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Research Article

Complete Controllability of Fractional Neutral Differential Systems in Abstract Space

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By using fractional power of operators and Sadovskii fixed point theorem, we study the complete controllability of fractional neutral differential systems in abstract space without involving the compactness of characteristic solution operators introduced by us.

1. Introduction

Recently, fractional differential systems have been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, and so forth (see [1–5]). There has been a great deal of interest in the solutions of fractional differential systems in analytic and numerical sense. One can see the monographs of Kilbas et al. [6], Miller and Ross [7], Podlubny [8], Lakshmikantham et al. [9], Tarasov [10], Wang et al. [11–13] and the survey of Agarwal et al. [14] and the reference therein. In order to study the fractional systems in the infinite dimensional space, the first important step is how to introduce a new concept of mild solutions. A pioneering work has been reported by EI-Borai [15] and Zhou and Jiao [16].

In recent years, controllability problems for various types of nonlinear fractional dynamical systems in infinite dimensional spaces have been considered in many publications. An extensive list of these publications focused on the complete and approximate controllability of the fractional dynamical systems can be found (see [17–34]). Although the controllability of fractional differential systems in abstract space has been discussed, Hernández et al. [35] point out that some papers on controllability of abstract control systems contain a similar technical error when the compactness of semigroup and other hypotheses is satisfied, more precisely, in this case the application of controllability results are restricted to the finite dimensional space. Ji et al. [32] find some conditions guaranteeing the controllability of impulsive differential system when the Banach space is nonseparable and evolution systems are not compact, by means of Möch fixed point theorem and the measure of noncompactness. Meanwhile, Wang et al. [19, 20] have researched the complete controllability of fractional evolution systems without involving the compactness of characteristic solution operators. Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last decades. Sakthivel and Ren [29] have established a new set of sufficient conditions for the complete controllability for a class of fractional order neutral systems with bounded delay under the natural assumption that the associated linear control is completely controllable. To the author's knowledge, there are few papers on the complete controllability of the abstract neutral fractional differential systems with unbounded delay.

In the present paper, we introduce a suitable concept of the mild solutions including characteristic solution operators $\varphi(\cdot)$ and $S(\cdot)$ which are associated with operators semigroup $\{T(t);\ t\ \geq\ 0\}$ and some probability density functions ξ_q. Then also without involving the compactness of characteristic solution operators, we obtain the controllability of the following abstract neutral fractional differential systems with unbounded delay:

$$\begin{aligned} {}^{c}D_t^q\left(x(t)+F\left(t,x_t\right)\right)+Ax(t)&=Cu(t)+G\left(t,x_t\right), \\ & t\in(0,a], \quad (1)\\ x_0(\vartheta)=\phi(\vartheta)\in B, \quad \vartheta&\in(-\infty,0], \end{aligned}$$

where the state variable $x(\cdot)$ takes values in Banach space X, $x_t:(-\infty,0]\to X$, $x_t(\vartheta)=x(t+\vartheta)$ belongs to some abstract phase space B, and B is the phase space to be specified later. The control function $u(\cdot)$ is given in $L^2([0,a];U)$, with U as a Banach spaces. C is a bounded linear operator from U to X. The operator $-A$ is a generator of a uniformly bounded analytic semigroup $\{T(t),\ t\geq 0\}$ in which X, $F,G:[0,a]\times B\to X$ are appropriate functions.

2. Preliminaries

Throughout this paper X will be a Banach space with norm $\|\cdot\|$ and Y is another Banach space, $L_b(X,Y)$ denote the space of bounded linear operators from X to Y. We also use $\|f\|_{L^p([0,a],R^+)}$ to denote the $L^p([0,a],R^+)$ of norm of f whenever $f\in L^p([0,a],R^+)$ for some p with $1\leq p<\infty$. Let $L^p([0,a],R^+)$ denote the Banach space of functions $f:[0,a]\to X$ which are Bochner integrable normed by $\|f\|_{L^p([0,a],R^+)}$. Let $-A:D(A)\to X$ be the infinitesimal generator of a uniformly bounded analytic semigroup $T(t)$. Let $0\in\rho(A)$, then it is possible to define the fractional power A^α, for $0<\alpha\leq 1$, as a closed linear operator on its domain $D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha=\left\|A^\alpha x\right\|,\quad x\in D\left(A^\alpha\right) \tag{2}$$

defines a norm on $D(A^\alpha)$. Hereafter we denote by X_α the Banach space $D(A^\alpha)$ normed with $\|x\|_\alpha$. Then for each $0<\alpha\leq 1$, X_α the Banach space, and $\|x\|_\alpha\hookrightarrow\|x\|_\beta$ for $0<\beta<\alpha\leq 1$ and the imbedding is compact whenever the resolvent operator of A is compact. For a uniformly bounded analytic semigroup $\{T(t);\ t\geq 0\}$ the following properties will be used:

(a) there is a $M\geq 0$ such that $\|T(t)\|\leq M$ for all $t\geq 0$.

(b) for any $\alpha\geq 0$, there exists a positive constant C_α such that

$$\left\|A^\alpha T(t)\right\|\leq\frac{C_\alpha}{t^\alpha},\quad 0<t\leq a. \tag{3}$$

For more details about the above preliminaries, we can refer to [16].

Although the semigroup $\{T(t);\ t\geq 0\}$ is only the uniformly bounded analytic semigroup but not compact, we can also give the definition of mild solution for our problem by using the similar method introduced in [36].

Definition 1. We say that a function $x(\cdot):(-\infty,a]\to X$ is a mild solution of the system (1) if $x_0=\phi$, the restriction of $x(\cdot)$ to the interval $[0,a]$ is continuous and for each $0\leq t\leq a$, the function $AS(t-s)F(s,x_s)$, $s\in[0,t]$ is integrable and satisfies the following integral equation:

$$\begin{aligned} x(t)=&\ \varphi(t)\left[\phi(0)+F(0,\phi)\right]-F\left(t,x_t\right)\\ &-\int_0^t(t-s)^{q-1}AS(t-s)F\left(s,x_s\right)ds\\ &+\int_0^t(t-s)^{q-1}S(t-s)\left[Cu(s)+G\left(s,x_s\right)\right]ds, \end{aligned} \tag{4}$$

where $\varphi(t)$ and $S(t)$ are called characteristic solution operators and are given by

$$\begin{aligned} \varphi(t)&=\int_0^\infty\xi_q(\theta)T\left(t^q\theta\right)d\theta,\\ S(t)&=q\int_0^\infty\theta\xi_q(\theta)T\left(t^q\theta\right)d\theta, \end{aligned} \tag{5}$$

and for $\theta\in(0,\infty)$, $\xi_q(\theta)=(1/q)\theta^{-1-1/q}\overline{w}_q(\theta^{-1/q})\geq 0$,

$$\overline{w}_q(\theta)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\vartheta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q). \tag{6}$$

Here, ξ_q is a probability density function defined on $(0,\infty)$, that is, $\xi_q(\theta)\geq 0$, $\theta\in(0,\infty)$, and $\int_0^\infty\xi_q(\theta)d\theta=1$.

Definition 2 (complete controllability). The fractional system (1) is said to be completely controllable on the interval $[0,a]$ if, for every initial function $\phi\in B$ and $x_1\in X$ there exists a control $u\in L^2([0,a],U)$ such that the mild solution $x(\cdot)$ of (1) satisfies $x(a)=x_1$.

The following results of $\varphi(t)$ and $S(t)$ will be used throughout this paper.

Lemma 3. *The operators $\varphi(t)$ and $S(t)$ have the following properties:*

(i) *for any fixed $t\geq 0$, $\varphi(t)$ and $S(t)$ are linear and bounded operators, that is, for any $x\in X$,*

$$\begin{aligned} \left\|\varphi(t)x\right\|&\leq M_0\|x\|,\\ \|S(t)x\|&\leq\frac{qM_0}{\Gamma(1+q)}\|x\|; \end{aligned} \tag{7}$$

(ii) *$\{\varphi(t),\ t\geq 0\}$ and $\{S(t),\ t\geq 0\}$ are strongly continuous and there exists M_1,M_2 such that $\|\varphi(t)\|\leq M_1$, $\|S(t)\|\leq M_2$ for any $t\in[0,a]$;*

(iii) *for $t\in[0,a]$ and any bounded subsets $D\subset X$, $t\to\{\varphi(t)x:x\in D\}$ and $t\to\{S(t)x:x\in D\}$ are equicontinuous if $\|T(t_2^q\theta)x-T(t_1^q\theta)x\|\to 0$ with respect to $x\in D$ as $t_2\to t_1$ for each fixed $\theta\in[0,\infty]$.*

The proof of Lemma 3 we can see in [33].

To end this section, we recall Kuratowski's measure of noncompactness, which will be used in the next section to study the complete controllability via the fixed points of condensing operator.

Definition 4. Let X be a Banach space and Ω_X the bounded set of X. The Kuratowski's measure of noncompactness is the map $\alpha : \Omega_X \to [0, \infty)$ defined by

$$\alpha(D) = \inf \left\{ d > 0 : D \subseteq \bigcup_{i=1}^{n} D_i,\ \operatorname{diam}(D_i) \le d \right\}, \quad (8)$$

here $D \in \Omega_X$.

One will use the following basic properties of the α measure and Sadovskii's fixed point theorem here (see [37–39]).

Lemma 5. *Let D_1 and D_2 be two bounded sets of a Banach space X. Then*

(i) $\alpha(D_1) = 0$ *if and only if* D_1 *is relatively compact;*

(ii) $\alpha(D_1) \le \alpha(D_2)$ *if* $D_1 \subseteq D_2$;

(iii) $\alpha(D_1 + D_2) \le \alpha(D_1) + \alpha(D_2)$.

Lemma 6 (sadovskii's fixed point theorem). *Let N be a condensing operator on a Banach space X, that is, N is continuous and takes bounded sets into bounded sets, and $\alpha(N(D)) < \alpha(D)$ for every bounded set D of X with $\alpha(D) > 0$. If $N(S) \subset S$ for a convex closed and bounded set S of X, then N has a fixed point in S.*

3. Complete Controllability Result

To study the system (1), we assume the function x_t represents the history of the state from $-\infty$ up to the present time t and $x_t : (-\infty, 0] \to X, x_t(\vartheta) = x(t + \vartheta)$ belongs to some abstract phase space B, which is defined axiomatically. In this article, we will employ an axiomatic definition of the phase space B introduced by Hale and Kato [40] and follow the terminology used in [41]. Thus, B will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_B$. We will assume that B satisfies the following axioms:

(A) If $x \in (-\infty, \sigma + a) \to X$, $a > 0$, is continuous on $[\sigma, \sigma + a]$ and $x_\sigma \in B$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold:

(i) x_t is in B;

(ii) $\|x(t)\| \le H\|x_t\|_B$;

(iii) $\|x_t\|_B \le K(t - \sigma)\sup\{\|x(t)\| : \sigma \le s \le t\} + M(t - \sigma)\|x_\sigma\|_B$.

Here $H \ge 0$ is a constant, $K, M : [0, +\infty) \to [0, +\infty)$, K is continuous and M is locally bounded, and H, K, M are independent of $x(t)$.

(B) For the function $x(\cdot)$ in (A), x_t is a B-valued continuous function on $[\sigma, \sigma + a]$.

(C) The space B is complete.

Now we give the basic assumptions on the system (1).

(H_0) (i) A generates a uniformly bounded analytic semigroup $\{T(t),\ t \ge 0\}$ in X; (ii) for all bounded subsets $D \subset X$ and $x \in D$, $\|T(t_2^q\theta)x - T(t_1^q\theta)x\| \to 0$ as $t_2 \to t_1$ for each fixed $\theta \in [0, \infty]$.

(H_1) $F: [0, a] \times B \to X$ is continuous function, and there exists a constant $\beta \in (0, 1)$ and $L, L_1 > 0$ such that the function F is X_β-valued and satisfies the Lipschitz condition:

$$\begin{aligned} &\left\| A^\beta F(s_1, \phi_1) - A^\beta F(s_2, \phi_2) \right\| \\ &\quad \le L\left(|s_1 - s_2| + \|\phi_1 - \phi_2\|_B\right), \end{aligned} \quad (9)$$

for $0 \le s_1, s_2 \le a$, $\phi_1, \phi_2 \in B$, and the inequality

$$\left\| A^\beta F(t, \phi) \right\| \le L_1\left(\|\phi\|_B + 1\right) \quad (10)$$

holds for $t \in [0, a]$, $\phi \in B$.

(H_2) The function $G : [0, a] \times B \to X$ satisfies the following conditions:

(i) for each $t \in [0, a]$, the function $G(t, \cdot) : B \to X$ is continuous and for each $\phi \in B$ the function $G(\cdot, \phi) : [0, a] \to X$ is strongly measureable;

(ii) for each positive number k, there is a positive function $g_k \in L^{1/q_1}([0, a])$, $0 < q_1 < q$ such that

$$\begin{aligned} &\sup_{\|\phi\|_B \le k} \|G(t, \phi)\| \le g_k(t), \\ &\liminf \frac{1}{k}\|g_k\|_{L^{1/q_1}[0,a]} = \gamma < \infty. \end{aligned} \quad (11)$$

(H_3) The linear operator C is bounded, W from U into X is defined by

$$Wu = \int_0^a (a - s)^{q-1} S(a - s)\, Cu(s)\, ds \quad (12)$$

and there exists a bounded invertible operator W^{-1} defined on $L^2([0, a]; U)/\ker W$ and there exist two positive constants $M_3, M_4 > 0$ such that $\|B\|_{L_b(U,X)} \le M_3$, $\|W^{-1}\|_{L_b(X, L^2([0,a],U)/\ker W)} \le M_4$.

(H_4) For all bounded subsets $D \subseteq X$, the set

$$\Pi_{h,\delta}(t) = \left\{ Q_{2,h,\delta} z(t) \mid z \in D \right\}, \quad (13)$$

where

$$\begin{aligned} Q_{2,h,\delta} z(t) = &\int_0^{t-h} \int_\delta^\infty (t - s)^{q-1} \theta \xi_q(\theta)\, qT\left((t - s)^q \theta\right) \\ &\times \left[Cu(s) + G\left(s, \bar{z}_s + y_s\right) d\theta\, ds \right. \end{aligned} \quad (14)$$

is relatively compact in X for arbitrary $h \in (0, t)$ and $\delta > 0$.

Theorem 7. *Let $\phi \in B$. If the assumptions (H_0)–(H_4) are satisfied, then the system* (1) *is controllable on interval* $[0,a]$ *provided that*

$$M_5 L K_a + \frac{C_{1-\beta}\Gamma(1+\beta)a^{q\beta}}{\beta\Gamma(1+q\beta)} L K_a < 1, \tag{15}$$

$$\begin{aligned}(1 + aM_2M_3M_4) &\\ \times \Bigg(L_1 M_5 K_a &+ M_2 \left(\left(\frac{1-q_1}{q-q_1} \right) a^{(q-q_1)/(1-q_1)} \right)^{1-q_1} K_a \gamma \\ &+ \frac{C_{1-\beta}\Gamma(1+\beta)a^{q\beta}}{\beta\Gamma(1+q\beta)} L_1 K_a \Bigg) < 1,\end{aligned} \tag{16}$$

where $M_5 = \|A^{-\beta}\|$, $K_a = \sup\{K(t) : 0 \le t \le a\}$ *and* $C_{1-\beta}$ *is from* (3).

Proof. Using the assumption (H_3), for arbitrary function $x(\cdot)$ define the control

$$\begin{aligned}u(t) = W^{-1}\Bigg[& x_1 - \varphi(a)(\phi(0) + F(0,\phi)) + F(a, x_a) \\ & + \int_0^a (a-s)^{q-1} AS(a-s) F(s, x_s)\, ds \\ & - \int_0^a (a-s)^{q-1} S(a-s) G(s, x_s)\, ds \Bigg](t).\end{aligned} \tag{17}$$

It will be shown that when using this control the operator P defined by

$$\begin{aligned}Px(t) = {} & \varphi(t)[\phi(0) + F(0,\phi)] - F(t, x_t) \\ & - \int_0^t (t-s)^{q-1} AS(t-s) F(s, x_s)\, ds \\ & + \int_0^t (t-s)^{q-1} S(t-s)[Cu(s) + G(s, x_s)]\, ds\end{aligned} \tag{18}$$

has a fixed point $x(\cdot)$. Then $x(\cdot)$ is a mild solution of system (1), and it is easy to verify that $x(a) = Px(a) = x_1$, which implies that the system is controllable.

Next we will prove that P has a fixed point using the fixed point theorem of Sadovskii [38].

Let $y(\cdot) : (-\infty, a] \to X$ be the function defined by

$$y(t) = \begin{cases} \varphi(t)\phi(0), & t \in [0,a], \\ \phi(t), & -\infty < t < 0, \end{cases} \tag{19}$$

then $y_0 = \phi$ and the map $t \to y_t$ is continuous. We can assume $N = \sup\{\|y_t\| : 0 \le t \le a\}$. For each $z \in C([0,a] : X), z(0) = 0$. We can denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} z(t), & 0 \le t \le a, \\ 0, & -\infty < t < 0. \end{cases} \tag{20}$$

If $x(\cdot)$ satisfies (18), we can decompose it as $x(t) = \bar{z}(t) + y(t)$, $0 \le t \le a$, which implies $x_t = \bar{z}_t + y_t$ for every $0 \le t \le a$ and the function $z(\cdot)$ satisfies

$$\begin{aligned}z(t) = {} & \varphi(t) F(0,\phi) - F(t, \bar{z}_t + y_t) \\ & - \int_0^t (t-s)^{q-1} AS(t-s) F(s, \bar{z}_s + y_s)\, ds \\ & + \int_0^t (t-s)^{q-1} S(t-s)[Cu(s) + G(s, \bar{z}_s + y_s)]\, ds.\end{aligned} \tag{21}$$

Moreover $\bar{z}_0 = 0$. Let Q be the operator on $C([0,a], X)$ defined by

$$\begin{aligned}Qz(t) = {} & \varphi(t) F(0,\phi) - F(t, \bar{z}_t + y_t) \\ & - \int_0^t (t-s)^{q-1} AS(t-s) F(s, \bar{z}_s + y_s)\, ds \\ & + \int_0^t (t-s)^{q-1} S(t-s)[Cu(s) + G(s, \bar{z}_s + y_s)]\, ds.\end{aligned} \tag{22}$$

Obviously the operator P has a fixed point is equivalent to Q has a fixed point, so it turns out to prove that Q has a fixed point. For each positive number k, let

$$B_k = \{z \in C([0,a] : X) : z(0) = 0, \|z(t)\| \le k, 0 \le t \le a\}, \tag{23}$$

then for each k, B_k is clearly a bounded closed convex set in $C([0,a] : X)$. Since by (3) and (10) the following relation holds:

$$\begin{aligned}\|AS(t-s) F(s, \bar{z}_s + y_s)\| &\le \left\| A^{1-\beta} S(t-s) A^{\beta} F(s, \bar{z}_s + y_s) \right\| \\ &\le \frac{C_{1-\beta} q \Gamma(1+\beta)}{\Gamma(1+q\beta)} (t-s)^{-(1-\beta)q} \\ &\quad \times L_1 (\|\bar{z}_s + y_s\|_B + 1) \\ &\le \frac{C_{1-\beta} q \Gamma(1+\beta)}{\Gamma(1+q\beta)} (t-s)^{-(1-\beta)q} \\ &\quad \times L_1 (\|\bar{z}_s\|_B + \|y_s\|_B + 1) \\ &\le \frac{C_{1-\beta} q \Gamma(1+\beta)}{\Gamma(1+q\beta)} (t-s)^{-(1-\beta)q} \\ &\quad \times L_1 (k K_a + N + 1),\end{aligned} \tag{24}$$

then from Bocher's theorem [42] it follows that $AS(t-s)F(s, \bar{z}_s + y_s)$ is integrable on $[0,a]$, so Q is well defined on B_k.

In order to make the following process clear we divide it into several steps.

Step 1. We claim that there exists a positive number k such that $Q(B_k) \subseteq B_k$.

If it is not true, then for each positive number k, there is a function $z_k(\cdot) \in B_k$, but $Qz_k \notin B_k$, that is, $\|Qz_k(t)\| > k$ for some $t \in [0, a]$. However, on the other hand, we have

$$\begin{aligned}
k &< \|Qz_k(t)\| \\
&= \Big\| \varphi(t) F(0,\phi) - F(t, \bar{z}_{k,t} + y_t) \\
&\quad - \int_0^t (t-s)^{q-1} \times AS(t-s) F(s, \bar{z}_{k,s} + y_s)\, ds \\
&\quad + \int_0^t (t-s)^{q-1} S(t-s) \times [Cu_k(s) + G(s, \bar{z}_{k,s} + y_s)]\, ds \Big\| \\
&= \Big\| \varphi(t) F(0,\phi) - F(t, \bar{z}_{k,t} + y_t) \\
&\quad - \int_0^t (t-s)^{q-1} AS(t-s) \times F(s, \bar{z}_{k,s} + y_s)\, ds \\
&\quad + \int_0^t (t-s)^{q-1} S(t-s) \times \Big[CW^{-1} \Big\{ x_1 - \varphi(a)[\phi(0) + F(0,\phi)] \\
&\qquad + F(a, \bar{z}_{k,a} + y_a) \\
&\qquad + \int_0^a (a-\tau)^{q-1} AS(a-\tau) \times F(\tau, \bar{z}_{k,\tau} + y_\tau)\, d\tau \\
&\qquad - \int_0^a (a-\tau)^{q-1} S(a-\tau) \times G(\tau, \bar{z}_{k,\tau} + y_\tau)\, d\tau \Big\}(s)\, ds \\
&\quad + \int_0^t (t-s)^{q-1} S(t-s) \times G(s, \bar{z}_{k,s} + y_s) \Big] ds \Big\| \\
&\le M_1 \|F(0,\phi)\| + \|F(t, \bar{z}_{k,t} + y_t)\| \\
&\quad + \int_0^t (t-s)^{q-1} \|AS(t-s) F(s, \bar{z}_{k,s} + y_s)\|\, ds \\
&\quad + \int_0^t M_2 M_3 M_4 \times \Big\{ \|x_1\| + M_1 \|\phi(0) \text{ is} + F(0,\phi)\| \\
&\qquad + \|F(a, \bar{z}_{k,a} + y_a)\| \\
&\qquad + \int_0^a (a-\tau)^{q-1} \times \|AS(a-\tau) F(\tau, \bar{z}_{k,\tau} + y_\tau)\|\, d\tau \\
&\qquad + \int_0^a M_2 (a-\tau)^{q-1} \times \|G(\tau, \bar{z}_{k,\tau} + y_\tau)\|\, d\tau \Big\}(s)\, ds \\
&\quad + \int_0^t M_2 (t-s)^{q-1} \|G(s, \bar{z}_{k,s} + y_s)\|\, ds,
\end{aligned} \tag{25}$$

where u_k is the corresponding control of x_k, $x_k = \bar{z}_k + y$. Since

$$\begin{aligned}
&\Big\| \int_0^t (t-s)^{q-1} AS(t-s) F(s, \bar{z}_{k,s} + y_s)\, ds \Big\| \\
&\quad \le \int_0^t (t-s)^{q-1} \|A^{1-\beta} S(t-s) A^{\beta} F(s, \bar{z}_{k,s} + y_s)\|\, ds \\
&\quad \le \frac{C_{1-\beta} q \Gamma(1+\beta)}{\Gamma(1+q\beta)} \times \int_0^t (t-s)^{q-1} (t-s)^{-(1-\beta)q} L_1(kK_a + N + 1) \\
&\quad \le \frac{C_{1-\beta} \Gamma(1+\beta) a^{q\beta}}{\beta \Gamma(1+q\beta)} L_1(kK_a + N + 1), \\
&\|F(t, \bar{z}_{k,t} + y_t)\| \\
&\quad = \|A^{-\beta} A^{\beta} F(t, \bar{z}_{k,t} + y_t)\| \\
&\quad \le M_5 L_1(kK_a + N + 1), \\
&\int_0^t \|(t-s)^{q-1} G(s, \bar{z}_{k,s} + y_s)\|\, ds \\
&\quad \le \int_0^t (t-s)^{q-1} g_{kK_a+N}(s)\, ds,
\end{aligned} \tag{26}$$

there holds

$$\begin{aligned}
k &< M_1 \|F(0,\phi)\| + M_5 L_1(kK_a + N + 1) \\
&\quad + \frac{C_{1-\beta} \Gamma(1+\beta) a^{q\beta}}{\beta \Gamma(1+q\beta)} L_1(kK_a + N + 1) \\
&\quad + a M_2 M_3 M_4 \Big\{ \|x_1\| + M_1 \|\phi(0) + F(0,\phi)\| \\
&\qquad + M_5 L_1(kK_a + N + 1) \\
&\qquad + \frac{C_{1-\beta} \Gamma(1+\beta) a^{q\beta}}{\beta \Gamma(1+q\beta)} L_1(kK_a + N + 1) \\
&\qquad + M_2 \int_0^a (a-\tau)^{q-1} g_{kK_a+N}(\tau)\, d\tau \Big\}
\end{aligned}$$

$$
\begin{aligned}
&+ M_2 \int_0^a (a-s)^{q-1} g_{kK_a+N}(s)\,ds \\
&= M_5 L_1 k K_a (1 + a M_2 M_3 M_4) \\
&\quad + M_2 (1 + a M_2 M_3 M_4) \int_0^a (a-s)^{q-1} g_{kK_a+N}(s)\,ds \\
&\quad + \frac{C_{1-\beta}\Gamma(1+\beta) a^{q\beta}}{\beta\Gamma(1+q\beta)} L_1 (kK_a + N + 1)(1 + a M_2 M_3 M_4) \\
&\quad + M_1 \|F(0,\phi)\| + M_5 L_1 N + M_5 L_1 + a M_2 M_3 M_4 \|x_1\| \\
&\quad + a M_2 M_3 M_4 \|\phi(0) + F(0,\phi)\| \\
&\quad + M_5 L_1 N a M_2 M_3 M_4 + M_5 L_1 a M_2 M_3 M_4 \\
&= M^* + (1 + a M_2 M_3 M_4) \\
&\quad \times \left[M_5 L_1 K_a k + M_2 \left(\int_0^a (a-s)^{(q-1)/(q-q_1)} ds \right)^{1-q_1} \right. \\
&\qquad \times \left(\int_0^a \left(g_{kK_a+N}(s)\right)^{1/q_1} ds \right)^{q_1} \\
&\qquad \left. + \frac{C_{1-\beta}\Gamma(1+\beta) a^{q\beta}}{\beta\Gamma(1+q\beta)} L_1 (kK_a + N + 1) \right] \\
&= M^* + (1 + a M_2 M_3 M_4) \\
&\quad \times \left[M_5 L_1 K_a k + M_2 \left(\frac{1-q_1}{q-q_1} a^{(q-q_1)/(1-q_1)} \right)^{1-q_1} \right. \\
&\qquad \times \|g_{kK_a+N}\|_{L^{1/q_1}[0,a]} \\
&\qquad \left. + \frac{C_{1-\beta}\Gamma(1+\beta) a^{q\beta}}{\beta\Gamma(1+q\beta)} L_1 (kK_a + N + 1) \right],
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
M^* = {} & M_1 \|F(0,\phi)\| + M_5 L_1 N + M_5 L_1 + a M_2 M_3 M_4 \|x_1\| \\
& + a M_2 M_3 M_4 \|\phi(0) + F(0,\phi)\| \\
& + M_5 L_1 N a M_2 M_3 M_4 + M_5 L_1 a M_2 M_3 M_4.
\end{aligned}
\tag{28}
$$

Dividing on both sides by k and taking the low limit, we get

$$
\begin{aligned}
&(1 + a M_2 M_3 M_4) \\
&\quad \times \left(L_1 M_0 K_a + M_2 \left(\left(\frac{1-q_1}{q-q_1} \right) a^{(q-q_1)/(1-q_1)} \right)^{1-q_1} K_a \gamma \right. \\
&\qquad \left. + \frac{C_{1-\beta}\Gamma(1+\beta) a^{q\beta}}{\beta\Gamma(1+q\beta)} L_1 K_a \right) \geq 1.
\end{aligned}
\tag{29}
$$

This contradicts (16). Hence for some positive number k, $QB_k \subseteq B_k$.

Now, we define operator Q_1 and Q_2 on B_k as

$$
\begin{aligned}
(Q_1 z)(t) &= \varphi(t) F(0,\phi) - F(t, \bar{z}_t + y_t) \\
&\quad - \int_0^t (t-s)^{q-1} A S(t-s) F(s, \bar{z}_s + y_s)\,ds, \\
(Q_2 z)(t) &= \int_0^t (t-s)^{q-1} S(t-s) \\
&\quad \times [Cu(s) + G(s, \bar{z}_s + y_s)]\,ds,
\end{aligned}
\tag{30}
$$

for all $t \in [0,a]$, respectively.

We prove that Q_1 is contraction, while Q_2 is completely continuous.

Step 2. Q_1 is contraction.

Let $z_1, z_2 \in B_k$. Then, for each $t \in [0,a]$, and by axiom (A)-(iii) and (15), we have

$$
\begin{aligned}
&\|Q_1 z_1(t) - Q_1 z_2(t)\| \\
&\quad = \left\| \varphi(t) F(0,\phi) - F(t, \bar{z}_{1,t} + y_t) \right. \\
&\qquad - \int_0^t (t-s)^{q-1} A S(t-s) F(s, \bar{z}_{1,s} + y_s)\,ds \\
&\qquad - \varphi(t) F(0,\phi) + F(t, \bar{z}_{2,t} + y_t) \\
&\qquad \left. + \int_0^t (t-s)^{q-1} A S(t-s) F(s, \bar{z}_{2,s} + y_s)\,ds \right\| \\
&\quad \leq \|F(t, \bar{z}_{1,t} + y_t) - F(t, \bar{z}_{2,t} + y_t)\| \\
&\qquad + \left\| \int_0^t (t-s)^{q-1} A S(t-s) \right. \\
&\qquad\quad \left. \times (F(s, \bar{z}_{1,s} + y_s) - F(s, \bar{z}_{2,s} + y_s))\,ds \right\| \\
&\quad \leq \|A^{-\beta} A^{\beta} F(t, \bar{z}_{1,t} + y_t) - A^{-\beta} A^{\beta} F(t, \bar{z}_{2,t} + y_t)\| \\
&\qquad + \int_0^t (t-s)^{q-1} \\
&\qquad \times \|A^{1-\beta} S(t-s) A^{\beta} (F(s, \bar{z}_{1,s} + y_s) - F(s, \bar{z}_{2,s} + y_s))\|\,ds \\
&\quad \leq M_5 L K_a \|\bar{z}_{1,t} - \bar{z}_{2,t}\|_B \\
&\qquad + \int_0^a (t-s)^{q-1} \frac{C_{1-\beta} q \Gamma(1+\beta)}{\Gamma(1+q\beta)} (t-s)^{-(1-\beta)q} \\
&\qquad\quad \times L \|\bar{z}_{1,s} - \bar{z}_{2,s}\|_B\,ds \\
&\quad \leq M_5 L K_a \sup_{0\leq s\leq a} \|z_1(s) - z_2(s)\| \\
&\qquad + \frac{C_{1-\beta}\Gamma(1+\beta) a^{q\beta}}{\beta\Gamma(1+q\beta)} L_1 K_a \sup_{0\leq s\leq a} \|z_1(s) - z_2(s)\|
\end{aligned}
$$

$$\leq \left(M_5 L K_a + \frac{C_{1-\beta}\Gamma(1+\beta) a^{q\beta}}{\beta\Gamma(1+q\beta)} L_1 K_a \right) \times \sup_{0\leq s\leq a} \| z_1(s) - z_2(s) \| . \tag{31}$$

Thus

$$\| Q_1 z_1(t) - Q_1 z_2(t) \| < \| z_1 - z_2 \| , \tag{32}$$

and Q_1 is contraction.

Step 3. Q_2 is completely continuous.

Let $\{z_n\} \subseteq B_k$ with $z_n \to z$ in B_k, then for each $s \in [0,a]$, $\overline{z}_{n,s} \to \overline{z}_s$, and by (H_1) and (H_2)-(i), we have

$$\begin{gathered} G(s, \overline{z}_{n,s} + y_s) - G(s, \overline{z}_s + y_s) \longrightarrow 0, \\ F(s, \overline{z}_{n,s} + y_s) - F(s, \overline{z}_s + y_s) \longrightarrow 0, \\ u_n(s) - u(s) \longrightarrow 0, \end{gathered} \tag{33}$$

as $n \to \infty$.

Since $\| G(s, \overline{z}_{n,s} + y_s) - G(s, \overline{z}_s + y_s) \| \leq 2 g_{kK_a + N(s)}$, then by the dominated convergence theorem we have

$$\begin{aligned} &\| Q_2 z_n(t) - Q_2 z(t) \| \\ &\quad = \sup_{0\leq t\leq a} \left\| \int_0^t (t-s)^{q-1} S(t-s) C u_n(s)\, ds \right. \\ &\qquad + \int_0^t (t-s)^{q-1} S(t-s) G(s, \overline{z}_{n,s} + y_s)\, ds \\ &\qquad - \int_0^t (t-s)^{q-1} S(t-s) C u(s)\, ds \\ &\qquad \left. - \int_0^t (t-s)^{q-1} S(t-s) G(s, \overline{z}_s + y_s)\, ds \right\| \\ &\quad \leq \left\| \int_0^t (t-s)^{q-1} S(t-s) (C u_n(s) - C u(s))\, ds \right\| \\ &\qquad + \left\| \int_0^t (t-s)^{q-1} S(t-s) \right. \\ &\qquad \left. \times (G(s, \overline{z}_{n,s} + y_s) - G(s, \overline{z}_s + y_s))\, ds \right\| \longrightarrow 0, \end{aligned} \tag{34}$$

as $n \to \infty$, that is, Q_2 is continuous.

Next we prove that the family $\{Q_2 z : z \in B_k\}$ is an equicontinuous family of functions. To do this, let $0 \leq t_1 < t_2 \leq a$, then

$$\begin{aligned} &\| Q_2 z(t_2) - Q_2 z(t_1) \| \\ &\quad = \left\| \int_0^{t_2} (t_2-s)^{q-1} S(t_2-s) C u(s)\, ds \right. \\ &\qquad + \int_0^{t_2} (t_2-s)^{q-1} S(t_2-s) G(s, \overline{z}_s + y_s)\, ds \\ &\qquad - \int_0^{t_1} (t_1-s)^{q-1} S(t_1-s) C u(s)\, ds \\ &\qquad \left. - \int_0^{t_1} (t_1-s)^{q-1} S(t_1-s) G(s, \overline{z}_s + y_s)\, ds \right\| \\ &\quad \leq M_3 \int_0^{t_1} (t_2-s)^{q-1} \| (S(t_2-s) - S(t_1-s)) u(s) \| \, ds \\ &\qquad + M_3 \int_0^{t_1} \left((t_2-s)^{q-1} - (t_1-s)^{q-1} \right) \\ &\qquad\qquad \times \| S(t_1-s) \| \| u(s) \| \, ds \\ &\qquad + \int_0^{t_1} (t_2-s)^{q-1} \\ &\qquad\qquad \times \| (S(t_2-s) - S(t_1-s)) G(s, \overline{z}_s + y_s) \| \, ds \\ &\qquad + \int_0^{t_1} \left((t_2-s)^{q-1} - (t_1-s)^{q-1} \right) \\ &\qquad\qquad \times \| S(t_1-s) \| \| G(s, \overline{z}_s + y_s) \| \, ds \\ &\qquad + M_3 \int_{t_1}^{t_2} (t_2-s)^{q-1} \| S(t_2-s) \| \| u(s) \| \, ds \\ &\qquad + \int_{t_1}^{t_2} (t_2-s)^{q-1} \| S(t_2-s) \| \| G(s, \overline{z}_s + y_s) \| \, ds. \end{aligned} \tag{35}$$

Noting that

$$\begin{aligned} \| u(s) \| &\leq M_4 \left[\| x_1 \| + M_1 \| \phi(0) + F(0,\phi) \| + \| F(a, x_a) \| \right. \\ &\qquad + \int_0^a (a-s)^{q-1} \| S(a-s) F(s, x_s) \| \, ds \\ &\qquad \left. + \int_0^a (a-s)^{q-1} \| S(a-s) G(s, x_s) \| \, ds \right] \\ &\leq M_4 \left[\| x_1 \| + M_1 \| \phi(0) + F(0,\phi) \| \right. \\ &\qquad + \| F(a, x_a) \| + M_5 L_1 (k K_a + N + 1) \\ &\qquad + \frac{C_{1-\beta}\Gamma(1+\beta) a^{q\beta}}{\beta\Gamma(1+q\beta)} L_1 (k K_a + N + 1) \end{aligned}$$

$$+ M_2\left(\frac{1-q_1}{q-q_1}a^{(q-q_1)/(1-q_1)}\right)^{1-q_1} \times \left\|g_{kK_a+N}\right\|_{L^{1/q_1}[0,a]}\Bigg],$$

$$\begin{aligned}
&\int_0^t \left\|(t-s)^{q-1}G\left(s,\overline{z}_{k,s}+y_s\right)\right\| ds \\
&\quad\le \int_0^t (t-s)^{q-1} g_{kK_a+N}(s)\, ds \\
&\quad\le \int_0^a (a-s)^{q-1} g_{kK_a+N}(s)\, ds \\
&\quad\le \left(\left(\frac{1-q_1}{q-q_1}\right)a^{(q-q_1)/(1-q_1)}\right)^{1-q_1}\left\|g_{kK_a+N}\right\|_{L^{1/q_1}[0,a]}.
\end{aligned} \tag{36}$$

We see that $\|Q_2z(t_2)-Q_2z(t_1)\|$ tends to zero independently of $z\in B_k$ as $t_2 \to t_1$ since for $t\in[0,a]$ and any bounded subsets $D\subset X$, $t \to \{S(t)x : x\in D\}$ is equicontinuous.

Hence, Q_2 maps B_k into an equicontinuous family functions.

It remains to prove that $V(t)=\{(Q_2z)(t): z\in B_k\}$ is relatively compact in X. let $0\le t\le a$ be fixed, $0<\epsilon<t$, for $z\in B_k$, we define $\Pi = Q_2B_k$ and $\Pi(t)=\{Q_2z(t)\mid z\in B_k\}$, for $t\in[0,a]$.

Clearly, $\Pi(0)=\{Q_2z(0)\mid z\in B_k\}=\{0\}$ is compact, and hence, it is only to consider $0<t\le a$. For each $h\in(0,t)$, $t\in(0,a]$, arbitrary $\delta>0$, define

$$\Pi_{h,\delta}(t)=\left\{Q_{2,h,\delta}z(t)\mid z\in B_k\right\}, \tag{37}$$

where

$$Q_{2,h,\delta}z(t) = \int_0^{t-h}\int_\delta^\infty (t-s)^{q-1}\theta\xi_q(\theta)\, qT\left((t-s)^q\theta\right) \times\left[Cu(s)+G\left(s,\overline{z}_s+y_s\right)\right] d\theta\, ds. \tag{38}$$

Then the sets $\{Q_{2,h,\delta}z(t)\mid z\in B_k\}$ are relatively compact in X since the condition (H_4). It comes from the following inequalities:

$$\begin{aligned}
&\left\|Q_2z(t)-Q_{2,h,\delta}z(t)\right\| \\
&\quad= \Bigg\|\int_0^t\int_0^\infty (t-s)^{q-1}\theta\xi_q(\theta)\, qT\left((t-s)^q\theta\right) \times\left[Cu(s)+G\left(s,\overline{z}_s+y_s\right)\right] d\theta\, ds \\
&\qquad -\int_0^{t-h}\int_\delta^\infty (t-s)^{q-1}\theta\xi_q(\theta)\, qT\left((t-s)^q\theta\right) \times\left[Cu(s)+G\left(s,\overline{z}_s+y_s\right)\right] d\theta\, ds\Bigg\| \\
&\quad= \Bigg\|\int_0^t\int_0^\delta (t-s)^{q-1}\theta\xi_q(\theta)\, qT\left((t-s)^q\theta\right) \times\left[Cu(s)+G\left(s,\overline{z}_s+y_s\right)\right] d\theta\, ds \\
&\qquad +\int_0^t\int_\delta^\infty (t-s)^{q-1}\theta\xi_q(\theta)\, qT\left((t-s)^q\theta\right) \times\left[Cu(s)+G\left(s,\overline{z}_s+y_s\right)\right] d\theta\, ds \\
&\qquad -\int_0^{t-h}\int_\delta^\infty (t-s)^{q-1}\theta\xi_q(\theta)\, qT\left((t-s)^q\theta\right) \\
&\qquad -\int_0^{t-h}\int_\delta^\infty (t-s)^{q-1}\theta\xi_q(\theta)\, qT\left((t-s)^q\theta\right) \times\left[Cu(s)+G\left(s,\overline{z}_s+y_s\right)\right] d\theta\, ds\Bigg\| \\
&\quad\le MM_3\|u(s)\|\, q\int_0^t (t-s)^{q-1}ds\int_0^\delta \theta\xi_q(\theta)\, d\theta \\
&\qquad + Mq\int_0^t (t-s)^{q-1}g_{kK_a+N}(s)\, ds\int_0^\delta \theta\xi_q(\theta)\, d\theta \\
&\qquad + MM_3\|u(s)\|\int_{t-h}^t (t-s)^{q-1}ds\cdot q\int_\delta^\infty \theta\xi_q(\theta)\, d\theta \\
&\qquad + M\int_{t-h}^t (t-s)^{q-1}g_{kK_a+N}(s)\, ds\cdot q\int_\delta^\infty \theta\xi_q(\theta)\, d\theta \\
&\quad\le \Bigg\{MM_3M_4\Bigg[\left\|x_1\right\| + M_1\left\|\phi(0)+F(0,\phi)\right\| + \left\|F\left(a,x_a\right)\right\| + M_5L_1\left(kK_a+N+1\right) \\
&\qquad + \frac{C_{1-\beta}\Gamma(1+\beta)a^{q\beta}}{\beta\Gamma(1+q\beta)}L_1\left(kK_a+N+1\right) + M\left(\left(\frac{1-q_1}{q-q_1}\right)a^{(q-q_1)/(1-q_1)}\right)^{1-q_1} \times\left\|g_{kK_a+N}\right\|_{L^{1/q_1}[0,a]}\Bigg]a^q \\
&\qquad + M\left(\left(\frac{1-q_1}{q-q_1}\right)a^{(q-q_1)/(1-q_1)}\right)^{1-q_1}\left\|g_{kK_a+N}\right\|_{L^{1/q_1}[0,a]}\Bigg\}q \\
&\qquad \times\int_0^\delta \theta\xi_q(\theta)\, d\theta + M\left(\left(\frac{1-q_1}{q-q_1}\right)h^{(q-q_1)/(1-q_1)}\right)^{1-q_1} \\
&\qquad \times\left\|g_{kK_a+N}\right\|_{L^{1/q_1}[0,a]}q\int_0^\infty \theta\xi_q(\theta)\, d\theta.
\end{aligned} \tag{39}$$

Therefore, $\Pi(t)=\{Q_2z(t)\mid z\in B_k\}$ is relatively compact in X for all $t\in[0,a]$.

Thus, the continuity of Q_2 and relatively compact of $\{Q_2z(t)\mid z\in B_k\}$ imply that Q_2 is a completely continuous operator.

These arguments enable us to conclude that $Q = Q_1 + Q_2$ is a condense mapping on B_k, and by the fixed point theorem of Sadovskii there exists a fixed point $z(\cdot)$ for Q on B_k. In fact, by Step 1–Step 3 and Lemma 3, we can conclude that $Q = Q_1 + Q_2$ is continuous and takes bounded sets into bounded sets. Meanwhile, it is easy to see $\alpha(Q_2(B_k)) = 0$ since $Q_2(B_k)$ is relatively compact. Since $Q_1(B_k)) \subseteq B_k$ and $\alpha(Q_2(B_k)) = 0$, we can obtain $\alpha(Q(B_k)) \leq \alpha(Q_1(B_k)) + \alpha(Q_2(B_k)) \leq \alpha(B_k)$ for every bounded set B_k of X with $\alpha(B_k) > 0$, that is, $Q = Q_1 + Q_2$ is a condense mapping on B_k. If we define $x(t) = z(t) + y(t), -\infty < t \leq a$, it is easy to see that $x(\cdot)$ is a mild solution of (1) satisfying $x_0 = \phi, x(a) = x_1$. Then the proof is completed. □

Remark 8. In order to describe various real-world problems in physical and engineering sciences subject to abrupt changes at certain instants during the evolution process, impulsive fractional differential equations always have been used in the system model. So we can also consider the complete controllability for (1) with impulses.

Remark 9. Since the complete controllability steers the systems to arbitrary final state while approximate controllability steers the system to arbitrary small neighborhood of final state. In view of the definition of approximate controllability in [28], we can deduce that the considered systems (1) is also approximate controllable on the interval $[0, a]$.

4. An Example

As an application of Theorem 7, we consider the following system:

$$\begin{aligned} &\frac{\partial^{2/3}}{\partial t^{2/3}}\left[z(t,x) + \int_{-\infty}^{t}\int_0^{\pi} b(s-t,y,x)\,z(s,y)\,dy\,ds\right] \\ &\quad - \frac{\partial^2}{\partial x^2} z(t,x) \\ &= Cu(t) + a_0(x)\,z(t,x) \\ &\quad + \int_{-\infty}^{t} a_1(s,t)\,z(s,x)\,ds + a_2(t,x), \\ &\qquad 0 \leq t \leq a, \quad 0 \leq x \leq \pi, \\ &z(t,0) = z(t,\pi) = 0, \quad z(\vartheta,x) = \phi(\vartheta,x), \quad \vartheta \leq 0. \end{aligned} \tag{40}$$

To write system (40) to the form of (1), let $X = L^2([0,\pi])$ and A defined by $Af = -f''$ with domain $D(A) = \{f(\cdot) \in X : f, f'$ absolutely continuous, $f'' \in X, f(0) = f(\pi) = 0\}$, the $-A$ generates a uniformly bounded analytic semigroup which satisfies the condition (H_0). Furthermore, A has a discrete spectrum, the eigenvalues are $-n^2$, $n \in N$, with the corresponding normalized eigenvectors $z_n(x) = (2/\pi)^{1/2}\sin(nx)$. Then the following properties hold.

(i) If $A \in D(A)$, then

$$Af = \sum_{n=1}^{\infty} n^2 \langle f, z_n\rangle z_n. \tag{41}$$

(ii) For each $f \in X$,

$$A^{-1/2} f = \sum_{n=1}^{\infty} \frac{1}{n}\langle f, z_n\rangle z_n. \tag{42}$$

In particular, $\|A^{-1/2}\| = 1$.

(iii) The operator $A^{1/2}$ is given by

$$A^{1/2} f = \sum_{n=1}^{\infty} n \langle f, z_n\rangle z_n \tag{43}$$

on the space $D(A^{1/2}) = \{f(\cdot) \in X, A^{1/2} f \in X\}$.

Here we take the phase space $B = C_0 \times L^2(g, X)$, which contains all classes of functions $\phi : (-\infty, 0] \to X$ such that ϕ is Lebesgue measurable and $g(\cdot)\|\phi(\cdot)\|^2$ is Lebesgue integrable on $(-\infty, 0)$ where $g : (-\infty, 0) \to R$ is a positive integrable function. The seminorm in B is defined by

$$\|\phi\|_B = \|\phi(0)\| + \left(\int_{-\infty}^{0} g(\vartheta)\,\|\phi(\vartheta)\|^2 d\vartheta\right)^{1/2}. \tag{44}$$

From [41], under some conditions B is a phase space verifying (A), (B), (C), and in this case $K(t) = 1 + (\int_{-t}^{0} g(\vartheta)d\vartheta)^{1/2}$ (see [41] for the details).

We assume the following conditions hold.

(a) The function b is measurable and $\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}(b^2(\vartheta, y, x)/g(\vartheta))dy\,d\vartheta\,dx < \infty$.

(b) The function $(\partial/\partial x)b(\vartheta, y, x)$ is measurable, $b(\vartheta, y, 0) = b(\vartheta, y, \pi) = 0$ and let $N_1 = \int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}(1/g(\vartheta))((\partial/\partial x)b(\vartheta, y, x))^2 dy\,d\vartheta\,dx < \infty$.

(c) The function $a_0(\cdot) \in L^{\infty}([0,\pi])$, $a(\cdot)$ is measurable, with $\int_{-\infty}^{0}(a_1^2(\vartheta))/g(\vartheta)d\vartheta < \infty$, the function $a_2(t,\cdot) \in L^2([0,\pi])$ for each $t \geq 0$ is measurable in t.

(d) The function ϕ defined by $\phi(\vartheta)(x) = \phi(\vartheta, x)$ belongs to B.

(e) The linear operator $W : U \to X$ is defined by

$$Wu = \int_0^{a} (a-s)^{-1/3} S(a-s)\,Cu(s)\,ds \tag{45}$$

and has a bounded invertible operator W^{-1} defined $L^2([0, a]); U)/\ker W$.

We define $F, G : [0,a] \times B \to X$ by $F(t,\phi) = Z_1(\phi)$ and $G(t,\phi) = Z_2(\phi) + h(t)$, where

$$\begin{aligned} Z_1(\phi) &= \int_{-\infty}^{0}\int_0^{\pi} b(\vartheta, y, x)\,\phi(\vartheta, x)\,dy\,d\vartheta, \\ Z_2(\phi) &= a_0(x)\,\phi(0,x) + \int_{-\infty}^{0} a_1(\vartheta)\,\phi(\vartheta, x)\,d\vartheta, \\ h(t) &= a_2(t,\cdot). \end{aligned} \tag{46}$$

From (a) and (c) it is clear that Z_1 and Z_2 are bounded linear operators on B. Furthermore, $Z_1(\phi) \in D(A^{1/2})$, and $\|A^{1/2}Z_1\| \le N_1$. In fact, from the definition of Z_1 and (b) it follows that $\langle Z_1(\phi), z_n\rangle = (1/n)(2/\pi)^{1/2}\langle Z(\phi), \mathrm{con}(nx)\rangle$, where $Z(\phi) = \int_{-\infty}^{0}\int_0^{\pi}(\partial/\partial x)b(\vartheta, y, x)\phi(\vartheta, x)d\vartheta$. From (b) we know that $Z : B \to X$ is a bounded linear operator with $\|Z\| \le N_1$. Hence $\|A^{1/2}Z_1(\phi)\| = \|Z(\phi)\|$, which implies the assertion. Therefore, from Theorem 7, the system (40) is completely controllable on $[0, a]$ under the above assumptions.

5. Conclusion

In this paper, by using the uniformly boundedness, analyticity, and equicontinuity of characteristic solution operators and the Sadovskii fixed point theorem, we obtained the complete controllability of the abstract neutral fractional differential systems with unbounded delay in a Banach space. It shows that the compactness of the characteristic solution operators can be weakened to equicontinuity. Our theorem guarantees the effectiveness of complete controllability results under some weakly compactness conditions.

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