Eur. Phys. J. C (2016) 76:582
DOI 10.1140/epjc/s10052-016-4414-0

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

(m, n)-String in (p, q)-string and (p, q)-five-brane background

Josef Klusoň[a]

Department of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

Received: 21 July 2016 / Accepted: 3 October 2016 / Published online: 27 October 2016

Abstract We study dynamics of (m, n)-string in (p, q)-five-brane and (p, q)-string background. We determine world-volume stress energy tensor and we analyze the dependence of the string's dynamics on the values of the charges (m, n) and the value of the angular momentum.

1 Introduction and summary

Low energy effective actions of superstring theories have reached a spectrum of solutions that preserve some fractions of supersymmetry; for a review see for example [1–4]. These objects have the property that they are sources of various form fields that are presented in supergravity theories. Further, fundamental string, D-brane, and NS5-brane solutions preserve one half of the space-time supersymmetries and can be considered as the building block of other solutions. For example, taking the intersection of these configurations we get backgrounds that preserve some fractions of supersymmetry [5]. Another possibility is to generate new solutions using the U-duality symmetry of M-theory (for a review see for example [6]), which is basically the symmetry of M-theory on its maximally supersymmetric toroidal compactifications. For example, M-theory compactified on a two torus possesses the $SL(2, Z)$ symmetry, which turns out to be the non-perturbative $SL(2, Z)$ duality of type IIB theory. More precisely, it is well known that the low effective action of type IIB supergravity written in an Einstein frame is invariant under $SL(2, R)$ duality. A special case of $SL(2, R)$ transformation is the S-duality transformation that roughly speaking transforms the theory at weak coupling to strong coupling. The fact that the type IIB supergravity action is invariant under this symmetry suggests the possibility to generate new supergravity solutions when we apply a $SL(2, R)$ rotation on known supergravity solutions, as for example fundamental string or NS5-brane backgrounds. Such a procedure was first used in a famous paper [7] where the manifestly $SL(2, R)$ covariant supergravity solution corresponding to a (p, q)-string was found. The extension of this analysis to the case of an NS5-brane was performed in [8] when the $SL(2, Z)$ covariant expression for supergravity solutions corresponding to the (p, q)-five brane was derived.[1] These backgrounds are very interesting and certainly deserve to be studied further. In particular, it is well known that the continuous classical symmetry group $SL(2, R)$ of type IIB supergravity cannot be a symmetry of the full string theory when non-perturbative effects break it to a discrete subgroup $SL(2, Z)$. To see this more clearly, note that the fundamental string carries one unit of NSNS two-form charge and hence this charge has to be quantized in integer units. On the other hand $SL(2, R)$ transformations map a fundamental string into a string with d units of this charge where d is an entry of the $SL(2, R)$ matrix. From this result we conclude that d has to be integer. In a similar way we can argue that the $SL(2, R)$ symmetry of the low energy effective action has to be broken to its $SL(2, Z)$ subgroup when a fundamental string is mapped under this duality to a (p, q)-string that carries charge p of NSNS two-form and charge q of the Ramond–Ramond two-form [9]. It was also shown in [9] that the type IIB string effective action together with the (p, q)-string action is covariant under $SL(2, R)$ transformations. However, the fact that the (p, q) string has to map to another (p', q')-string where p', q' are integers suggests that the full symmetry group of the combined action breaks to $SL(2, Z)$. On the other hand, solutions found in [7,8] were determined using the $SL(2, R)$ matrices so that it is interesting to analyze the problem of an (m, n)-string probe in such a background and this is precisely the aim of this paper.

We begin with the D1-brane action that we rewrite into a manifestly covariant $SL(2, Z)$ form; for a related analysis see [10] and for a very elegant formulation of the manifestly $SL(2, Z)$ covariant superstring, see [11,12]. Now using the

[a] e-mail: klu@physics.muni.cz

[1] For a non-extremal form of these solutions, see [17].

fact that (p, q)-five and fundamental string solutions were derived using $SL(2, R)$ transformations we can map the problem of the dynamics of the (m, n)-string in this background to the problem of the analysis of the (m', n')-string in the original NS5-brane and fundamental string background with the crucial exception that the harmonic functions that define these solutions have constant factors that differ from the factors that define NS5-brane and fundamental string solutions. It is also important to stress that now (m', n') are not integers but depend on p, q and also on asymptotic values of the dilaton and Ramond–Ramond zero-form. We think that this is not a quite satisfactory resort and one can ask the questions whether it would be possible to find (p, q)-string and five-brane backgrounds that are derived from the NS5-brane and fundamental string background through manifest $SL(2, Z)$ transformations when the probe (m, n)-string will transform in an appropriate way. This problem is currently under study and we return to it in the near future. We rather focus on the dynamics of the probe (m, n)-string in the backgrounds [7,8], following the very nice analysis introduced in [14]. Using a manifest $SL(2, Z)$ covariant formulation of a probe (m, n)-string we can analyze the time evolution of the homogeneous time-dependent string in a given background. We determine the components of the world-sheet stress energy tensor and study its time evolution. The properties of this stress energy tensor and the dynamics of the probe depend on the values of m, n and hence our results can be considered as a generalization of the analysis performed in [14].

As the next step we analyze the dynamics of the probe (m, n)-string in the background of (p, q)-macroscopic string. Thanks to the form of the solution [7] we formulate this problem as the analysis of the dynamics of (m', n')-string in the background of fundamental string. This problem was studied previously in [15] but we focus on a different aspect of the dynamics of the probe. Explicitly we will be interested in the behavior of the probe where the difference between its energy and the rest energy is small. We find that the potential is flat, which is in agreement with the fact that the string probe in the fundamental string background can form a marginal bound state with the strings that are sources of this background. We also analyze the situation with a non-zero angular momentum and we find that there is a potential barrier that does not allow the probe string to move towards to the horizon. These results are in agreement with the analysis performed in [15].

The organization of this paper is as follows. In the next section (Sect. 2) we review $SL(2, R)$ duality of the type IIB low energy effective action. We also introduce a manifestly $SL(2, R)$ covariant action for (m, n)-string. In Sect. 3 we study the dynamics of this string in the background of a (p, q)-five brane. Finally in Sect. 4 we study the dynamics of the (m, n)-string in the background of a (p, q)-string.

2 $SL(2, R)$-Covariance of type IIB low energy effective action

The type IIB theory has two three-form field strengths $H = \mathrm{d}B$, $F = \mathrm{d}C^{(2)}$, where H corresponds to the NSNS three-form, while F belongs to the RR sector and does not couple to the usual string world-sheet. Type IIB theory has also two scalar fields, which can be combined into a complex field $\tau = \chi + \mathrm{i}e^{-\Phi}$. The dilaton Φ is in the NSNS sector, while χ belongs to the RR sector. The other Bose fields are the metric $g_{\mu\nu}$ and the self-dual five-form field strength F_5, which we set zero in this paper. Then it is possible to write down a covariant form of the bosonic part of type IIB effective action,

$$S_{\text{IIB}} = \frac{1}{2\tilde{\kappa}_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g}\bigg(R + \frac{1}{4}\mathrm{Tr}(\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}) - \frac{1}{12}\mathbf{H}^T_{\mu\nu\sigma}\mathcal{M}\mathbf{H}^{\mu\nu\sigma}\bigg), \tag{1}$$

where $\tilde{\kappa}_{10}^2 = \frac{1}{4\pi}(4\pi^2\alpha')^4$ and where we have combined $B, C^{(2)}$ into

$$\mathbf{H} = \mathrm{d}\mathbf{B} = \begin{pmatrix} \mathrm{d}B \\ \mathrm{d}C^{(2)} \end{pmatrix}, \tag{2}$$

and where

$$\mathcal{M} = e^{\Phi}\begin{pmatrix} \tau\tau^* & \chi \\ \chi & 1 \end{pmatrix} = e^{\Phi}\begin{pmatrix} \chi^2 + e^{-2\Phi} & \chi \\ \chi & 1 \end{pmatrix}. \tag{3}$$

The action (1) has manifest invariance under the global $SL(2, R)$ transformation

$$\hat{\mathcal{M}} = \Lambda\mathcal{M}\Lambda^T, \hat{\mathbf{B}} = (\Lambda^T)^{-1}\mathbf{B}, \tag{4}$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5}$$

It is well known that all string theories contain the fundamental string and the magnetic dual NS5-brane as solutions of the equations of motion of its low energy effective actions. Using the manifest $SL(2, R)$ covariance of type IIB effective action it is possible to derive solutions corresponding to the (p, q)-five brane [8] and fundamental string [7]. It will be certainly interesting to analyze the properties of a given background with the help of the appropriate probe, which will be a probe (m, n)-string. For that reason we introduce a manifestly covariant form of the (m, n)-string action.

2.1 (m, n)-String action

In this section we formulate the action for the (m, n)-string. Even if such a formulation is well known [9–13] we derive this action in a slightly different way with the help of the Hamiltonian formalism which will also be useful for the analysis of the dynamics of the probe (m,n)-string in (p,q)-five and (p,q)-string background.

To begin with we introduce an action for n coincident D1-branes in a general background,

$$
\begin{aligned}
S &= -nT_{D1}\int d\tau d\sigma e^{-\Phi}\sqrt{-\det\mathbf{A}} \\
&\quad + nT_{D1}\int d\tau d\sigma((b_{\tau\sigma}+2\pi\alpha'\mathcal{F}_{\tau\sigma})\chi + c_{\tau\sigma}), \\
\mathbf{A}_{\alpha\beta} &= G_{MN}\partial_\alpha x^M\partial_\beta x^N + 2\pi\alpha'\mathcal{F}_{\alpha\beta} + B_{MN}\partial_\alpha x^M\partial_\beta x^N, \\
\mathcal{F}_{\alpha\beta} &= \partial_\alpha A_\beta - \partial_\beta A_\alpha,
\end{aligned}
\tag{6}
$$

where $x^M, M, N = 0, 1, \ldots, 9$ are embedding coordinates of the D1-brane in the background that is specified by the metric G_{MN} and NSNS two-form $B_{MN} = -B_{NM}$ together with Ramond–Ramond two-form $C^{(2)}_{MN} = -C^{(2)}_{NM}$. Note that we use capital letters G_{MN} for the string frame metric, while g_{MN} corresponds to the Einstein-frame metric. We further consider a background with a non-trivial dilaton Φ and RR zero-form χ. Further, $\sigma^\alpha = (\tau,\sigma)$ are world-sheet coordinates and $b_{\tau\sigma}, c_{\tau\sigma}$ are pull-backs of B_{MN} and C_{MN} to the world-volume of D1-brane. Explicitly,

$$
b_{\alpha\beta} \equiv B_{MN}\partial_\alpha x^M\partial_\beta x^N, \quad c_{\tau\sigma} = C^{(2)}_{MN}\partial_\tau x^M\partial_\sigma x^N. \tag{7}
$$

Finally, $T_{D1} = \frac{1}{2\pi\alpha'}$ is the D1-brane tension and $A_\alpha, \alpha = \tau,\sigma$ is a two dimensional gauge field that propagates on the world-sheet of the D1-brane.

It is useful to rewrite the action (6) in the form

$$
\begin{aligned}
S &= -nT_{D1}\int d\tau d\sigma e^{-\Phi}\sqrt{-\det g - (2\pi\alpha'\mathcal{F}_{\tau\sigma}+b_{\tau\sigma})^2} \\
&\quad + nT_{D1}\int d\tau d\sigma((b_{\tau\sigma}+2\pi\alpha'\mathcal{F}_{\tau\sigma})\chi + c_{\tau\sigma}),
\end{aligned}
\tag{8}
$$

where $g_{\alpha\beta} = G_{MN}\partial_\alpha x^M\partial_\beta x^N, \det g = g_{\tau\tau}g_{\sigma\sigma} - (g_{\tau\sigma})^2$. Now we proceed to the Hamiltonian formulation of the theory defined by the action (8). First of all we derive the conjugate momenta to x^M and A_α from (8)

$$
\begin{aligned}
p_M &= \frac{\delta L}{\delta\partial_\tau x^M} = nT_{D1}\frac{e^{-\Phi}}{\sqrt{-\det g - (2\pi\alpha' F_{\tau\sigma}+b_{\tau\sigma})^2}} \\
&\quad \times(G_{MN}\partial_\alpha x^N g^{\alpha\tau}\det g + (2\pi\alpha' F_{\tau\sigma}+b_{\tau\sigma})B_{MN}\partial_\sigma x^N) \\
&\quad + nT_{D1}(\chi B_{MN}\partial_\sigma x^N + C^{(2)}_{MN}\partial_\sigma x^N), \\
\pi^\sigma &= \frac{\delta L}{\delta\partial_\tau A_\sigma} = \frac{ne^{-\Phi}(2\pi\alpha' F_{\tau\sigma}+b_{\tau\sigma})}{\sqrt{-\det g - (2\pi\alpha' F_{\tau\sigma}+b_{\tau\sigma})^2}} + n\chi\ , \\
\pi^\tau &= \frac{\delta L}{\delta\partial_\tau A_\tau} \approx 0,
\end{aligned}
\tag{9}
$$

and hence

$$
\begin{aligned}
\Pi_M &\equiv p_M - \frac{\pi^\sigma}{(2\pi\alpha')}B_{MN}\partial_\sigma x^N - nT_{D1}C^{(2)}_{MN}\partial_\sigma x^N) = \\
&= nT_{D1}\frac{e^{-\Phi}}{\sqrt{-\det g - (2\pi\alpha' F_{\tau\sigma}+b_{\tau\sigma})^2}}G_{MN}\partial_\alpha x^N g^{\alpha\tau}\det g.
\end{aligned}
\tag{10}
$$

Using these relations it is easy to see that the bare Hamiltonian is equal to

$$
H_B = \int d\sigma(p_M\partial_\tau x^M + \pi^\sigma\partial_\tau A_\sigma - \mathcal{L}) = \int d\sigma\pi^\sigma\partial_\sigma A_\tau, \tag{11}
$$

while we have three primary constraints,

$$
\begin{aligned}
&\pi^\tau \approx 0, \quad \mathcal{H}_\sigma \equiv p_M\partial_\sigma x^M \approx 0, \\
&\mathcal{H}_\tau \equiv \frac{1}{T_{D1}}\Pi_M G^{MN}\Pi_N \\
&\quad + T_{D1}\left(n^2 e^{-2\Phi} + \left(\pi^\sigma - n\chi\right)^2\right)g_{\sigma\sigma} \approx 0.
\end{aligned}
\tag{12}
$$

Including these primary constraints in the definition of the Hamiltonian we obtain an extended Hamiltonian in the form

$$
H = \int d\sigma(\lambda_\tau\mathcal{H}_\tau + \lambda_\sigma\mathcal{H}_\sigma - A_\tau\partial_\sigma\pi^\sigma + v_\tau\pi^\tau), \tag{13}
$$

where $\lambda_\tau, \lambda_\sigma, v_\tau$ are Lagrange multipliers corresponding to the primary constraints $\mathcal{H}_\tau \approx 0$, $\mathcal{H}_\sigma \approx 0, \pi^\tau \approx 0$. Now we have to check the stability of all constraints. The requirement of the preservation of the primary constraint $\pi^\tau \approx 0$ implies the secondary constraint,

$$
\mathcal{G} = \partial_\sigma\pi^\sigma \approx 0. \tag{14}
$$

In the case of the constraints $\mathcal{H}_\tau, \mathcal{H}_\sigma$ we can easily show in the same way as in [16] that the constraints $\mathcal{H}_\tau, \mathcal{H}_\sigma$ are first class constraints and hence they are preserved during the time evolution.

An action for (m,n)-string is derived when we fix the gauge generated by $\mathcal{G}$ with the gauge fixing function $A_\sigma =$ const. Then the fixing of the gauge implies that $\pi^\sigma = f(\tau)$, but the equation of motion for π^σ implies that $\partial_\tau\pi^\sigma = 0$ and hence $\pi^\sigma = m$, where m is an integer that counts the number of fundamental strings bound to n D1-branes. After this partial gauge fixing the Hamiltonian density has the form

$$
\mathcal{H}_{(m,n)} = \int d\sigma(\lambda_\tau\mathcal{H}_\tau + \lambda_\sigma\mathcal{H}_\sigma). \tag{15}
$$

In order to find the (m,n)-string action we derive the Lagrangian density corresponding to the Hamiltonian (15). Explicitly, from (15) we obtain equations of motion for x^M

$$\partial_\tau x^M = \left\{x^M, H_{(m,n)}\right\} = 2\lambda_\tau \frac{1}{T_{D1}} G^{MN}\Pi_N + \lambda_\sigma \partial_\sigma x^M \tag{16}$$

and hence

$$\begin{aligned}\mathcal{L}_{(m,n)} &= p_M \partial_\tau x^M - \mathcal{H}_{(m,n)} \\ &= \frac{1}{2\pi\alpha'}\left(\frac{1}{4\lambda_\tau}(g_{\tau\tau} - 2\lambda_\sigma g_{\tau\sigma} + \lambda_\sigma^2 g_{\sigma\sigma}) \right. \\ &\quad - \lambda_\tau (n^2 \mathrm{e}^{-2\Phi} \\ &\quad \left. +(m-n\chi)^2) g_{\sigma\sigma} + m b_{\tau\sigma} + n c_{\tau\sigma}\right).\end{aligned} \tag{17}$$

As the final step we solve the equations of motion for λ_τ and λ_σ, which follow from (17), and we obtain

$$\lambda_\sigma = \frac{g_{\tau\sigma}}{g_{\sigma\sigma}}, \quad \lambda_\tau = \frac{1}{2g_{\sigma\sigma}\sqrt{n^2\mathrm{e}^{-2\Phi} + (m-n\chi)^2}}\sqrt{-\det g}. \tag{18}$$

Inserting this result into the Lagrangian density (17) we obtain the action in manifestly covariant $SL(2,R)$ form,

$$\begin{aligned}S = &-T_{D1}\int \mathrm{d}\tau \mathrm{d}\sigma (\sqrt{\mathbf{m}^T \mathcal{M}^{-1}\mathbf{m}}\sqrt{-\det g_{MN}\partial_\alpha x^M \partial_\beta x^N} \\ &+ T_{D1}\int \mathrm{d}\tau\mathrm{d}\sigma \mathbf{m}^T \mathbf{B}_{MN}\partial_\tau x^M \partial_\sigma x^N,\end{aligned} \tag{19}$$

where

$$\mathbf{m} = \begin{pmatrix} m \\ n\end{pmatrix}, \mathbf{B}_{MN} = \begin{pmatrix} B_{MN} \\ C^{(2)}_{MN}\end{pmatrix}, \tag{20}$$

and where $g_{MN} = \mathrm{e}^{-\Phi/2}G_{MN}$ is an Einstein-frame metric. Since $\hat{\mathbf{B}} = (\Lambda^T)^{-1}\mathbf{B}$ we see that $\mathbf{m}$ transforms as

$$\hat{\mathbf{m}} = \Lambda \mathbf{m} \tag{21}$$

in order for the action (19) to be manifestly $SL(2,R)$ covariant. On the other hand, since m, n count the number of fundamental strings and D1-branes and hence have to be integers, we find that the non-perturbative duality group of type IIB superstring theory is $SL(2,Z)$, which will have an important consequence for the analysis of the dynamics of (m,n)-string in (p,q)-five-brane and (p,q)-fundamental string background.

3 (m,n)-string in the background of (p,q)-five brane

We would like to analyze the dynamics of the (m,n)-string in the background of a (p,q)-five brane that has the form [8]

$$\begin{aligned}\mathrm{d}s_E^2 &= (1 + \frac{Q_{(p,q)}}{r^2})^{-1/4}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &\quad +(1+\frac{Q_{(p,q)}}{r^2})^{3/4}\mathrm{d}x^m \mathrm{d}x^m, \\ \lambda &= \chi + i\mathrm{e}^{-\Phi} \\ &= \frac{\chi_0 \triangle_{(p,q)}A_{(p,q)} + pq\mathrm{e}^{-\Phi_0}(A_{(p,q)}-1) + i\triangle_{(p,q)}A^{1/2}_{(p,q)}\mathrm{e}^{-\Phi_0}}{p^2\mathrm{e}^{-\Phi_0} + A_{(p,q)}\mathrm{e}^{\Phi_0}(\chi_0 p + q)^2}, \\ H &= \mathrm{d}B = 2p(2\pi\alpha')^2\epsilon_3, \\ F &= \mathrm{d}C_2 = 2q(2\pi\alpha')^2\epsilon_3,\end{aligned} \tag{22}$$

where

$$Q_{(p,q)} = \sqrt{\triangle_{(p,q)}}2\pi\alpha' = \sqrt{\mathrm{e}^{-\Phi_0}p^2 + (q+p\chi_0)^2\mathrm{e}^{\Phi_0}}2\pi\alpha', \tag{23}$$

and where ϵ_3 is the volume form of the three sphere when we express the line element of the transverse space $\mathrm{d}x_m\mathrm{d}x^m$ as $\mathrm{d}x_m \mathrm{d}x^m = \mathrm{d}r^2 + r^2\mathrm{d}\Omega_3$. Note also that $x^\mu, \mu = 0,\ldots,5$ labels the directions along the world-volume of the (p,q)-five brane. Further $A_{(p,q)}$ is defined as

$$A_{(p,q)} = \left(1 + \frac{Q_{(p,q)}}{r^2}\right)^{-1}, \tag{24}$$

and $\mathrm{d}s_E^2$ means that this line element is expressed in an Einstein-frame metric. Let us now consider the probe (m,n)-string action (19) in a given background. The analysis of this problem simplifies considerably when we realize how the solution (22) was determined. Following [8] and [7] we introduce the $SL(2,R)$ matrix

$$\Lambda = \triangle_{(p,q)}^{-1/2}\begin{pmatrix} \mathrm{e}^{-\Phi_0}p + \chi_0 \mathrm{e}^{\Phi_0}(q+p\chi_0) & -(q+p\chi_0)+\chi_0 p \\ \mathrm{e}^{\Phi_0}(q+p\chi_0) & p\end{pmatrix}, \tag{25}$$

where

$$\triangle_{(p,q)} = \mathrm{e}^{-\Phi_0}p^2 + (q+p\chi_0)^2\mathrm{e}^{\Phi_0}, \tag{26}$$

and where χ_0 and Φ_0 are asymptotic values of the fields Φ and χ. Note that the inverse matrix has the form

$$\Lambda^{-1} = \triangle_{(p,q)}^{-1/2}\begin{pmatrix} p & q \\ -\mathrm{e}^{\Phi_0}(q+p\chi_0) & \mathrm{e}^{-\Phi_0}p + \chi_0\mathrm{e}^{\Phi_0}(q+p\chi_0)\end{pmatrix}. \tag{27}$$

Now with the help of this matrix we can write $\mathcal{M}$ as [8]

$$\mathcal{M} = \Lambda(p,q)\begin{pmatrix} \sqrt{A_{(p,q)}} & 0 \\ 0 & \frac{1}{\sqrt{A_{(p,q)}}}\end{pmatrix}\Lambda^T(p,q), \tag{28}$$

so that

$$\mathbf{m}^T \mathcal{M}^{-1}\mathbf{m} = m'^2 \frac{1}{\sqrt{A_{(p,q)}}} + n'^2 \sqrt{A_{(p,q)}} , \tag{29}$$

where

$$\mathbf{m}' = \begin{pmatrix} m' \\ n' \end{pmatrix} = \Lambda^{-1}(p,q)$$
$$\mathbf{m} = \triangle_{(p,q)}^{-1/2} \begin{pmatrix} pm + qn \\ \mathrm{e}^{\Phi_0}(q + p\chi_0)(-m + n\chi_0) + \mathrm{e}^{-\Phi_0} pn \end{pmatrix}. \tag{30}$$

It is interesting that, for the special values of m, n equal to

$$m = -q, n = p, \tag{31}$$

we obtain

$$\mathbf{m}' = \triangle^{-1/2} \begin{pmatrix} 0 \\ \mathrm{e}^{\Phi_0}(q + p\chi_0)^2 + \mathrm{e}^{-\Phi_0} p^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \triangle_{(p,q)}^{1/2} \end{pmatrix}. \tag{32}$$

Since $m' = 0$ we can interpret this configuration as a pure D1-brane which, however, does not have integer charge. We also see from (30) that in order to find a configuration with $n' = 0$ we have to require that $\Phi_0 = 0 = \chi_0$ and set $m = p, n = q$,

$$\mathbf{m}' = \triangle^{-1/2} \begin{pmatrix} p^2 + q^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{p^2 + q^2} \\ 0 \end{pmatrix}. \tag{33}$$

Generally we see that the action for the probe (m, n)-string in a (p, q)-five-brane background is equivalent to the action of (m', n')-string in an NS5-brane background with the important exception that the harmonic function has the factor $Q_{(p,q)}$ (23) instead of the standard one, which corresponds to the number of NS5-branes. Note also that m', n' depend on m, n, p, q and the moduli Φ_0 and χ_0 as follows from (30). The crucial point, however, is that m', n' are not integers, which suggests inconsistency with the (p, q)-five-brane background. We come to this important observation later.

Let us now return to the analysis of the dynamics of the probe (m, n)-string in this background. It is convenient to impose the static gauge

$$x^0 = \tau, \quad x^1 = \sigma \tag{34}$$

and introduce spherical coordinates in the transverse space $\mathbf{R}^4$

$$x^1 = r \cos\psi, x_1 = r \sin\psi \cos\theta,$$
$$x^3 = r \sin\psi \sin\theta \cos\phi, x^4 = r \sin\psi \sin\theta \sin\phi, \tag{35}$$

so that the volume element of Ω_3 is equal to

$$\mathrm{d}\Omega_3 = \sin^2\psi \sin\theta \mathrm{d}\psi \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{36}$$

Using these equations we see that we have the following components of the RR and NSNS two-forms:

$$B_{\psi\phi} = 2p(2\pi\alpha')^2 \sin^2\phi \cos\theta,$$
$$C^{(2)}_{\psi\phi} = 2q(2\pi\alpha')^2 \sin^2\psi \cos\theta. \tag{37}$$

Now we would like to derive the components of the stress energy tensor $T_{\alpha\beta}$ for the gauge fixed theory. To do this we temporarily replace the fixed two dimensional metric $\eta_{\alpha\beta}$ with two dimensional metric $\gamma_{\alpha\beta}$ and write the gauge fixed action in the form

$$S_{\text{fixed}} = -T_{D1} \int \mathrm{d}\tau \mathrm{d}\sigma (\sqrt{\mathbf{m}^T \mathcal{M}^{-1}\mathbf{m}} A_{(p,q)}^{1/4} \sqrt{-\det \mathbf{A}_{\alpha\beta}} + S_{WZ}, \tag{38}$$

where

$$\mathbf{A}_{\alpha\beta} = \gamma_{\alpha\beta} + \frac{1}{A_{(p,q)}} \delta_{mn} \partial_\alpha x^m \partial_\beta x^n + \delta_{\alpha\beta} \partial_\alpha x^\alpha \partial_\beta x^\beta, \tag{39}$$

where $x^\alpha, \alpha = 1, \ldots, 5$ labels the coordinates along the world-volume of the (p, q)-five brane. Then we define the components of the two dimensional stress energy tensor as

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-\det\gamma}} \frac{\delta S_{\text{fixed}}}{\delta\gamma^{\alpha\beta}}$$
$$= -\frac{T_{D1}}{\sqrt{-\det\gamma}} \gamma_{\alpha\gamma} (\mathbf{A}^{-1})^{\gamma\delta} \gamma_{\delta\beta} \sqrt{-\det\mathbf{A}} \sqrt{\mathbf{m}^T \mathcal{M}^{-1}\mathbf{m}} A_{(p,q)}^{1/4}. \tag{40}$$

Now we return back to the flat metric $\gamma_{\alpha\beta} \to \eta_{\alpha\beta}$ and consider pure time-dependent ansatz. As a result we obtain the following components of the world-sheet stress energy tensor:

$$T_{\tau\tau} = \frac{T_{D1} \sqrt{\mathbf{m}^T \mathcal{M}^{-1}\mathbf{m}} A_{(p,q)}^{1/4}}{\sqrt{1 - \frac{1}{A_{(p,q)}} \partial_\tau x^m \partial_\tau x_m - \partial_\tau x^\alpha \partial_\tau x_\alpha}}, \quad T_{\tau\sigma} = 0,$$
$$T_{\sigma\sigma} = -T_{D1} \sqrt{\mathbf{m}^T \mathcal{M}^{-1}\mathbf{m}} A_{(p,q)}^{1/4} \times \sqrt{1 - \frac{1}{A_{(p,q)}} \partial_\tau x^m \partial_\tau x_m - \partial_\tau x^\alpha \partial_\tau x_\alpha}, \tag{41}$$

which are a generalization of the components of the stress energy tensor of a Dp-brane moving in an NS5-brane background as found in [14].

3.1 Gauge fixing in Hamiltonian formalism

Now we proceed to the analysis of dynamics of the probe (m, n)-string in (p, q)-five-brane background. It turns out that it is useful to perform this analysis in the canonical approach when we impose the static gauge using the two gauge fixing functions

$$\mathcal{G}_\tau = x^0 - \tau \approx 0, \quad \mathcal{G}_\sigma = x^1 - \sigma \approx 0. \tag{42}$$

These constraints have non-zero Poisson brackets with $\mathcal{H}_\tau \approx 0, \mathcal{H}_\sigma \approx 0$ so that they are the second class constraints. As a result $\mathcal{H}_\tau, \mathcal{H}_\sigma$ vanish strongly and can be solved for p_0 and p_1, respectively, where we can relate $-p_0$ with the Hamiltonian density of gauge fixed theory $\mathcal{H}_{\text{fix}}$. To see this note that the action has the form

$$S = \int d\tau d\sigma (p_M \partial_\tau x^M - \mathcal{H}) = \int d\tau d\sigma (p_i \partial_\tau x^i + p_0)$$
$$= \int d\tau d\sigma (p_i \partial_\tau x^i - \mathcal{H}_{\text{fix}}). \tag{43}$$

Now from $\mathcal{H}_\sigma = 0$ we obtain $p_1 = -(p_i \partial_\sigma x^i)$ and from $\mathcal{H}_\tau$ we find

$$\mathcal{H}_{\text{fix}} = \sqrt{-g_{00}\left(\Pi_1 g^{11}\Pi_1 + \Pi_i g^{ij}\Pi_j + T_{D1}^2(m'^2 e^{\Phi} + n'^2 e^{-\Phi})(g_{11} + g_{ij}\partial_\sigma x^i \partial_\sigma x^j)\right)} - \frac{1}{2\pi\alpha'}\mathbf{m}'^T \mathbf{B}_{0M}\partial_\sigma x^M, \tag{44}$$

where $i, j = 2, \ldots, 9$. The analysis simplifies further when we presume that the embedding modes depend on τ only so that the Hamiltonian density (44) reduces to

$$\mathcal{H}_{\text{fix}}^2 = A_{(p,q)}^{1/4}\left(p^\alpha p_\alpha A_{(p,q)}^{-1/4} + A_{(p,q)}^{3/4}\left(p_r^2 + \frac{1}{r^2}p_\psi^2 + \frac{1}{r^2\sin^2\psi}p_\theta^2 + \frac{1}{r^2\sin^2\psi\sin^2\theta}p_\phi^2\right) + T_{D1}^2(m'^2 + n'^2 A_{(p,q)})A_{(p,q)}^{-1/4}\right) \equiv \mathcal{K}, \tag{45}$$

where $p_\alpha, \alpha = 2, 3, 4, 5$ denotes the momenta along the world-volume of the (p, q)-five branes. Since they are conserved we restrict ourselves to the case when $p_\alpha = 0$. At the same time we find that p_ϕ is conserved as well and we denote this constant as $p_\phi = L$. On the other hand the equations of motion for θ, p_θ have the form

$$\dot{\theta} = \{\theta, H_{\text{fix}}\} = \frac{A_{(p,q)}p_\theta}{r^2\sin^2\psi\sqrt{\mathcal{K}}},$$
$$\dot{p}_\theta = \{p_\theta, H_{\text{fix}}\} = \frac{A_{(p,q)}\sin\theta\cos\theta}{r^2\sin^2\psi\sin^3\theta\sqrt{\mathcal{K}}}p_\psi^2. \tag{46}$$

We see that this equation has a solution when $\theta = \frac{\pi}{2}$ and $p_\theta = 0$. In the same way we find that $p_\psi = 0$, $\psi = \frac{\pi}{2}$ solve the equations of motion. Finally we proceed to the analysis of the time evolution of r. The equation of motion for r gives

$$\dot{r} = \{r, H_{\text{fix}}\} = \frac{A_{(p,q)}p_r}{\sqrt{\mathcal{K}}}. \tag{47}$$

To proceed, we use the fact that the Hamiltonian density $\mathcal{H}_{\text{fix}}$ is conserved and we denote its constant value by E. Then we can solve $\mathcal{H}_{\text{fix}} = E$ for p_r as

$$p_r = \sqrt{\frac{E^2 - \frac{A_{(p,q)}L^2}{r^2} - T_{D1}^2(m'^2 + n'^2 A_{(p,q)})}{A_{(p,q)}}}, \tag{48}$$

so that from (47) we obtain

$$\dot{r}^2 = A_{(p,q)} - \frac{A_{(p,q)}^2}{E^2}\left(\frac{L^2}{r^2} + T_{D1}^2 n'^2\right) - \frac{A_{(p,q)}T_{D1}^2}{E^2}m'^2. \tag{49}$$

As a check note that the first two terms on the right side in (49) coincide with the expression that governs the dynamic of the Dp-brane in an NS5-brane background [14] while the last one, which is proportional to m'^2, corresponds to the dynamics of the fundamental string in this background. For the next purposes we also determine the equation of motion for ϕ

$$\dot{\phi} = \{\phi, H_{\text{fix}}\} = \frac{A_{(p,q)}L}{r^2 E}. \tag{50}$$

3.2 The Case $L = 0$

We first of all consider the case of the vanishing angular momentum $p_\theta = L = 0$. Then Eq. (50) implies that ϕ is a constant while Eq. (49) has the form

$$\dot{r}^2 = A_{(p,q)} - \frac{A_{(p,q)}^2 T_{D1}^2}{E^2}n'^2 - \frac{A_{(p,q)}T_{D1}^2}{E^2}m'^2. \tag{51}$$

Now we will analyze this expression in more detail. First of all the solution of this equation is restricted to the region where the right side is non-negative. Since

$$A_{(p,q)} = \left(1 + \frac{Q_{(p,q)}}{r^2}\right)^{-1}, \tag{52}$$

we obtain

$$\frac{Q_{(p,q)}}{r^2} > \frac{T_{D1}^2 n'^2}{E^2(1 - \frac{T_{D1}^2}{E^2}m'^2)} - 1. \tag{53}$$

Note that for $m'=0$ this result agrees with the result derived in [14]. We see that this condition is empty when

$$E^2 > T_{D1}^2(n'^2+m'^2), \tag{54}$$

which has a clear physical meaning. It corresponds to the situation when the total energy is greater than the asymptotic tension of the (m',n') string and the given string can escape to infinity. Note that for $E^2 < T_{D1}^2(n'^2+m'^2)$ the (m',n') string cannot escape from the attraction from the five brane.

We also determine the components of the stress energy tensor (41) for this configuration. Using (51) we easily find

$$\begin{aligned}
T_{\tau\tau} &= \frac{T_{D1}\sqrt{m'^2+n'^2A_{(p,q)}}}{\sqrt{1-\frac{1}{A_{(p,q)}}\dot{r}^2}} = E, \quad T_{\tau\sigma}=0,\\
T_{\sigma\sigma} &= -T_{D1}\sqrt{m'^2+n'^2A_{(p,q)}}\sqrt{1-\frac{1}{A_{(p,q)}}\dot{r}^2}\\
&= -\frac{T_{D1}^2}{E}(m'^2+n'^2A_{(p,q)})\,.
\end{aligned} \tag{55}$$

From $T_{\sigma\sigma}=\mathcal{P}$ we see that the contribution from the D1-brane to the pressure goes to zero when we approach the core of the five-brane background, while the string like contribution is constant. This is an analog of the well-known fact that the fundamental string can make the bound state with the NS5-brane.

Let us now consider such an energy interval when the entire trajectory is in the region when $Q_{(p,q)} \gg r^2$. Then the equation for $\dot{r}$ has the form

$$\dot{r}^2 = \frac{r^2}{Q_{(p,q)}}\left(1-\frac{T_{D1}^2m'^2}{E^2}\right) - \frac{r^4}{Q_{(p,q)}^2}\frac{T_{D1}^2}{E^2}n'^2, \tag{56}$$

which has the solution

$$r = \frac{1}{n'}\sqrt{Q_{(p,q)}\frac{E^2}{T_{D1}^2}-m'^2}\,\frac{1}{\cosh\sqrt{\left(1-\frac{T_{D1}^2}{E^2}m'^2\right)\frac{1}{Q_{(p,q)}}}t}, \tag{57}$$

where we chosen the initial condition that for $t=0$ the (m',n')-string is at the point of the maximal value corresponding to $\dot{r}=0$. From the previous expression we see that this result is valid in the case of $m'=0$. On the other hand the case $n'=0$ has to be analyzed separately in Eq. (56) and we obtain the result

$$r = r_0 \mathrm{e}^{\pm\sqrt{\frac{1}{Q_{(p,q)}}(1-\frac{T_{D1}^2}{E^2}m'^2)}t}, \tag{58}$$

where the $-$ sign corresponds to the m'-string moving towards to the world-volume of the five brane, while $+$ corresponds to the situation when the m'-string leaves it. Again, this result is the manifestation of the fact that the fundamental string can form a marginal bound state with NS5-brane. However, in our case this situation is not so clear due to the fact that m' is not an integer and depends on the asymptotic values of Φ_0 and χ_0. On the other hand it is clear that the equation of motion (51) possesses the constant solution $r=$ const in the case when $n'=0$ on condition that

$$E^2 = T_{D1}^2m'^2 = T_{D1}^2(p^2+q^2). \tag{59}$$

This is a rather puzzling result that shows the difficulty with the background solution (22). To see this in more detail let us imagine that we have a configuration of the background NS5-brane and a probe fundamental string. Under a $SL(2,Z)$ transformation these two objects transform differently. Explicitly, since the NS5-brane is a magnetically charged object with respect to the NSNS two-form it transforms in the same way as in (4). Then the (p,q)-five brane arises from the NS5-brane through the following $SL(2,Z)$ transformation:

$$\begin{pmatrix}\hat{Q}_{NS5}\\ \hat{Q}_{D5}\end{pmatrix} = \begin{pmatrix} p & -c\\ q & a\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}, \tag{60}$$

so that

$$\Lambda = \begin{pmatrix} a & -q\\ c & p\end{pmatrix}. \tag{61}$$

On the other hand we know that the fundamental string transforms under the $SL(2,Z)$ transformation as in (21). Then we find that for Λ given in (61) we obtain an (a,c)-string where $(ap+qc=1)$. Since NS5-brane and fundamental string form a marginal bound state the previous arguments suggest that such a bound state exists also for the (p,q)-five brane and the (a,c)-fundamental string. Then the condition given in (59) is not in agreement with this claim. In other words the condition (59) says that there should exist a marginal bound state between (p,q)-five brane and (p,q)-fundamental string, which is not consistent with the $SL(2,Z)$ duality of the type IIB string theory as was argued above. This is fact suggests that the (p,q)-five-brane background is not consistent from the probe point of view. Stated differently, the method of the constructions of (p,q)-five-brane and fundamental string backgrounds that was used in [7,8] in fact does not lead to the correct form of the background from the string probe point of view. We mean that the resolution of this paradox can be found when we construct the background (p,q)-five-brane solution with the help of an $SL(2,Z)$ transformation rather than the procedure used in

[8], which was based on the $SL(2, R)$ transformation. This question is now under active investigation and we hope to report our results soon.

3.3 The case $L \neq 0$

Let us now consider the case of non-zero angular momentum L. Following [14] we rewrite the equation of motion for (49) in the form

$$\dot{r}^2 + \frac{A^2_{(p,q)}}{E^2}\left(\frac{L^2}{r^2} + T^2_{D1}n'^2\right) + A_{(p,q)}\frac{T^2_{D1}}{E^2}m'^2 - A_{(p,q)} = 0, \tag{62}$$

which can be interpreted as the equation of conserved energy for a particle with mass $m = 2$ that moves in the effective potential $V_{\text{eff}}(r)$,

$$V_{\text{eff}} = \frac{A^2_{(p,q)}}{E^2}\left(\frac{L^2}{r^2} + T^2_{D1}n'^2\right) + A_{(p,q)}\frac{T^2_{D1}}{E^2}m'^2 - A_{(p,q)} \tag{63}$$

with zero energy. Now following [14] we will analyze the behavior of this potential for different values of r. For small r we obtain

$$V_{\text{eff}} = \frac{r^2}{Q_{(p,q)}}\left(\frac{L^2}{Q_{(p,q)}E^2} + \frac{T^2_{D1}}{Q_{(p,q)}E^2}m'^2 - 1\right). \tag{64}$$

On the other hand for large r we have

$$V_{\text{eff}} = \frac{T^2_{D1}}{E^2}n'^2 - 1. \tag{65}$$

Now we see that for $E < T_{D1}n'$ the potential V_{eff} approaches a positive value for $r \to \infty$ and since the particle has zero energy we find that it cannot escape to infinity. Further, in order to have trajectories with non-zero r we have to require that the potential approaches zero from below, which implies

$$\frac{L^2}{Q_{(p,q)}} < E^2 - \frac{T^2_{D1}m'^2}{Q_{(p,q)}}. \tag{66}$$

In fact, if this condition were not satisfied then the only solution would be $r = 0$.

Let us now explicitly find the solution of the equation of motion in the throat region when $A_{(p,q)} = \frac{r^2}{Q_{(p,q)}}$. Then Eq. (49) has the form

$$\dot{r}^2 = \frac{r^2}{Q_{(p,q)}}\left(1 - \frac{T^2_{D1}}{E^2}m'^2 - \frac{L^2}{Q_{(p,q)}E^2}\right) - \frac{r^4}{Q^2_{(p,q)}E^2}T^2_{D1}n'^2, \tag{67}$$

which has the solution

$$r = \frac{Q_{(p,q)}E}{T_{D1}}\sqrt{1 - m'^2\frac{T^2_{D1}}{E^2} - \frac{L^2}{Q_{(p,q)}E^2}} \times \frac{1}{\cosh\sqrt{1 - \frac{T^2_{D1}}{E^2}m'^2 - \frac{L^2}{Q_{(p,q)}E^2}}t}. \tag{68}$$

We see that the non-zero angular momentum slows down the decrease of r. Further, the equation of motion for ϕ implies

$$\phi = \frac{L}{EQ_{(p,q)}}t. \tag{69}$$

In other words, the previous solution describes an (m', n')-string that moves towards to the world-volume of the background five brane which, however, also circles around them.

As the next example we consider the situation when $n' = 0$. In this case we find the potential to be of the form

$$V_{\text{eff}} = \frac{A^2_{(p,q)}}{E^2}\frac{L^2}{r^2} + A_{(p,q)}\frac{T^2_{D1}}{E^2}m'^2 - A_{(p,q)}, \tag{70}$$

which in the throat region simplifies as

$$V_{\text{eff}} = A_{(p,q)}\left(\frac{L^2}{Q_{(p,q)}E^2} + \frac{T^2_{D1}}{E^2}m'^2 - 1\right) \tag{71}$$

and it vanishes identically when

$$E^2 = \frac{L^2}{Q_{(p,q)}} + T^2_{D1}m'^2. \tag{72}$$

In other words it is possible to find a string that rotates around the five brane for any values of r.

The situation is different when $E > T_{D1}n'$, which means that the potential is negative for $r \to \infty$. Further, if we again have (66) we see that we approach the point $r = 0$ from below and hence there is no potential barrier. In this case we have a possibility of a particle that starts at $r = 0$ for $t = -\infty$ and escapes to infinity in a time reverse process. On the other hand the situation is different when the bound (66) is not satisfied. Let us imagine that we have an (m, n)-string initially at a large distance from the (p, q)-five brane. The probe moves towards the (p, q)-five brane until it reaches the point when the effective potential vanishes, that is, at

$$r^2_{\text{min}} = \frac{L^2 - E^2Q_{(p,q)} - T^2_{D1}m'^2Q_{(p,q)}}{E^2 - T^2_{D1}m'^2 - T^2_{D1}n'^2}. \tag{73}$$

Following [14] we can interpret this process as a scattering of the (m, n)-string from the collection of (p, q)-five branes. Since the analysis is completely the same as in [14] we will not repeat it here.

4 (m, n)-String in (p, q)-string background

In this section we consider the dynamics of an (m, n)-string in the macroscopic (p, q)-string background [7],

$$\begin{aligned} ds_E^2 &= H_{pq}^{-3/4}[-dt^2 + dy^2] + H_{pq}^{1/4} dx_m dx^m, \\ H_{pq} &= 1 + \frac{(2\pi)^6 \alpha'^3 \triangle_{(p,q)}^{1/2}}{r^6 \Omega_7} \equiv 1 + \frac{\alpha}{r^6}, \\ \mathbf{B} &= (\Lambda_{p,q}^{-1})^T \begin{pmatrix} (H_{pq}^{-1} - 1) \\ 0 \end{pmatrix}, \\ \Lambda &= \triangle_{(p,q)}^{-1/2} \begin{pmatrix} p & -e^{-\Phi_0} q + \chi_0 e^{\Phi_0}(p - q\chi_0) \\ q & e^{\Phi_0}(p - q\chi_0) \end{pmatrix}, \\ \mathcal{M} &= \Lambda \begin{pmatrix} H_{pq}^{1/2} & 0 \\ 0 & H_{pq}^{-1/2} \end{pmatrix} \Lambda^T, \\ \triangle_{(p,q)} &= e^{-\Phi_0} q^2 + (p - q\chi_0)^2 e^{\Phi_0}. \end{aligned} \tag{74}$$

The Hamiltonian density for the time-dependent world-sheet modes has the form

$$\mathcal{H}_{\text{fix}} = \sqrt{\Pi_m \Pi^m H_{pq}^{-1} + H_{pq}^{-2} T_{D1}^2 (m'^2 + n'^2 H_{pq})} - T_{D1} m' (H_{pq}^{-1} - 1), \tag{75}$$

where $\mathbf{m}'$ is equal to

$$\mathbf{m}' = \begin{pmatrix} m' \\ n' \end{pmatrix} = \triangle_{(p,q)}^{-1/2} \begin{pmatrix} e^{\Phi_0}(p - q\chi_0)(m - \chi_0 n) + e^{-\Phi_0} q n \\ -qm + pn \end{pmatrix}. \tag{76}$$

Clearly for $m = p$ and $n = q$ we obtain

$$\mathbf{m}' = \triangle_{(p,q)}^{-1/2} \begin{pmatrix} e^{\Phi_0}(p - q\chi_0)^2 + e^{-\Phi_0} q n \\ 0 \end{pmatrix} = \begin{pmatrix} \triangle_{(p,q)}^{1/2} \\ 0 \end{pmatrix}, \tag{77}$$

with the following physical interpretation. As we know the (p, q)-string solution was derived through an $SL(2, R)$ transformation from the fundamental string background. Then clearly a fundamental string in a macroscopic string background maps to the same object under an $SL(2, Z)$ transformation. On the other hand from (77) we see that this is not exactly true, since the probe string does not carry integer charge. We again leave the resolution of this paradox for future research.

It is also useful to find the components of the stress energy tensor for the (m, n)-string in static gauge. As in the previous section we temporarily replace the fixed two dimensional metric $\eta_{\alpha\beta}$ with the two dimensional metric $\gamma_{\alpha\beta}$ and write the gauge fixed action in the form

$$S_{\text{fixed}} = -T_{D1} \int d\tau d\sigma \left(\sqrt{\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m}} H_{pq}^{-3/4} \sqrt{-\det \mathbf{A}} - \sqrt{-\gamma} (\Lambda_{pq}^{-1} \mathbf{m})^T \begin{pmatrix} \frac{1}{H_{pq}} - 1 \\ 0 \end{pmatrix} \right), \tag{78}$$

where

$$\mathbf{A}_{\alpha\beta} = \gamma_{\alpha\beta} + H_{pq} \delta_{mn} \partial_\alpha x^m \partial_\beta x^n. \tag{79}$$

Then the components of two dimensional stress energy tensor have the form

$$\begin{aligned} T_{\alpha\beta} &= -\frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{fixed}}}{\delta \gamma^{\alpha\beta}} = \\ &= -\frac{T_{D1}}{\sqrt{-\gamma}} \gamma_{\alpha\gamma} (\mathbf{A}^{-1})^{\gamma\delta} \gamma_{\delta\beta} \sqrt{-\det \mathbf{A}} \sqrt{\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m}} H_{pq}^{-3/4} \\ &\quad + \gamma_{\alpha\beta} (\Lambda_{pq}^{-1} \mathbf{m})^T \begin{pmatrix} \frac{1}{H_{pq}} - 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{80}$$

Finally we return to the flat metric $\gamma_{\alpha\beta} \to \eta_{\alpha\beta}$ and consider a pure time-dependent ansatz. Then we obtain

$$\begin{aligned} T_{\tau\tau} &= \frac{T_{D1} \sqrt{m'^2 + n'^2 H_{pq}}}{H_{pq} \sqrt{1 - H_{pq}(\dot{r}^2 + r^2 \dot{\phi}^2)}} + m' T_{D1} (1 - \frac{1}{H_{pq}}), \\ T_{\tau\sigma} &= 0, \\ T_{\sigma\sigma} &= -T_{D1} \sqrt{m'^2 + n'^2 H_{pq}} \frac{1}{H_{pq}} \sqrt{1 - H_{pq}(\dot{r}^2 + r^2 \dot{\phi}^2)} \\ &\quad - m' T_{D1} \left(1 - \frac{1}{H_{pq}} \right), \end{aligned} \tag{81}$$

where we also introduced spherical coordinates in the transverse $\mathbf{R}^8$ space and considered the dynamics of the probe in a two dimensional plane with radial variable r and angular variable ϕ. As a result the Hamiltonian density (75) simplifies considerably:

$$\mathcal{H}_{\text{fix}} = \frac{1}{H_{pq}} \left(\sqrt{H_{pq}(p_r^2 + \frac{1}{r^2} p_\phi^2 + T_{D1}^2 n'^2) + T_{D1}^2 m'^2} + T_{D1} m' (H_{pq} - 1) \right). \tag{82}$$

Note also that the equation of motion for ϕ has the form

$$\dot{\phi} = \{\phi, H_{\text{fix}}\} = \frac{L}{r^2 (H_{pq} E - T_{D1} m' (H_{pq} - 1))}, \tag{83}$$

where we used the fact that $p_\phi = L$ and $\mathcal{H}_{\text{fix}} = E$ are conserved. With the help of these results we obtain the following components of the stress energy tensor (81):

$$T_{\tau\tau} = E,$$
$$T_{\sigma\sigma} = \mathcal{P} = -\frac{T_{D1}^2(m'^2 + n'^2 H_{pq})}{H_{pq}(E - T_{D1}m') + T_{D1}m'} - T_{D1}m'\left(1 - \frac{1}{H_{pq}}\right), \quad (84)$$

where $\mathcal{P}$ is the pressure on the world-volume of the (m, n)-string.

Let us now proceed to the analysis of dynamics of this probe string. From H_{fix} we derive the equation of motion,

$$\dot{r} = \{r, H_{\text{fix}}\} = \frac{p_r}{\sqrt{H_{pq1}(p_r^2 + \frac{1}{r^2}L^2 + T_{D1}^2 n'^2) + T_{D1}^2 m'^2}}. \quad (85)$$

On the other hand from the fact that the energy density E is conserved we can express p_r as

$$p_r^2 = \frac{1}{H_{pq}}\left((H_{pq}(E - T_{D1}m') + m'T_{D1})^2 - T_{D1}^2 m'^2\right) - \frac{L^2}{r^2} - T_{D1}^2 n'^2, \quad (86)$$

so that (85) has the form

$$\dot{r}^2 = \frac{1}{H_{pq}}\left(1 - \frac{(T_{D1}^2 n'^2 + \frac{L^2}{r^2})H_{pq} + T_{D1}^2 m'^2}{(H_{pq}(E - T_{D1}m') + T_{D1}m')^2}\right). \quad (87)$$

We can again rewrite this equation into the more suggestive form

$$\dot{r}^2 + V_{\text{eff}} = 0, \quad (88)$$

where

$$V_{\text{eff}} = \frac{1}{H_{pq}}\left(\frac{(T_{D1}^2 n'^2 + \frac{L^2}{r^2})H_{pq} + T_{D1}^2 m'^2}{(H_{pq}(E - T_{D1}m') + T_{D1}m')^2} - 1\right). \quad (89)$$

We see that the equation above corresponds to the massive particle with mass $m = 2$ moving in the potential V_{eff} with zero energy so that much interesting information as regards the particle's trajectory follows from the properties of the given potential. As in the previous section we start with the case of the zero angular momentum.

4.1 The case $L = 0$

We see that for $r \to \infty$ we have $H_{pq} \to 1$ and hence

$$V_{\text{eff}} \to \frac{T_{D1}^2 m'^2 + T_{D1}^2 n'^2}{E^2} - 1, \quad (90)$$

while for small r we obtain

$$V_{\text{eff}} = \frac{r^6}{\alpha}\left(\frac{n'^2 T_{D1}^2 r^6}{\alpha(E - T_{D1}m')^2} - 1\right), \quad (91)$$

so that V_{eff} approaches the point $r = 0$ from below. As a result we have two qualitative different behaviors of the probe (m', n') string in this background. It follows from (90) that for $E^2 < T_{D1}^2(m'^2 + n'^2)$ V_{eff} approaches a positive constant for large r. Then the probe string cannot escape to infinity and it moves in the bounded region around the (p, q)-string background. On the other hand for $E^2 > T_{D1}^2(m'^2 + n^2)$ the potential is negative for all values and hence the probe string can move to infinity. Let us first consider the case when $n' = 0$. This case corresponds to the situation of the motion of fundamental string in the background of a collection of fundamental strings. We can expect that it is possible to form a marginal bound state of $N + m'$ fundamental strings. In fact, for $E - T_{D1}m' = \epsilon \ll 1$ we find that the effective potential has the form

$$V_{\text{eff}} = -2\frac{\epsilon}{T_{D1}m'} \quad (92)$$

and we see that it is flat. As a result we find a time dependence $r \sim \pm\epsilon t$, which means a very slow movement of the probe string. This is a confirmation of the claim that the probe string can form marginal bound state with N fundamental strings. Note that this approximation is valid on condition that

$$\frac{H_{pq}\epsilon}{T_{D1}m'} \ll 1, \quad (93)$$

which implies $r^6 \gg \frac{\alpha\epsilon}{T_{D1}m'}$, which can be obeyed in the whole region in the limit $\epsilon \to 0$. Finally note also that the pressure is equal to

$$\mathcal{P} = -2T_{D1}m' + H_{pq}\epsilon + \frac{T_{D1}}{H_{pq}}m', \quad (94)$$

which has the following physical explanation. Consider the initial configuration when the m'-string is sitting at infinity when $H_{pq} = 1$ and consequently $\mathcal{P} = -T_{D1}m' + \epsilon = E - 2T_{D1}m'$. The string moves slowly to the horizon when $H_{pq} \to \infty$ and hence the pressure approaches the value $\mathcal{P} \to -m'T_{D1}$ at the horizon in the limit $\epsilon \to 0$.

To see this more clearly let us consider the case of the near horizon limit when $H_{pq}\epsilon \gg T_{D1}m'$ when ϵ is small but finite. In this case we find that the leading order behavior of the effective potential is

$$V_{\text{eff}} = -\frac{r^6}{\alpha} \quad (95)$$

and hence we have the differential equation

$$\frac{dr}{dt} = \pm \frac{r^3}{\sqrt{\alpha}}. \quad (96)$$

The + sign corresponds to the string moving from the horizon when the approximation we use quickly breaks. The sign − corresponds to the string moving to the horizon and for this possibility we find the solution

$$r = \frac{r_0}{\sqrt{1 + \frac{2r_0^2}{\sqrt{\alpha}} t}}, \quad r_0^6 \ll \alpha. \quad (97)$$

We see that the probe string approaches horizon at asymptotic time $t \to \infty$. Observe that this behavior does not depend on the value of the energy of the string probe.

4.2 The case $L \neq 0$

Now we would like to analyze the behavior of the potential in the case $n' = 0$ and $L \neq 0$ and in the limit $E - T_{D1}m' = \epsilon \ll 1$. In this case we find the effective potential in the form

$$V_{\rm eff} = -2\frac{\epsilon}{T_{D1}m'} + \frac{L^2}{r^2 T_{D1}^2 m'^2}\left(1 - \frac{2\epsilon}{T_{D1}m'}\right) - \frac{2\alpha}{r^8}\frac{L^2 \epsilon}{T_{D1}^3 m'^3}. \quad (98)$$

From this form of the effective potential we can deduce the existence of the potential barrier, since there are two points where $V_{\rm eff}$ vanishes. We find these points as follows. We presume that the first root corresponds to the root when we neglect the term proportional to r^{-8}. Then we solve the quadratic equation with the solution

$$r_+ = \frac{L}{\sqrt{2T_{D1}m'\epsilon}}. \quad (99)$$

We see that it has a very large value, which justifies our assumption. The second root corresponds to the situation when we neglect the constant term in (98) and we obtain

$$r_- = \left(\frac{2\alpha\epsilon}{T_{D1}m'}\right)^{1/6}, \quad (100)$$

which is much smaller than r_+ again in agreement with our assumptions. The physical picture is the following. If we have a probe m'-string with $E - T_{D1}m' \ll 1$ far away from the background (p, q)-string then it moves towards it, however, it cannot cross the horizon. Rather it approaches the distance given by r_+ and then it is deflected. On the other hand the m'-string that is initially in the region below r_- will spirally move towards the the horizon. This situation is similar to the case of the $(m, 1)$-string studied in [15] and we will not repeat it here.

Acknowledgements This work was supported by the Grant Agency of the Czech Republic under the Grant P201/12/G028.

References

1. M.J. Duff, R.R. Khuri, J.X. Lu, String solitons. Phys. Rept. **259**, 213 (1995). doi:10.1016/0370-1573(95)00002-X. arXiv:hep-th/9412184
2. D.J. Smith, Intersecting brane solutions in string and M theory. Class. Quant. Grav. **20**, R233 (2003). doi:10.1088/0264-9381/20/9/203. arXiv:hep-th/0210157
3. K.S. Stelle, Lectures on supergravity p-branes, in *Trieste 1996, High energy physics and cosmology*, pp. 287–339. arXiv:hep-th/9701088
4. T. Ortin, *Gravity and Strings* (Cambridge University Press, Cambridge, 2015)
5. A.A. Tseytlin, Harmonic superpositions of M-branes. Nucl. Phys. B **475**, 149 (1996). doi:10.1016/0550-3213(96)00328-8. arXiv:hep-th/9604035
6. N.A. Obers, B. Pioline, U duality and M theory. Phys. Rept. **318**, 113 (1999). doi:10.1016/S0370-1573(99)00004-6. arXiv:hep-th/9809039
7. J.H. Schwarz, An SL(2,Z) multiplet of type IIB superstrings. Phys. Lett. B **360**, 13 (1995) [Phys. Lett. B **364** (1995) 252]. doi:10.1016/0370-2693(95)01138-G. arXiv:hep-th/9508143
8. J.X. Lu, S. Roy, An SL(2, Z) multiplet of type IIB super five-branes. Phys. Lett. B **428**, 289 (1998). doi:10.1016/S0370-2693(98)00435-3. arXiv:hep-th/9802080
9. A.A. Tseytlin, Selfduality of Born–Infeld action and Dirichlet three-brane of type IIB superstring theory. Nucl. Phys. B **469**, 51 (1996). doi:10.1016/0550-3213(96)00173-3. arXiv:hep-th/9602064
10. Y. Lozano, D-brane dualities as canonical transformations. Phys. Lett. B **399**, 233 (1997). doi:10.1016/S0370-2693(97)00292-X. arXiv:hep-th/9701186
11. M. Cederwall, P.K. Townsend, The Manifestly Sl(2, Z) covariant superstring. JHEP **9709**, 003 (1997). doi:10.1088/1126-6708/1997/09/003. arXiv:hep-th/9709002
12. P.K. Townsend, Membrane tension and manifest IIB S duality. Phys. Lett. B **409**, 131 (1997). doi:10.1016/S0370-2693(97)00862-9. arXiv:hep-th/9705160
13. E.A. Bergshoeff, M. de Roo, S.F. Kerstan, T. Ortin, F. Riccioni, SL(2, R)-invariant IIB Brane Actions. JHEP **0702**, 007 (2007). doi:10.1088/1126-6708/2007/02/007. arXiv:hep-th/0611036
14. D. Kutasov, D-brane dynamics near NS5-branes. arXiv:hep-th/0405058
15. D. Bak, S.J. Rey, H.U. Yee, Exactly soluble dynamics of (p, q) string near macroscopic fundamental strings. JHEP **0412**, 008 (2004). doi:10.1088/1126-6708/2004/12/008. arXiv:hep-th/0411099
16. J. Kluson, Integrability of D1-brane on group manifold. JHEP **1409**, 159 (2014). doi:10.1007/JHEP09(2014)159. arXiv:1407.7665 [hep-th]
17. P. Bueno, T. Ortin, C.S. Shahbazi, Non-extremal branes. Phys. Lett. B **743**, 301 (2015). doi:10.1016/j.physletb.2015.02.070. arXiv:1412.5547 [hep-th]