Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 418091, 11 pages
doi:10.1155/2012/418091

Research Article

$\mathcal{W}$-Stability of Multistable Nonlinear Discrete-Time Systems

Zhishuai Ding,[1] Guifang Cheng,[1,2] and Xiaowu Mu[1]

[1] *Department of Mathematics, Zhengzhou University, Zhengzhou, Hennan 450001, China*
[2] *Physical Engineering College, Zhengzhou University, Zhengzhou, Hennan 450001, China*

Correspondence should be addressed to Guifang Cheng, gfcheng@zzu.edu.cn

Received 8 June 2011; Accepted 28 October 2011

Academic Editor: Baodong Zheng

Motivated by the importance and application of discrete dynamical systems, this paper presents a new Lyapunov characterization which is an extension of conventional Lyapunov characterization for multistable discrete-time nonlinear systems. Based on a new type stability notion of $\mathcal{W}$-stability introduced by D. Efimov, the estimates of solution and the Lyapunov stability theorem and converse theorem are proposed for multi-stable discrete-time nonlinear systems.

1. Introduction

Theory of discrete-time systems is rapidly developed and widely applied to various fields (see three remarkable books [1–3]). In paper [4], motivated by a continuous second-order predator-prey ecological system of Lotka-Volterra type, Efimov introduces a new type notion, $\mathcal{W}$-stable and presents Lyapunov characterization for multistable continuous nonlinear systems. For ecological systems, they usually considered that evolution and translation of populations is continuous. Thus continuous models [5, 6] are considered in many references. However, according to observing the translation process of population change, discrete models are better to represent ecological systems [7, 8]. Thus it is meaningful to study $\mathcal{W}$-stability for discrete dynamical systems. This paper extends stability results given by Efimov about continuous multistable systems to discrete-time multistable systems.

Stability analysis is one of the main issues for research of control systems theory. A rapid progress has been made in local or global stability analysis for unique equilibrium [9], trajectories [10], close invariant set [11, 12], part of state variable [13], and so forth. In papers [14, 15], Sontag and Wang introduce Lyapunov characterization of input to state stability for continuous systems. Paper [16] by Jiang and Wang presents the property of input to state stability for discrete-time systems. Converse Lyapunov theorem is presented in paper [11] by Lin et al. and paper [17] by Jiang and Wang for continuous and discrete-time systems, respectively. In recent years, multistable systems have attracted considerable attention [8, 18–21]. There have many methods to deal with stability problem for multistable systems. Two popular modern approaches are based on density functions [19] and monotone systems [20]. The former approach substitutes conventional Lyapunov function with density function for establishing stability of stable set. The latter approach develops some constructive conditions based on monotone systems for establishing stability of the set of equilibriums. The above approaches are effective to handle the stability problem of multistable systems. The stability results obtained according to the above approaches are based on conventional stable notions. However, in the areas of theoretical biology and engineering, many systems that represent models are called multistable systems. The set of all invariant solutions of those systems contains stable subset and unstable subset.

In this paper, we firstly introduce the notion of $\mathcal{W}$-stability to discrete-time multistable systems. Using some important approaches and techniques of stability analysis in papers [4, 11, 14–18], new Lyapunov characterizations are proposed for discrete-time multistable systems. Based on notions of $\mathcal{W}$-Lyapunov function and weak $\mathcal{W}$-Lyapunov function, the relation of two functions is presented, and a converse Lyapunov theorem is proved. Our main contribution is that Lyapunov characterizations presented in this paper contain the conventional Lyapunov characterization and it should be extensively applied.

The rest of the paper is organized as follows. Problem statement and mathematical preliminary are presented in Section 2. Stability results of multistable discrete-time nonlinear systems are proposed in Section 3. Finally, a brief conclusion is provided to summarize the paper in the final section.

2. Problem Statement and Mathematical Preliminary

Consider the following discrete nonlinear system:

$$x(k+1) = f(x(k)), \tag{2.1}$$

where $x(k) \in \mathbf{R}^n$ is the system state vector at time instant k, $k \in \mathbf{Z}_+ = \{0, 1, 2, \ldots\}$, $f : \mathbf{R}^n \to \mathbf{R}^n$ is locally Lipschitz continuous. $x(k, x_0)$ denotes the solution for any initial value $x_0 = x(k_0)$, and $k_0 \in \mathbf{Z}_+$. $\|x\|$ denotes the Euclidean norm of vector x.

Let $\mathcal{M}$ be a nonempty subset of $\mathbf{R}^n$. The set is called (forward) invariant for system (2.1) if

$$x_0 \in \mathcal{M} \Longrightarrow x(k, x_0) \in \mathcal{M}, \quad \forall k \in \mathbf{Z}_+. \tag{2.2}$$

An invariant set is called minimal if it does not contain other smaller invariant sets. The distance of the set $\mathcal{M}$ from a point $x \in \mathbf{R}^n$ is defined as

$$\|x\|_{\mathcal{M}} = \operatorname{dist}(x, \mathcal{M}) = \inf_{\xi \in \mathcal{M}} \|x - \xi\|. \tag{2.3}$$

An invariant set $\mathcal{A}$ is said to be a locally attracting set if there exists an open neighborhood $\mathcal{U}$ of $\mathcal{A}$ such that, for any $x_0 \in \mathcal{U}$, $\lim_{k \to +\infty} \|\mathrm{x}(k, x_0)\|_{\mathcal{A}} = 0$. An invariant set $\mathcal{R}$ is said to be a locally repelling set if exists an open neighborhood $\mathcal{U}$ of $\mathcal{R}$ such that, for any $x_0 \in \mathcal{U} - \mathcal{R}$, $\lim_{k\to+\infty}\|x(k,x_0)\|_{\mathcal{R}} > 0$.

Let $\mathcal{W} \subset \mathbf{R}^n$ be the set of all invariant solutions of system (2.1). Clearly, it is an invariant set. Assume the set $\mathcal{W}$ is a closed and compact set and satisfies $\mathcal{W} = \mathcal{A} \cup \mathcal{R}$, where $\mathcal{A}$ and $\mathcal{R}$ denote attracting set and repelling set, respectively.

We first introduce the notion of $\mathcal{W}$-asymptotical stability [4] for continuous nonlinear systems to discrete-time nonlinear systems (2.1).

Definition 2.1. The system (2.1) is called $\mathcal{W}$-stable with respect to $\mathcal{W}$ if, for some given constant $R \geq 0$ and for each $R \leq \varepsilon \leq +\infty$, there exists $0 \leq \delta = \delta(\varepsilon) < +\infty$, such that when $\|x_0\|_{\mathcal{W}} \leq \delta$, it holds

$$\|x(k,x_0)\|_{\mathcal{W}} \leq \varepsilon, \quad \forall k \in \mathbf{Z}_+. \tag{2.4}$$

Remark 2.2. It is possible to exist unstable equilibriums in the set $\mathcal{W}$ which contains all invariant solutions. When the trajectories of system initiate from a neighborhood of unstable equilibriums, it cannot ensure the trajectories in this neighborhood. Thus, the definition of $\mathcal{W}$-stability need the existence $R \geq 0$ such that $\varepsilon \geq R$. If $\mathcal{R}$ is not empty then $R > 0$. Particularly, when $\mathcal{R}$ is empty (it implies $R = 0$), $\mathcal{W}$-stability is reduced to conventional stability (see [1, 2, 9]). The constant can be related with the radius of the set $\mathcal{W}$ [4].

Definition 2.3. The system (2.1) is called $\mathcal{W}$-asymptotically stable with respect to $\mathcal{W}$ if

(i) it is $\mathcal{W}$-stable;

(ii) it satisfies $\mathcal{W}$-attracting property. There is a positive constant c, such that, for all $\|x_0\|_{\mathcal{W}} \leq c$, $\lim_{k\to+\infty}\|x(k,x_0)\|_{\mathcal{W}} = 0$; that is, for each $0 < \eta < +\infty$ and $\|x_0\|_{\mathcal{W}} \leq c$, there exists $T = T(x_0,\eta) \in \mathbf{Z}_+$ such that

$$\|x(k,x_0)\|_{\mathcal{W}} < \eta, \quad \forall k \geq T(x_0,\eta). \tag{2.5}$$

Remark 2.4. $T(x_0,\eta)$ is dependent on x_0, and η. $T(\cdot,\eta)$ is different for different initial values; that is, there exists no uniform time of convergence to $\mathcal{W}$ of the trajectories which start from the neighborhood of different initial values due to the presence of unstable equilibriums.

Example 2.5. Consider a second-order discrete system

$$\begin{gathered} x(k+1) = x(k) + hy(k), \\ y(k+1) = y(k) + hy(k)\left(-y(k) + x(k) - x^3(k)\right). \end{gathered} \tag{2.6}$$

The set of all invariant solutions is $\mathcal{W} = \{(0,0),(1,0),(-1,0)\}$. Let $h = 0.1$. Simulation results are shown in Figure 1.

According to Figure 1, we can get that $(0,0)$ is unstable and $(1,0)$ and $(-1,0)$ is asymptotically stable. Thus $\mathcal{A} = \{(1,0),(-1,0)\}$, and $\mathcal{R} = \{(0,0)\}$. System (2.6) is $\mathcal{W}$-asymptotically stable by Definition 2.3.

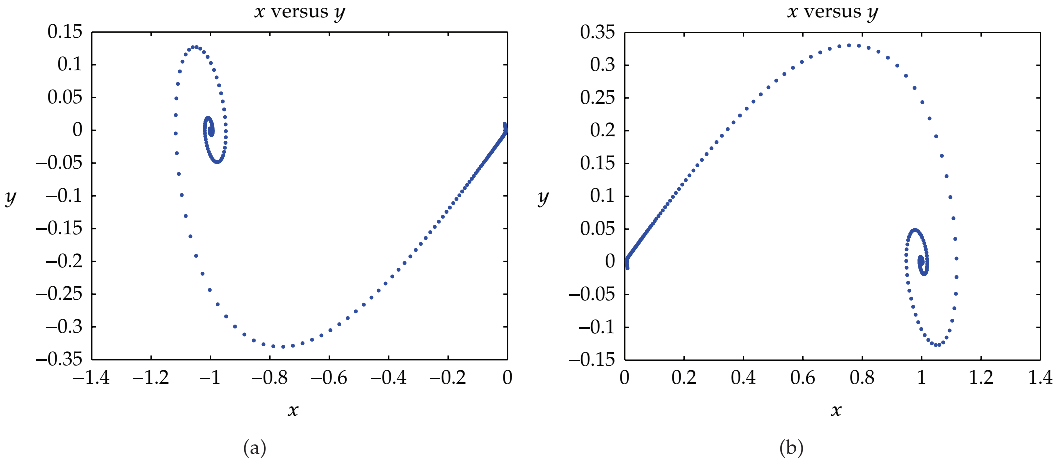

Figure 1: $\mathcal{W}$-asymptotical stability of system (2.6) with different initial values. (a) and (b) show phase portraits of system (2.6) with initial value $(-0.01, 0.01)$ and $(0.01, -0.01)$, respectively.

Example 2.6. Consider a second-order predator-prey system with a prey refuge in the following form:

$$x(k+1) = x(k) + hx(k)\left(\beta_1 - b_1 x(k) - a_1(1-m)y(k)\right),$$
$$y(k+1) = y(k) + hy(k)\left(\beta_2 - \frac{a_2 y(k)}{(1-m)x(k)}\right), \tag{2.7}$$

where $x(k)$, $y(k)$ represent the prey and predator density, parameters β_1 and β_2 are the intrinsic growth rates of the prey and the predator, respectively. h is the step size. $m \in [0,1)$ is a refuge protecting coefficient of the prey. The rest coefficients are positive constants. Let $m = 1/2, h = 0.1, \beta_1 = \beta_2 = 2$, and $a_1 = a_2 = b = 1$. The set of all invariant solutions of system (2.7) is $\mathcal{W} = \{(0,0), (2,0), (4/3, 4/3)\}$. Simulation results are shown in Figure 2.

According to Figure 2, the trajectories of solutions from the neighborhood of points (0,0) and (2,0) are convergent to point $(4/3, 4/3)$. We have $(0,0)$ and $(2,0)$ are unstable and $(4/3, 4/3)$ is asymptotically stable. System (2.7) is $\mathcal{W}$-asymptotical stable.

Remark 2.7. For system (2.7), there have been many important and interesting results, such as the global stability, periodic solutions, almost periodic solutions, and chaos (see [4–8, 14, 15, 22]). Here our interesting focus is on $\mathcal{W}$-asymptotical stability of the set of all invariant solutions.

Throughout this paper, assume $\mathcal{A}$ and $\mathcal{R}$ are not empty. In this case, we can exclude an open set containing $\mathcal{R}$ from admissible set of initial value. Then there exists a uniform convergent time T in the definition of attracting property. That is, $\mathcal{W}$-attracting property is similar to the definition of attracting property in [9, 18]. Define a hyper-surface

$$\Sigma = \{x \in \mathbf{R}^n : \|x\|_{\mathcal{R}} - \|x\|_{\mathcal{A}} = 0\}. \tag{2.8}$$

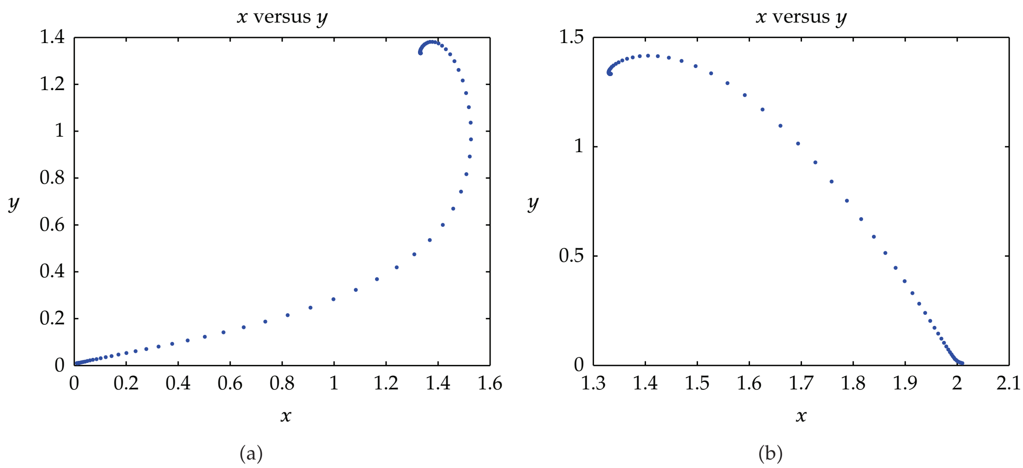

Figure 2: $\mathcal{W}$-asymptotical stability of system (2.7) with different initial values. (a) and (b) show phase portraits of system (2.7) with initial value (0.01,0.01) and (2.01,0.01), respectively.

Then we obtain

$$\Sigma^- = \{x \in \mathbf{R}^n : \|x\|_{\mathcal{R}} - \|x\|_{\mathcal{A}} < 0\}, \qquad \Sigma^+ = \{x \in \mathbf{R}^n : \|x\|_{\mathcal{R}} - \|x\|_{\mathcal{A}} > 0\}. \tag{2.9}$$

Let an open neighborhood $\mathcal{U}$ of $\mathcal{R}$ be $\mathcal{U} = \mathcal{B}_\rho = \{x \in \mathbf{R}^n : \|x\|_{\mathcal{R}} < \rho\}$. We can choose some ρ^* which ensure the properties $\mathcal{B}_\rho \subset \Sigma^-$, $\mathcal{B}_\rho \cap \Sigma^+ = \emptyset$ hold only for $\rho < \rho^*$.

Definition 2.8. System (2.1) satisfies $\mathcal{W}$-attracting property. Choose any open set $\mathcal{B}_\rho$ of $\mathcal{R}$. For each $\eta > 0$ and $r > 0$ there exists $T = T(r, \eta) \in [0, +\infty)$ such that for any $x_0 \in \{x : \|x\|_{\mathcal{W}} < r\} - \mathcal{B}_\rho$

$$\|x(k, x_0)\|_{\mathcal{W}} < \eta, \qquad \forall k \geq T(r, \eta). \tag{2.10}$$

3. Stability Results

3.1. Estimates of Solution

Lemma 3.1. *The system* (2.1) *is $\mathcal{W}$-stable with respect to $\mathcal{W}$ for some given $R \geq 0$ if and only if for any constant $r \geq 0$ there exists a class $\mathcal{K}$ function α such that for any $\|x_0\|_{\mathcal{W}} \leq r$*

$$\|x(k, x_0)\|_{\mathcal{W}} \leq \alpha\left(\|x_0\|_{\mathcal{W}} + \widetilde{R}\right), \quad \forall k \in \mathbf{Z}_+, \tag{3.1}$$

where $\widetilde{R} = \alpha^{-1}(R)$.

Proof. Sufficiency. For system (2.1), when $\|x_0\|_{\mathcal{W}} \leq r$, (3.1) holds for any $k \in \mathbf{Z}_+$. For each $\varepsilon = \alpha(r + \widetilde{R}) \geq R$, choose $\delta = \min\{r, \alpha^{-1}(\varepsilon) - \widetilde{R}\}$, when $\|x_0\|_{\mathcal{W}} \leq \delta$, $\|x(k, x_0)\|_{\mathcal{W}} \leq \varepsilon$ holds for any $k \in \mathbf{Z}_+$. $\mathcal{W}$-stability is ensured.

Necessity. Assume system (2.1) is $\mathcal{W}$-stable, that is, for some given $R \geq 0$ and for each $\varepsilon \geq 0$, there exists $\delta \geq 0$, such that

$$\|x_0\|_{\mathcal{W}} \leq \delta \Longrightarrow \|x(k, x_0)\|_{\mathcal{W}} \leq \varepsilon + R, \quad k \in \mathbf{Z}_+. \tag{3.2}$$

For fixed ε and R, let $\overline{\delta}(\varepsilon)$ be the supremun of all applicable $\delta(\varepsilon)$. Clearly, the function $\overline{\delta}(r)$ is positive and nondecreasing. So there exist a class $\mathcal{K}$ function φ and a constant $\lambda \in (0,1)$ such that $\varphi(r) \leq \varphi(r+R) \leq \lambda(\overline{\delta}(r) + \widetilde{R})$, where $\widetilde{R} = \varphi(R)$. Let $\alpha(r) = \varphi^{-1}(r)$. Then α is a function of class $\mathcal{K}$. Let $c = \lim_{r \to +\infty} \alpha(r)$. Given $\|x_0\|_{\mathcal{W}} \leq c - \widetilde{R}$, let $\varepsilon + R = \alpha(\|x_0\|_{\mathcal{W}} + \widetilde{R})$. Then we have $\|x_0\|_{\mathcal{W}} \leq \varphi(\varepsilon + R) - \widetilde{R} \leq \overline{\delta}(\varepsilon)$ and $\|x(k, x_0)\|_{\mathcal{W}} \leq \alpha(\|x_0\|_{\mathcal{W}} + \widetilde{R}) = \varepsilon + R$. Thus $\mathcal{W}$-stability implies the property as in inequality (3.1). □

Remark 3.2. The construction approach of class $\mathcal{K}$ function α is similar to its in [9, 11]. When $R = 0$, the result of Lemma 3.1 is the same as the corresponding result in [9, 11].

Lemma 3.3. *The system* (2.1) *is $\mathcal{W}$-asymptotically stable with respect to $\mathcal{W}$ if and only if for some constant $R \geq 0$ and any constant $r \geq 0$ there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function μ such that for any $\|x_0\|_{\mathcal{W}} \leq r$*

$$\|x(k, x_0)\|_{\mathcal{W}} < \beta(\|x_0\|_{\mathcal{W}} + R, k), \quad \forall k \in \mathbf{Z}_+. \tag{3.3}$$

Proof. Sufficiency. Suppose there is a class $\mathcal{KL}$ function β such that inequality (3.3) is satisfied. With fixed $k = k_1$, we have

$$\|x(k, x_0)\|_{\mathcal{W}} < \beta(\|x_0\|_{\mathcal{W}} + R, k_1). \tag{3.4}$$

By Lemma 3.1, system (2.1) is $\mathcal{W}$-stable. For any $\|x_0\|_{\mathcal{W}} \leq \zeta$, it yields

$$\|x(k, x_0)\|_{\mathcal{W}} < \beta(\zeta + R, k). \tag{3.5}$$

It implies $\|x(k, x_0)\|_{\mathcal{W}} \to 0$ as $k \to +\infty$. Attracting property is satisfied. Thus, system (2.1) is $\mathcal{W}$-asymptotically stable.

Necessity. Suppose that system (2.1) is $\mathcal{W}$-asymptotically stable. According to Lemma 3.1, there exists a class $\mathcal{K}$ function α such that for any $\|x_0\|_{\mathcal{W}} \leq r$

$$\|x(k, x_0)\|_{\mathcal{W}} \leq \alpha\left(\|x_0\|_{\mathcal{W}} + \widetilde{R}\right). \tag{3.6}$$

Moreover, choose an arbitrary small constant ρ which satisfies $0 < \rho < \rho^*$. For any $x_0 \in \{x : \|x\|_{\mathcal{W}} \leq r\} - \mathcal{B}_\rho$ and given $\eta > 0$, there exists $T(r, \eta)$ such that

$$\|x(k, x_0)\|_{\mathcal{W}} < \eta, \quad \forall k \geq T(r, \eta). \tag{3.7}$$

Let $\overline{T}(r,\eta)$ be the infimun of $T(r,\eta)$. The function $\overline{T}(r,\eta)$ is nonnegative in r and nonincreasing in η, and $\overline{T}(r,\eta) = 0$ for all $\eta > \alpha(r+R)$. Let

$$\psi_r(\eta) = \frac{2}{\eta}\int_{\eta/2}^{\eta} \overline{T}(r,s)ds > \overline{T}(r,\eta). \tag{3.8}$$

The function $\psi_r(\eta)$ is positive and has the following properties:

(i) for each fixed r, $\psi_r(\eta)$ is continuous, strictly decreasing, and $\psi_r(\eta) \to 0$ as $\eta \to +\infty$;

(ii) for each fixed η, $\psi_r(\eta)$ is strictly increasing in r.

Take $\chi_r(\eta) = \psi_r^{-1}(\eta)$. Then $\chi_r(\eta)$ also satisfies the above two properties and $\overline{T}(r,\chi_r(\eta)) < \psi_r(\chi_r(\eta)) = \eta$. So

$$\|x(k,x_0)\|_{\mathcal{W}} \le \chi_r(k), \quad \forall k \in \mathbf{Z}_+,\ \forall x_0 \in \{\|x_0\|_{\mathcal{W}} \le r\} - \mathcal{B}_\rho. \tag{3.9}$$

According to inequalities (3.6) and (3.9) we get

$$\|x(k,x_0)\|_{\mathcal{W}} \le \sqrt[2]{\alpha\left(\|x_0\|_{\mathcal{W}} + \widetilde{R}\right)\chi_r(k)} = \beta(\|x_0\|_{\mathcal{W}} + R, k). \tag{3.10}$$

Thus, according to arbitrariness of ρ, there exists a class $\mathcal{KL}$ function β such that inequality (3.3) is satisfied. □

3.2. Stability Theorem

Definition 3.4. A continuous function $\mathcal{W} : \mathbf{R}^n \to \mathbf{R}$ is a $\mathcal{W}$-Lyapunov function with respect to $\mathcal{W}$ for system (2.1) if

(i) there exist class $\mathcal{K}$ functions α_1 and α_2 and a constant $R \ge 0$ such that for any $k \in \mathbf{Z}_+$ and $x_0 \in \mathbf{R}^n$

$$\alpha_1(\|x_0\|_{\mathcal{W}}) \le \mathcal{W}(x(k,x_0)) \le \alpha_2(\|x_0\|_{\mathcal{W}} + R); \tag{3.11}$$

(ii) there exists a class $\mathcal{K}$ function α_3 such that for any $k \in \mathbf{Z}_+$ and $x_0 \in \mathbf{R}^n$

$$\mathcal{W}\big(f(x(k,x_0))\big) - \mathcal{W}(x(k,x_0)) \le -\alpha_3(\|x_0\|_{\mathcal{W}}). \tag{3.12}$$

Assume system (2.1) has output $y(k) = h(x(k))$, where $h : \mathbf{R}^n \to \mathbf{R}^m$ is a continuous function. The system (2.1) with output $y(x(k))$ is $\mathcal{W}$-detectable if for any $k \in \mathbf{Z}_+$ and $x_0 \in \mathbf{R}^n$

$$y(x(k,x_0)) = h(x(k,x_0)) \equiv 0 \Longrightarrow \|x(k,x_0)\|_{\mathcal{W}} \longrightarrow 0, \quad k \longrightarrow +\infty. \tag{3.13}$$

Definition 3.5. A continuous function $\mathcal{W} : \mathbf{R}^n \to \mathbf{R}$ is a weak $\mathcal{W}$-Lyapunov function with respect to $\mathcal{W}$ for system (2.1) if

(i) there exist class $\mathcal{K}$ functions α_1 and α_2 and a constant $R \geq 0$ such that for any $k \in \mathbf{Z}_+$ and $x_0 \in \mathbf{R}^n$

$$\alpha_1(\|x_0\|_{\mathcal{W}}) \leq \mathcal{W}(x(k, x_0)) \leq \alpha_2(\|x_0\|_{\mathcal{W}} + R); \tag{3.14}$$

(ii) there exists a continuous function $\alpha_3 : \mathbf{R}^m \to [0, \infty)$, with $\alpha_3(0) = 0$ and $\alpha_3(s) > 0$ for all $s \neq 0$ such that for any $\in \mathbf{Z}_+$ and $x_0 \in \mathbf{R}^n$

$$\mathcal{W}(f(x(k, x_0))) - \mathcal{W}(x(k, x_0)) \leq -\alpha_3(\|h(x(k, x_0))\|) \leq 0; \tag{3.15}$$

(iii) system (2.1) with output $y(x(k))$ is $\mathcal{W}$-detectable.

Theorem 3.6. *Weak $\mathcal{W}$-Lyapunov function implies $\mathcal{W}$-asymptotical stability.*

Proof. Suppose system (2.1) has a weak $\mathcal{W}$-Lyapunov function. Using inequality (3.15), we have

$$\mathcal{W}(f(x(k, x_0))) - \mathcal{W}(x(k, x_0)) \leq 0. \tag{3.16}$$

It implies $\mathcal{W}$ is bounded and

$$\alpha_1(\|x_0\|_{\mathcal{W}}) \leq \mathcal{W}(x(k, x_0)) \leq \mathcal{W}(x_0) \leq \alpha_2(\|x_0\|_{\mathcal{W}} + R), \quad \forall k \in \mathbf{Z}_+. \tag{3.17}$$

Thus

$$\|x(k, x_0)\|_{\mathcal{W}} \leq \alpha_1^{-1}(\alpha_2(\|x_0\|_{\mathcal{W}} + R)). \tag{3.18}$$

According to Lemma 3.1, system (2.1) is $\mathcal{W}$-stable with respect to $\mathcal{W}$.

Since the compactness of set $\mathcal{W}$, there exist a class $\mathcal{K}$ function μ and a positive constant c such that

$$\|x(k)\| \leq \mu(\|x(k)\|_{\mathcal{W}} + c), \tag{3.19}$$

which shows $x(k)$ is bounded. Thus, for any solution of system (2.1) we can find a forward invariant attracting compact set $\Omega(x(0))$. Choosing $x_0 \in \Omega(x(0))$, we have $x(k, x_0) \in \Omega(x(0))$ for any $k \in \mathbf{Z}_+$.

Furthermore, by $\mathcal{W}(f(x(k, x_0))) - \mathcal{W}(x(k, x_0)) \leq 0$ we get $\mathcal{W}(x(k, x_0))$ is nonincreasing. However, by $\mathcal{W}(x(k, x_0)) \geq \alpha_1(\|x_0\|_{\mathcal{W}})$ we obtain $\mathcal{W}(x(k))$ is nondecreasing. Thus, there exists a positive constant d such that

$$\mathcal{W}(x(k, x_0)) = d, \tag{3.20}$$

which implies

$$\mathcal{W}(f(x(k, x_0))) - \mathcal{W}(x(k, x_0)) = 0. \tag{3.21}$$

Then $\alpha_3(h(x(k,x_0))) = 0$. Using the detectability of system (2.1) with $y(x(k))$, we have $\|x(k,x_0)\|_{\mathcal{W}} \to 0$ as $k \to +\infty$. That is, $\mathcal{W}$-attracting property holds.

The proof is completed. □

Lemma 3.7 is given by Sontag in [15] which is useful for proof of Theorem 3.8.

Lemma 3.7. *Assume that β is a function of class $\mathcal{KL}$. Then there exist two class $\mathcal{K}_\infty$ functions φ_1 and φ_2 such that*

$$\beta(s,r) \le \varphi_1\left(\varphi_2(s)e^{-r}\right), \quad \forall s \ge 0,\ \forall r \ge 0. \tag{3.22}$$

Theorem 3.8. *Considering system* (2.1)*, the following is equivalent:*

(a) *there exists a $\mathcal{W}$-Lyapunov function;*

(b) *there exists a weak $\mathcal{W}$-Lyapunov function;*

(c) *there is $\mathcal{W}$-asymptotically stable.*

Proof. (a)⇒(b). Let $h(x(k)) = \alpha_3(\mathcal{W}(x(k,x_0)))$. Because α_3 is a class $\mathcal{K}$ function, $h(x(k))$ is continuous and inequality (3.15) is satisfied. Furthermore, when $h(x(k,x_0)) \equiv 0$, we have

$$\|x(k,x_0)\|_{\mathcal{W}} \longrightarrow 0, \quad k \longrightarrow +\infty. \tag{3.23}$$

The Property (iii) of Definition 3.5 holds. Thus (a) implies (b).

(b)⇒(c). The proof is given in Theorem 3.6.

(c)⇒(a). Assume that system (2.1) is $\mathcal{W}$-asymptotically stable. According to Lemma 3.3, there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function μ such that for any $x_0 \in \mathbf{R}^n$

$$\|x(k,x_0)\|_{\mathcal{W}} \le \beta(\|x_0\|_{\mathcal{W}} + R, k), \quad \forall k \in \mathbf{Z}_+. \tag{3.24}$$

By Lemma 3.1, there exist two class $\mathcal{K}_\infty$ functions φ_1 and φ_2 such that

$$\|x(k,x_0)\|_{\mathcal{W}} \le \beta(\|x_0\|_{\mathcal{W}} + R, k) \le \varphi_1\left(\varphi_2(\|x_0\|_{\mathcal{W}} + R)e^{-k}\right). \tag{3.25}$$

Let $\theta = \varphi_1^{-1}$. We have

$$\theta(\|x(k,x_0)\|_{\mathcal{W}}) \le \varphi_2(\|x_0\|_{\mathcal{W}} + R)e^{-k}. \tag{3.26}$$

Define $\mathcal{W}(x(k,x_0)) = \sum_{k=0}^{\infty}\theta(\|x(k+k_0,x_0)\|_{\mathcal{W}})$. Clearly, $\mathcal{W}(x(k,x_0))$ is continuous since φ_1 is a class $\mathcal{K}_\infty$ function. By the definition of $\mathcal{W}$, it yields

$$\theta(\|x_0\|_{\mathcal{W}}) \le \mathcal{W}(x(k,x_0)) \le \sum_{k=0}^{\infty}\varphi_2(\|x_0\|_{\mathcal{W}} + R)e^{-k} \le \frac{e}{e-1}\varphi_2(\|x_0\|_{\mathcal{W}} + R). \tag{3.27}$$

$\mathcal{W}$ satisfies the property of inequality (3.11) due to inequality (3.27).

In the following we show $\mathcal{W}(x(k))$ satisfies the property as in inequality (3.12). Arbitrarily choose $k_0 \in \mathbf{Z}_+$ and $x_0 = x(k_0) \in \mathbf{R}^n$. Then $x(k_0+1, x(k_0)) = f(x(k_0, x(k_0)))$ and

$$\mathcal{W}(x(k_0+1, x(k_0+1))) = \sum_{k=0}^{\infty} \theta(\|x(k+k_0+1, x(k_0+1))\|_{\mathcal{W}}). \tag{3.28}$$

Since function $f(x(k))$ is Lipschitz continuous, the solution of system (2.1) is unique for arbitrary initial value. Then we have

$$x(k+k_0+1, x(k_0+1)) = x(k+k_0+1, x(k_0)), \quad \forall k \in \mathbf{Z}_+. \tag{3.29}$$

Considering (3.28), we can get

$$\begin{aligned} \mathcal{W}(x(k_0+1, x(k_0+1))) &= \sum_{k=0}^{\infty} \theta(\|x(k+k_0+1, x(k_0+1))\|_{\mathcal{W}}) \\ &= \sum_{k=0}^{\infty} \theta(\|x(k+k_0+1, x(k_0))\|_{\mathcal{W}}) \\ &= \sum_{k=1}^{\infty} \theta(\|x(k+k_0, x(k_0))\|_{\mathcal{W}}) \\ &= \sum_{k=0}^{\infty} \theta(\|x(k+k_0, x(k_0))\|_{\mathcal{W}}) - \theta(\|x(k_0, x(k_0))\|_{\mathcal{W}}) \\ &\leq \mathcal{W}(x(k_0, x(k_0))) - \theta(\|x(k_0, x(k_0))\|_{\mathcal{W}}). \end{aligned} \tag{3.30}$$

That is,

$$\mathcal{W}(x(k_0+1, x(k_0, x(k_0)))) - \mathcal{W}(x(k_0, x(k_0))) \leq -\theta(\|x_0\|_{\mathcal{W}}). \tag{3.31}$$

Due to the arbitrariness of k_0, we have

$$\mathcal{W}(f(x(k, x_0))) - \mathcal{W}(x(k, x_0)) \leq -\theta(\|x_0\|_{\mathcal{W}}). \tag{3.32}$$

Function $\mathcal{W}$ satisfies the property as in inequality (3.12). □

4. Conclusion

We conclude with a brief discussion. The notion of $\mathcal{W}$-stability introduced by Efimov is different from conventional notion of stability. It is required to consider the set of all invariant solutions of systems. However, the set of all invariant solutions can contain not only stable invariant solutions but also unstable invariant solutions. If it does not contain unstable invariant solutions, $\mathcal{W}$-stability is conventional stability. Thus our results should have an more extensive application than those corresponding results in the sense of conventional stability.

Acknowledgment

This work was supported by the Natural Science Foundation of China under the Contract no. 60874006.

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