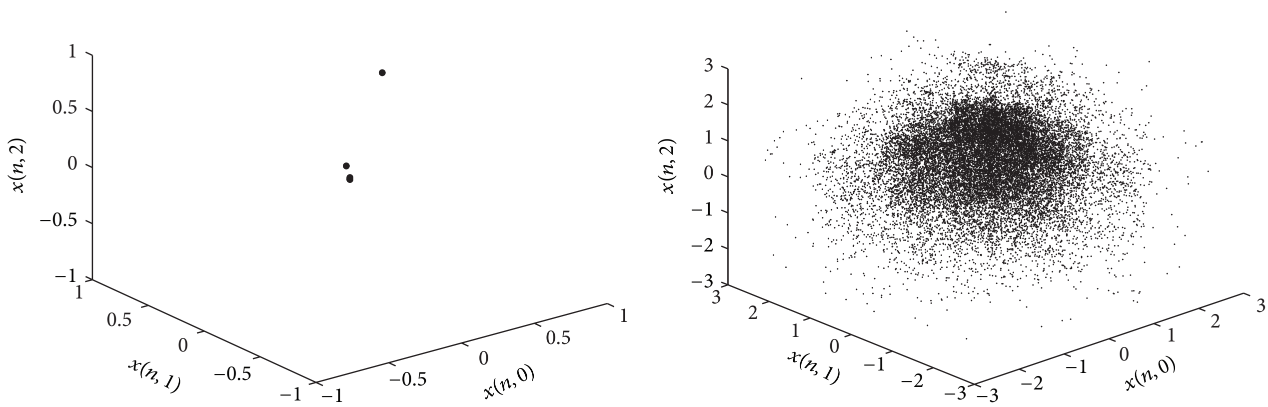
$$\begin{aligned} &\|DF(x)\| \\ &= \max\left\{\|DF(x)\omega\| : \omega \in \mathbf{R}^{k+1},\ \|\omega\| = 1\right\} \\ &\leq \max\left\{\left|f_x(\alpha(j))\right| + \left|f_y(\alpha(j))\right|,\ 0 \leq j \leq k-1, \right. \\ &\qquad \left. \left|f_x(\alpha(k))\right| + \left|f_y(\alpha(k))\varphi'(x(0))\right|\right\} \\ &\leq \max\left\{L, \left|f_x(\alpha(k))\right| + \left|f_y(\alpha(k))\varphi'(x(0))\right|\right\} = M. \end{aligned} \quad (23)$$

Now, we prove that $G(x)$ satisfies condition (ii) in Lemma 5. By (iii), $G(0) = G(\bar{\xi}) = 0$, where $\bar{\xi} := \underbrace{(\xi, \xi, \ldots, \xi)}_{k+1}^T \in (a,b)^{k+1}$. Furthermore, it follows from condition (ii) that $G \in C^1([-r,r]^{k+1} \cup [a,b]^{k+1}, R^{k+1})$ and

$$DG(x) = \begin{pmatrix} g'(x(0)) & 0 & \cdots & 0 \\ 0 & g'(x(1)) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & g'(x(k)) \end{pmatrix}. \quad (24)$$

Obviously, $DG(x)$ is an invertible map, and it follows from condition (ii) that its inverse is

$$(DG(x))^{-1} = \begin{pmatrix} (g'(x(0)))^{-1} & 0 & \cdots & 0 \\ 0 & (g'(x(1)))^{-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (g'(x(k)))^{-1} \end{pmatrix}. \quad (25)$$



(a) (b)

Figure 1: Computer simulation results, where $k = 2$, $n = 0, 1, \ldots, 20000$, and the initial value is $x(0,0) = 0.3$, $x(0,1) = 0.1$, and $x(0,2) = 0.8$. (a) Simple dynamical behaviors for the original system (35); (b) simulation results of the system (29) for $\mu = 4$, which shows that there is a dense orbit around the origin and then there are complex dynamical behaviors in (29).

Hence, for any $x \in [-r,r]^{k+1} \cup [a,b]^{k+1}$, one can obtain that

$$\left\|(DG(x))^{-1}\right\| \le \frac{1}{N}. \tag{26}$$

Therefore, $DG(x)$ is an invertible linear map. Hence,

$$\begin{aligned}\|G(x) - G(y)\| &= \max\{|g(x(i)) - g(y(i))| : 0 \le i \le k\} \\ &\ge N\|x - y\|, \quad \forall x, y \in [-r,r]^{k+1} \cup [a,b]^{k+1}.\end{aligned} \tag{27}$$

In summary, both F and G meet all the conditions in Lemma 5. So this theorem holds. □

Theorem 8. *Assume that*

(1) $f(0,0) = 0$ *and* $f \in C^1([-r,r]^2, R)$;

(2) $g \in C^1([-a,a] \cup [b,r], R)$, *for* $0 < a < b < r$, *and* $g'(x) \neq 0$, *for any* $x \in [-a,a] \cup [b,r]$;

(3) $g(0) = 0$ *and there is a point* $\xi \in (b,r)$ *satisfying* $g(\xi) = 0$;

(4) $\varphi \in C^1([-r,r,-r,r])$ *and* $\varphi(0) = 0$.

Then, for

$$\mu > \mu_0 := \max\left\{\frac{Ma + r}{Na}, \frac{Mr}{N(\xi - b)}, \frac{Mr}{N(r - \xi)}\right\} \tag{28}$$

and for any neighborhood U *of* $x = 0$, *there exist a Cantor set* $\Lambda \subset U^{k+1}$ *and a perfect and compact invariant set* $E \subset \mathbf{R}^{k+1}$ *containing* Λ *such that*

$$\begin{aligned}x(n+1,m) = f(x(n,m), x(n,m+1)) + \mu g(x(n,m)), \\ n \ge 0,\ 0 \le m \le k,\end{aligned} \tag{29}$$

with (4) *being chaotic on* E *in the sense of both Li-Yorke and Devaney, where* $M = \max\{L, |f_x(x,y)| + |f_y(x,y)\varphi'(x(0))| : x, y, x(0) \in [-r,r]\}$; L *is defined by* (16), *and* $N = \min\{|g'(x)| : x \in [-a,a] \cup [b,r]\}$.

Proof. The system induced by system (29) is

$$x_{n+1} = F(x_n) + \mu G(x_n), \quad n \ge 0, \tag{30}$$

where F and G are defined by (8) and (20), respectively. Similar to the proof of Theorem 7, it can be proved that F and G meet all the conditions of Lemma 6. Hence, Theorem 8 holds by Lemma 6. □

4. An Example

Consider the controlled system (29) with (4), which is a special case of the discrete heat equation (see (1.3) in [12]):

$$\begin{aligned}u(n+1,m) = \alpha u(n,m-1) + \beta u(n,m) \\ + \gamma u(n,m+1), \quad \alpha, \beta, \gamma \in \mathbf{R},\end{aligned} \tag{31}$$

where $u(n,m)$ denotes the temperature at time n and position m of the rod. In system (29),

$$\begin{aligned}f(x,y) &= \frac{1}{12}x + \frac{1}{12}y, \\ \varphi(x) &= x^2, \\ g(x) &= \begin{cases} 2x, & x \in \left[-\frac{1}{3}, \frac{1}{3}\right], \\ x - \frac{4}{5}, & x \in \left[\frac{1}{2}, 1\right], \\ \frac{1}{3}\cos x, & \text{otherwise}. \end{cases}\end{aligned} \tag{32}$$

By Corollary 5.1 [6], the original system

$$\begin{aligned}x(n+1,m) = \frac{1}{12}x(n,m) + \frac{1}{12}x(n,m+1), \\ n \ge 0,\ 0 \le m \le k,\end{aligned} \tag{33}$$

is stable near the origin (see Figure 1(a)). In addition, f, g, and φ satisfy all the conditions of Theorem 8 with $r = 1$,

$a = 1/3$, $b = 1/2$, $\xi = 4/5$, $L = 1/6$, $N = 1$, $M = 1/4$. Therefore, it follows from Theorem 8 that system (29) with (4) is chaotic in the sense of both Li-Yorke and Devaney for $\mu > \mu_0 = 13/4$.

We take $k = 2$, $\mu = 4$ for computer simulation. The simulation result is shown in Figure 1(b), which indicates that system (29) with (4) has a dense orbit around the origin and then has very complicated dynamical behaviors near the origin.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research was supported by the RFDP of Higher Education of China (Grant 20100131110024), the NSFC (Grants 11126120, 11101246), the NNSF of Shandong Province (Grant ZR2011AM002), and the RFDP of Henan Polytechnic University (Grant B2011-032).

References

[1] H. Gang and Q. Z. Qu Zhilin, "Controlling spatiotemporal chaos in coupled map lattice systems," *Physical Review Letters*, vol. 72, no. 1, pp. 68–71, 1994.

[2] F. H. Willeboordse, "Time-delayed map as a model for open fluid flow," *Chaos*, vol. 2, no. 3, pp. 423–426, 1992.

[3] G. Chen and S. T. Liu, "On spatial periodic orbits and spatial chaos," *International Journal of Bifurcation and Chaos*, vol. 13, no. 4, pp. 935–941, 2003.

[4] G. Chen, C. Tian, and Y. Shi, "Stability and chaos in 2-D discrete systems," *Chaos, Solitons & Fractals*, vol. 25, no. 3, pp. 637–647, 2005.

[5] Y. Shi, "Chaos in first-order partial difference equations," *Journal of Difference Equations and Applications*, vol. 14, no. 2, pp. 109–126, 2008.

[6] Y. Shi, P. Yu, and G. Chen, "Chaotification of discrete dynamical systems in Banach spaces," *International Journal of Bifurcation and Chaos*, vol. 16, no. 9, pp. 2615–2636, 2006.

[7] W. Liang, Y. Shi, and C. Zhang, "Chaotification for a class of first-order partial difference equations," *International Journal of Bifurcation and Chaos*, vol. 18, no. 3, pp. 717–733, 2008.

[8] Z. Li, "Chaotification for linear delay difference equations," *Advances in Difference Equations*, vol. 2013, article 459, 11 pages, 2013.

[9] T. Y. Li and J. A. Yorke, "Period three implies chaos," *The American Mathematical Monthly*, vol. 82, no. 10, pp. 985–992, 1975.

[10] R. L. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley Studies in Nonlinearity, Addison-Wesley, Redwood City, Calif, USA, 2nd edition, 1989.

[11] W. Huang and X. Ye, "Devaney's chaos or 2-scattering implies Li-Yorke's chaos," *Topology and Its Applications*, vol. 117, no. 3, pp. 259–272, 2002.

[12] S. S. Cheng, *Partial Difference Equations*, vol. 3 of *Advances in Discrete Mathematics and Applications*, Taylor & Francis, London, UK, 2003.