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Higher dimensional loop quantum cosmology

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Abstract Loop quantum cosmology (LQC) is the symmetric sector of loop quantum gravity. In this paper, we generalize the structure of loop quantum cosmology to the theories with arbitrary spacetime dimensions. The isotropic and homogeneous cosmological model in $n + 1$ dimensions is quantized by the loop quantization method. Interestingly, we find that the underlying quantum theories are divided into two qualitatively different sectors according to spacetime dimensions. The effective Hamiltonian and modified dynamical equations of $n+1$ dimensional LQC are obtained. Moreover, our results indicate that the classical big bang singularity is resolved in arbitrary spacetime dimensions by a quantum bounce. We also briefly discuss the similarities and differences between the $n + 1$ dimensional model and the $3 + 1$ dimensional one. Our model serves as a first example of higher dimensional loop quantum cosmology and offers the possibility to investigate quantum gravity effects in higher dimensional cosmology.

1 Introduction

Higher dimensional spacetime are a subject of great interest as regards grand unified theories. Historically, the first higher dimensional theories is the famous Kaluza–Klein theory, trying to unify the 4 dimensional general relativity (GR) and Maxwell theory [1]. Recent theoretical developments reveal that higher dimensions are preferred by many theories, such as the string/M theories [2], the AdS/CFT correspondence [3], the brane world scenario [4,5], and so on. In the past decades, many aspects of these higher dimensional theories have extensively been studied, particularly on the issues related to black holes and cosmology. In fact, higher dimensional cosmology received increasing attention, and it has become a rather active field with fruitful results. For instance, some of the higher dimensional cosmological models can naturally explain the accelerated expansions of Universe [6,7].

Loop quantum gravity (LQG) is a quantum gravity theory trying to quantize GR with nonperturbative techniques [8–11]. Many issues of LQG have been explored in the past 30 years. Among these issues, loop quantum cosmology (LQC), which is the cosmological application of LQG, has received particularly interest. Recently, LQC has become one of the most thriving and fruitful directions of LQG [12–16]. One of the most attractive features of this theory is that LQC is a singularity free theory. In LQC, the cosmological singularity, which is inevitable in classical GR is naturally replaced by a quantum bounce [17,18]. Although LQC is a fruitful theory, nowadays all the discussions are still limited to 4 spacetime dimensions. Recently, LQC has been generalized to the $2 + 1$ dimensional case [19]. Hence it is naturally to ask if it is possible to generalize the structure of LQC to the higher spacetime dimensions.

However, this is not an easy task, essentially because LQG is a quantization scheme based on the connection dynamics formalism. The $SU(2)$ connection dynamics is only well defined in 3 and 4 dimensions and thus cannot be directly generalized to the higher dimensional gravity theories. Fortunately, this difficulty has been overcome by Thiemann et al. in a series of papers [20–23]. The main idea of [20] is that in $n+1$ dimensional GR, in order to obtain a well-defined connection dynamics, one should adopt $SO(n+1)$ connections A_a^{IJ} rather than the speculated $SO(n)$ connections. With this higher dimensional connection dynamics in hand, Thiemann et al. successfully generalize the LQG to arbitrary spacetime dimensions. Thus the purpose of the present paper is to investigate the issue of the $n + 1$ dimensional LQC in this generalized LQG framework. Note that the $2+1$ dimensional case is already studied in [19]. Therefore this paper will be devoted to the LQC with $n \geq 3$.

This paper is organized as follows: After a brief introduction, in Sect. 2, we first review the classical connection formalism of $n + 1$ dimensional LQG, and then we use it to

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derive the cosmological Hamiltonian through the symmetric reduction procedure. In Sect. 3 we give a detailed construction of the quantum theory of $n+1$ dimensional LQC and derive the difference equation which represents the dynamical evolution of the $n+1$ dimensional Universe. Then we briefly discuss the singularity resolution issue in Sect. 4. The effective Hamiltonian and the modified effective dynamical equations are obtained in Sects. 5 and 6, respectively. Some conclusions are given in the last section.

2 Classical theory

To make this paper self-contained and also convenient for the reader, we first review some basic elements of classical $n+1$ dimensional gravity concerned in this paper. The connection dynamics of $n+1$ dimensional gravity with the gauge group $SO(n+1)$ or $SO(1,n)$ is obtained in [20]. The Ashtekar formalism of $n+1$ dimensional gravity constitutes a $SO(1,n)$(or $SO(n+1)$) connection A_a^{IJ} and a group value densitized vector π^b_{IJ} defined on an oriented n dimensional manifold S, where $a, b = 1, 2 \ldots n$ is for the spatial indices and $I, J = 1, 2, 3 \ldots n$ denotes $SO(1,n)$ group indices. The commutation relation for the canonical conjugate pairs satisfies

$$\{A_{aIJ}(x), \pi^{bKL}(y)\} = 2\kappa\gamma\delta^K_{[I}\delta^L_{J]}\delta^b_a\delta(x,y) \tag{2.1}$$

where $\kappa = 8\pi G$ and γ is a nonzero real number. Here the π^{bKL} satisfy the "simplicity constraint" [20] and can be written as $\pi^{bKL} = 2n^{[K}E^{b|L]} = 2\sqrt{h}h^{ab}n^{[K}e_a^{L]}$, where the spatial metric reads $h_{ab} = e^i_a e_{bi}$, n^K is a normal which satisfies $e^K_a n_K = 0$ and $n^K n_K = -1$ for $SO(1,n)$ (for the case $SO(n+1)$, $n^K n_K = 1$). Moreover, the densitized vector E^a_I satisfies $hh^{ab} = E^a_I E^{bI}$, where h is the determinant of the spatial metric h_{ab}. A^{IJ}_a is a $SO(1,n)$ connection defined as $A^{IJ}_a = \Gamma^{IJ}_a + \gamma K^{IJ}_a$, here Γ^{IJ}_a and K^{IJ}_a are the n dimensional spin connection and extrinsic curvature, respectively. Besides the simplicity constraint, the $n+1$ dimensional gravity has three constraints similar to 3 + 1 dimensional general relativity [20,22],

$$G^{IJ} = \mathcal{D}_a\pi^{aIJ} = \partial_a\pi^{aIJ} + 2A_a^{[I}{}_K\pi^{a|K|J]}, \tag{2.2}$$

$$V_a = \frac{1}{2\gamma}F_{abIJ}\pi^{bIJ}, \tag{2.3}$$

$$\begin{aligned} H_{gr} = \frac{1}{2\kappa\sqrt{h}}\Big(& F_{abIJ}\pi^{aIK}\pi^b{}_K{}^J \\ & + 4\bar{D}^{aIJ}_T(F^{-1})_{aIJ,bKL}\bar{D}^{bKL}_T \\ & - 2(1+\gamma^2)K_{aI}K_{bJ}E^{a[I}E^{b|J]}\Big) \end{aligned} \tag{2.4}$$

where $F_{abIJ} \equiv 2\partial_{[a}A_{b]IJ} + 2A_{a[I|K}A_b{}^K{}_{|J]}$ is the curvature of the connection A_{aIJ}, and $\bar{D}^{aIJ}_T = \frac{\gamma}{4}F^{aIJ,bKL}\bar{K}^T_{bKL}$ with $\bar{K}^T_{bKL}$ are the transverse and traceless part of the extrinsic curvature K_{bKL}. Moreover, we have $[F\cdot F^{-1}]^{aIJ}_{bKL} = \delta^a_b\bar{\eta}^I_{[K}\bar{\eta}^J_{L]}$ with $\bar{\eta}^I_J = \delta^I_J - n^I n_J$.

Now let us consider the $n+1$ dimensional isotropic and homogeneous $k=0$ Universe. Its line element is described by the $n+1$ dimensional Friedmann–Robertson–Walker (FRW) metric,

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + a^2(t)\mathrm{d}\Omega^2 \tag{2.5}$$

where a is the scale factor and $\mathrm{d}\Omega^2$ is the n dimensional sphere. We choose a fiducial Euclidean metric ${}^oq_{ab}$ on the spatial slice of the isotropic observers and introduce a pair of fiducial orthonormal basis elements as $({}^oe^a_I, {}^o\omega^I_a)$ such that ${}^oq_{ab} = {}^o\omega^I_a{}^o\omega^I_b$. The physical spatial metric is related to the fiducial one by $q_{ab} = a^{2o}q_{ab}$. Then the densitized vector can be expressed as $E^a_I = pV_0^{-\frac{n-1}{n}}\sqrt{{}^0q}{}^oe^a_I$. Thus the π^{aIJ} and spin connection A^{IJ}_b, respectively, reduce to

$$\pi^{aIJ} = 2pV_0^{-\frac{n-1}{n}}\sqrt{{}^0q}{}^on^{[I o}e^{a|J]} = pV_0^{-\frac{n-1}{n}}\sqrt{{}^0q}{}^o\pi^{aIJ}, \tag{2.6}$$

$$A^{IJ}_b = 2cV_0^{-\frac{1}{n}}{}^on^{[I o}\omega^{J]}_b = cV_0^{-\frac{1}{n}}{}^o\Omega^{IJ}_b. \tag{2.7}$$

In the following, for simplicity, we will fix the fiducial volume $V_0 = 1$. By using the classical expression π^{aIJ}, A^{IJ}_b and cosmological line elements (2.5), one can easily find

$$p = a^{n-1}, \quad c = \gamma\dot{a}. \tag{2.8}$$

These canonical variables satisfy the commutation relation as follows:

$$\{c, p\} = \frac{\kappa\gamma}{n}. \tag{2.9}$$

For our cosmological case, the Gaussian and diffeomorphism constraints are satisfied automatically. On the other hand, for the Hamiltonian constraint, we first note that in the isotropic cosmological situation, the extrinsic curvature only has a diagonal part. Hence the transverse traceless part of the extrinsic curvature $\bar{K}^T_{bKL}$ is identical to zero. Therefore the second term of the Hamiltonian constraint is vanishing. Moreover, the spin connection Γ is also zero for our homogeneous Universe. With this fact in mind, a simple and straightforward calculation shows the $KKEE$ term to be proportional to the $F\pi\pi$ term. Combining all the above ingredients, the Hamiltonian constraint (2.4) reduces to

$$H_{gr} = -\frac{1}{2\kappa\gamma^2}F_{abIJ}\frac{\pi^{aIK}\pi^b{}_K{}^J}{\sqrt{h}}. \tag{2.10}$$

Now, as in the 3 + 1 dimensional LQC, we also consider a minimally coupled massless scalar field ϕ as our matter field. The total Hamiltonian now reads

$$H_{\text{Total}} = -\frac{1}{2\kappa\gamma^2}F_{abIJ}\frac{\pi^{aIK}\pi^b{}_K{}^J}{\sqrt{h}} + \frac{p^2_\phi}{2\sqrt{h}} \tag{2.11}$$

where the p_ϕ by definition is the conjugate momentum of the massless scalar field ϕ. The Poisson bracket between scalar field ϕ and conjugate momentum p_ϕ reads $\{\phi, p_\phi\} = 1$. In the cosmological model we considered in this paper, this Hamiltonian therefore reduces to

$$H_{\text{Total}} = -\frac{n(n-1)}{2\kappa\gamma^2} c^2 p^{\frac{n-2}{n-1}} + \frac{p_\phi^2}{2p^{\frac{n}{n-1}}}. \tag{2.12}$$

At the classical level, this $SO(1, n)$ connection dynamics formalism is equivalent to the $n+1$ dimensional Arnowitt–Deser–Misner (ADM) formalism [20]. In the cosmological situation, the ADM formalism will lead to the classical Friedmann equation. Thus as a consistent check of our symmetric reduction procedure, we need to reproduce the $n+1$ dimensional Friedmann equation from our Hamiltonian (2.12) and commutation relation (2.9). To this aim, we calculate the equation of motion for p, which reads

$$\dot{p} = \{p, H_{\text{Total}}\} = \frac{n-1}{\gamma} c p^{\frac{n-2}{n-1}}. \tag{2.13}$$

By using the Hamiltonian constraint we successfully reproduce the classical $n+1$ dimensional Friedmann equation,

$$\begin{aligned} H^2 &= \left(\frac{\dot{p}}{(n-1)p}\right)^2 = \frac{1}{\gamma^2 p^2} c^2 p^{\frac{2(n-2)}{n-1}} \\ &= \frac{2\kappa}{n(n-1)} \frac{p_\phi^2}{2p^{\frac{2n}{n-1}}} = \frac{2\kappa}{n(n-1)} \rho \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} H &= \frac{\dot{a}}{a}, \\ \rho &= \frac{p_\phi^2}{2V^2} = \frac{p_\phi^2}{2p^{\frac{2n}{n-1}}} \end{aligned} \tag{2.15}$$

are the Hubble parameter and the matter density in $n+1$ dimensions, respectively. Moreover, another dynamical equation, namely the so-called Raychaudhuri equation which evolves with a second-order time derivative of the scale factor a can be obtained by combining the continuity equation in $n+1$ dimensions, $\dot{\rho} + nH(\rho + p) = 0$, with the Friedmann equation (2.14)

$$\frac{\ddot{a}}{a} = \frac{2\kappa}{n(n-1)} \rho - \frac{\kappa}{n-1}(\rho + p). \tag{2.16}$$

3 Quantum theory

Now we come to the issue of quantizing the cosmological model, we first need to construct the quantum kinematical Hilbert space of $n+1$ dimensional cosmology by mimicking the $3+1$ dimensional loop quantum cosmology. These quantum kinematical Hilbert spaces are constituted by the so-called polymer-like quantization for the geometric part, while the Schrödinger representation is adopted for the massless scalar field part. The resulting kinematical Hilbert space for the geometry part reads $\mathcal{H}_{\text{kin}}^{\text{gr}} \equiv L^2(R_{\text{Bohr}}, \mathrm{d}\mu_H)$, where R_{Bohr} and $\mathrm{d}\mu_H$ are, respectively, the Bohr compactification of the real line R and the corresponding Haar measure on it [12]. On the other hand, following the standard treatment of LQC, we choose the Schrödinger representation for the massless scalar field [15]. Thus the kinematical Hilbert space for the matter field part is defined in the usual way: $\mathcal{H}_{\text{kin}}^{\text{sc}} \equiv L^2(R, \mathrm{d}\mu)$. Hence the whole Hilbert space of $n+1$ dimensional loop quantum cosmology takes the form of a direct product, $\mathcal{H}_{\text{kin}} := \mathcal{H}_{\text{kin}}^{\text{gr}} \otimes \mathcal{H}_{\text{kin}}^{\text{sc}}$. Now let $|\mu\rangle$ be the eigenstates of $\hat{p}$ in the kinematical Hilbert space $\mathcal{H}_{\text{kin}}^{\text{gr}}$ such that $\hat{p}|\mu\rangle = \frac{4\pi G\gamma\hbar\mu}{n}|\mu\rangle = \frac{\hbar\kappa\gamma}{2n}\mu|\mu\rangle$. These eigenstates $|\mu_i\rangle$ obey the orthonormal condition $\langle\mu_i|\mu_j\rangle = \delta_{\mu_i,\mu_j}$ with δ_{μ_i,μ_j} being the Kronecker delta function rather than the Dirac delta function. In $n+1$ dimensional quantum gravity, the $n-1$ dimensional area operator is quantized just like their counterparts in $3+1$ dimensions, the discrete spectrum of this $n-1$ dimensional operator reads [22]

$$\begin{aligned} \Delta_n &= \kappa\hbar\gamma \sum_I \sqrt{I(I+n-1)} \\ &= 8\pi\gamma(\ell_{\text{p}})^{n-1} \sum_I \sqrt{I(I+n-1)} \end{aligned} \tag{3.1}$$

where I is an integer and $\ell_{\text{p}} = \sqrt[n-1]{G\hbar}$ is the Planck length. The interpretation of I is that for every edge, we can associate a simple representation of $SO(n+1)$, which is labeled by its corresponding highest weight $\Lambda = (I, 0, 0, \ldots)$ with I being an integer. This equation tells us the existence of minimal area gap, which is given by

$$\Delta_n = \sqrt{n}k\hbar\gamma \equiv 8\sqrt{n}\pi\gamma(\ell_{\text{p}})^{n-1}. \tag{3.2}$$

Note that the quantization of area refers to physical geometries in $3+1$ dimensional LQC [17], and we generalize this argument to our $n+1$ dimensional LQC. We take the $n-1$ dimensional cube, every vertex of the cube has $n-1$ edges, and the holonomy loop $\square_{ij}$ is constituted by its arbitrary two edges from one vertex. Now we should shrink the holonomy loop $\square_{ij}$ till the $n-1$ dimensional area of the cube, which is measured by the physical metric q_{ab}, reaches the value of a minimal $n-1$ dimensional area Δ_n. Since the physical $n-1$ dimensional area of the elementary cell is $|p|$ and each side of $\square_{ij}$ is λ times the edge of the elementary cell, in order to compare with $3+1$ dimensions, we also use a specific function $\bar{\mu}(p)$ to denote λ, and similar to that in [17], we have

$$\bar{\mu}^{n-1}(p)|p| = \Delta_n \equiv 8\sqrt{n}\pi\gamma(\ell_{\text{p}})^{n-1}. \tag{3.3}$$

It is easy to see that, when $n = 3$, the above formulation goes back to the famous $\bar{\mu}$ scheme in $3+1$ dimensions. For convenience of studying the quantum dynamics, we define the following new variables:

$$v := \frac{2(n-1)\Delta_n}{\hbar\kappa\gamma}\bar{\mu}^{-n}, \quad b := \bar{\mu}c,$$

where $\bar{\mu} = (\frac{\Delta_n}{|p|})^{\frac{1}{n-1}}$ with Δ_n being a minimum nonzero eigenvalue of the n dimensional area operator [14]. It is easy to verify that these new variables satisfy the commutation relation $\{b, v\} = \frac{2}{\hbar}$. It turns out that the eigenstates of $\hat{v}$ also constitute an orthonormal basis in the kinematical Hilbert space $\mathcal{H}^{gr}_{kin}$. We denote $|\phi, v\rangle$ as the generalized orthonormal basis for the whole kinematical Hilbert space $\mathcal{H}_{kin}$. For simplicity, in the following, $|\phi, v\rangle$ will be abbreviated as $|v\rangle$.

The action of volume operator $\hat{V}$ on this basis $|v\rangle$ reads

$$\hat{V}|v\rangle = \frac{\hbar\kappa\gamma(\Delta_n)^{\frac{1}{n-1}}}{2(n-1)}|v||v\rangle = \frac{(\Delta_n)^{\frac{n}{n-1}}}{4(n-1)\sqrt{3}}|v||v\rangle$$
$$= \frac{4\pi\gamma(\Delta_n)^{\frac{1}{n-1}}}{(n-1)}|v|\ell_p^{n-1}|v\rangle. \quad (3.4)$$

The Hamiltonian constraint needs to be reformulated in terms of these (b, v) variables as

$$H_T = -\frac{n(n-1)}{2\kappa\gamma^2}c^2 p^{\frac{n-2}{n-1}} + \frac{p_\phi^2}{2p^{\frac{n}{n-1}}}$$
$$= -\frac{n\hbar}{4\gamma(\Delta_n)^{\frac{1}{n-1}}}b^2|v| + \left(\frac{2(n-1)}{\hbar\kappa\gamma(\Delta_n)^{\frac{1}{n-1}}}\right)\frac{p_\phi^2}{2|v|}. \quad (3.5)$$

Note that we adapt the polymer representation for geometric part. In the quantum theory, the connection should be replaced by a well-defined holonomy operator. For a given edge e with length $\bar{\mu}$, the holonomy is defined as [22]

$$h_e^{\bar{\mu}}(A) := P\exp\left(\int_e A_a^{IJ}\tau_{IJ}\dot{e}^a\right) \quad (3.6)$$

where $\dot{e}^a$ is the tangent of the edge e. In our cosmological setting, we take the edge $\dot{e}^a = {}^o\dot{e}^a_{KL}$ such that ${}^o\dot{e}^a_{KL}{}^o\Omega_a^{IJ} = \delta^I_{[K}\delta^J_{L]}$. Recall that in our cosmological case, we have $A_a^{IJ} = c^o\Omega_b^{IJ}$; the holonomy then reads

$$h_{IJ}^{\bar{\mu}} = \exp(\bar{\mu}c\tau_{IJ}) = \cos\left(\frac{\bar{\mu}c}{2}\right) + 2\tau_{IJ}\sin\left(\frac{\bar{\mu}c}{2}\right) \quad (3.7)$$

where $\tau_{IJ} = -\frac{i}{4}[\gamma_I, \gamma_J]$ with γ_I being the gamma matrices constitutes a representation of $SO(1,n)$ [22]. On the other hand, similarly to the case in 3 + 1 dimensions, in order to express the curvature, we first note that $h_e(A) = I + \epsilon\dot{e}^a A_a^{IJ}\tau_{IJ} + O(\epsilon^2)$ [22]. For a given loop with area $Ar_\Box \to 0$, the curvature can be expressed through the holonomy as

$$F_{abIJ} = -2\lim_{Ar_\Box\to 0}\mathrm{Tr}\left(\frac{(h_\Box^{\bar{\mu}})_{KL,MN} - 1}{\bar{\mu}^2}\right){}^o\Omega_a^{KL\,o}\Omega_b^{MN}\tau_{IJ}$$
$$= 2\frac{\sin^2(\bar{\mu}c)}{\bar{\mu}^2}{}^o\Omega^{[I}_{aK}{}^o\Omega_b^{K|J]} \quad (3.8)$$

where we consider a square $\Box$ and

$$(h_\Box^{\bar{\mu}})_{IJ,KL} = h_{IJ}h_{KL}h_{IJ}^{-1}h_{KL}^{-1} \quad (3.9)$$

denotes the holonomy along a closed loop $\Box$. Every edge of the square has length $\lambda(V_0)^{\frac{1}{n}}$ with respect to the fiducial metric and $Ar_\Box$ denotes the area of the square.

Now our task is to implement the Hamiltonian constraint at the quantum level. With this purpose, we first need to rewrite the Hamiltonian constraint in a suitable manner. This is inevitable because the expression of the classical Hamiltonian constraint involves the inverse of the determinate of the n-metric and thus cannot be promoted as a well-defined operator on the kinematical Hilbert space. In 3 + 1 dimensional case, this difficulty can be overcome by using the well-known classical identity $\frac{1}{2}\epsilon^{ijk}\frac{\epsilon_{abc}E_j^b E_k^c}{\sqrt{q}} = \frac{1}{\kappa\gamma}\{A_a^i, V\}$ [9]. Generalization of this expression to $n+1$ dimensions is highly nontrivial. The most interesting point is that the treatment of the quantity $\frac{\pi^{[a|IK}\pi_K^{b]J}}{\sqrt{h}}$ can be divided into two different sectors according to spacetime dimensions, namely, an even dimensional sector and an odd dimensional sector [22]. First we note that

$$\pi_{aIJ}(x) := -\frac{n-1}{2\kappa\gamma\sqrt{h}}\{A_{aIJ}, V(x)\}; \quad (3.10)$$

now the quantity $\frac{\pi^{[a|IK}\pi_K^{b]J}}{\sqrt{h}}$ appearing in the Hamiltonian constraint can be constructed with these basic building blocks.

3.1 Even dimensional sector

For the case of the spacetime dimensions $n+1$ being even, we let $s = \frac{(n-1)}{2}$, and we note that we have the following classical identity [22]:

$$\frac{\pi^{[a|IK}\pi_K^{b]J}}{\sqrt{h}} = \frac{1}{4(n-2)!}\epsilon^{abca_1b_1\ldots a_{s-1}b_{s-1}}\epsilon^{IJKLI_1J_1\ldots I_{s-1}J_{s-1}}$$
$$= \pi_{cKL}\pi_{a_1I_1K_1}\pi^{K_1}_{b_1J_1}\cdots\pi_{a_{s-1}I_{s-1}K_{s-1}}\pi^{K_{s-1}}_{b_{s-1}J_{s-1}}\sqrt{h}^{n-2}. \quad (3.11)$$

Since in the quantum theory, the connection should be replaced by the well-defined holonomy operator, we can rewrite the Hamiltonian constraint as follows:

$$H_{gr} = -\frac{1}{2\kappa\gamma^2}\int d\Sigma F_{abIJ}\frac{\pi^{[a|IK}\pi_K^{b]J}}{\sqrt{h}}$$
$$= -\frac{1}{8(n-2)!\kappa^{n-1}\gamma^n}$$
$$\times\epsilon^{abca_1b_1\cdots a_{s-1}b_{s-1}}\epsilon^{IJKLI_1J_1\cdots I_{s-1}J_{s-1}}(n-1)^{n-2}$$
$$\times\frac{1}{2^{n-2}}\int d\Sigma\left(F_{abIJ}\{A_{cKL}, V\}\{A_{a_1I_1K_1}, V\}\right.$$
$$\left.\times\{A^{K_1}_{b_1J_1}, V\}\cdots\{A_{a_{s-1}I_{s-1}K_{s-1}}, V\}\{A^{K_{s-1}}_{b_{s-1}J_{s-1}}, V\}\right). \quad (3.12)$$

At the quantum level, the connection is not a well-defined operator, thus we replace it by the holonomy. To this aim, first we observe that $\{A_a^{IJ}, V\}\tau_{IJ} = \{c\tau_{IJ}, V\}^o\Omega_a^{IJ} = -\frac{1}{\bar{\mu}}h_{IJ}\{h_{IJ}^{-1}, V\}^o\Omega_a^{IJ}$. Moreover, since we have the following identity:

$$\sqrt{{}^0q} = \det({}^o\Omega_a^{IJ}) = \frac{1}{2n!}\epsilon^{abca_1b_1\ldots a_{s-1}b_{s-1}}\epsilon^{IJKLI_1J_1\ldots I_{s-1}J_{s-1}} \times \left({}^o\Omega_{aIM}{}^o\Omega_{bJ}^M{}^o\Omega_{cKL}{}^o\Omega_{a_1I_1K_1} \times{}^o\Omega_{b_1J_1}^{K_1}\ldots{}^o\Omega_{a_{s-1}I_{s-1}K_{s-1}}{}^o\Omega_{b_{s-1}J_{s-1}}^{K_{s-1}}\right) \tag{3.13}$$

according our convention, the spatial integral of the above equation gives $\int d\Sigma\sqrt{{}^0q} = V_0 = 1$. Combining these facts with Eq. (3.8) and using the commutator to replace the Poisson bracket, we obtain the exact expression of the Hamiltonian constraint:

$$\begin{aligned}\hat{H}_{\rm gr} &= \frac{(-1)^{n-2}n(n-1)^{n-1}}{2(i\hbar)^{n-2}\kappa^{n-1}\gamma^n\bar{\mu}^n}\sin^2(\bar{\mu}c) \\ &\quad\times\left(\sin\left(\frac{\bar{\mu}c}{2}\right)\hat{V}\cos\left(\frac{\bar{\mu}c}{2}\right) - \cos\left(\frac{\bar{\mu}c}{2}\right)\hat{V}\sin\left(\frac{\bar{\mu}c}{2}\right)\right)^{n-2} \\ &= \sin(\bar{\mu}c)\hat{F}\sin(\bar{\mu}c)\end{aligned} \tag{3.14}$$

where the action of $\hat{F}$ on a quantum state $\Psi(v)$ is defined by

$$\begin{aligned}\hat{F}\Psi(v) &= -\frac{n\hbar}{2^n\gamma(\Delta_n)^{\frac{1}{n-1}}}|v|\left(|v-1|-|v+1|\right)^{n-2}\Psi(v) \\ &\equiv F(v)\Psi(v).\end{aligned} \tag{3.15}$$

Interestingly, when $n = 3$, the above Hamiltonian operator reads

$$\hat{H}_{\rm gr} = \frac{6i}{\hbar\kappa^2\gamma^3\bar{\mu}^3}\sin^2(\bar{\mu}c)\left(\sin\left(\frac{\bar{\mu}c}{2}\right)\hat{V}\cos\left(\frac{\bar{\mu}c}{2}\right) - \cos\left(\frac{\bar{\mu}c}{2}\right)\hat{V}\sin\left(\frac{\bar{\mu}c}{2}\right)\right), \tag{3.16}$$

which has exactly the same form as the 3 + 1 dimensional LQC Hamiltonian operator [17]. The action of the Hamiltonian operator $\hat{H}_{\rm gr}$ on a quantum state $\Psi(v) \in \mathcal{H}_{\rm kin}$ led to a similar difference equation as that in the case of the 3 + 1 dimensional LQC,

$$\begin{aligned}\hat{H}_{\rm gr}\Psi(v) &= f_+(v)\Psi(v+4) + f_0(v)\Psi(v) \\ &\quad+ f_-(v)\Psi(v-4)\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}f_+(v) &= -\frac{1}{4}F(v+2) \\ &= \frac{n\hbar}{2^{n+2}\gamma(\Delta_n)^{\frac{1}{n-1}}}(|v+2|)\left(|v+1|-|v+3|\right)^{n-2} \\ f_-(v) &= -\frac{1}{4}F(v-2) \\ f_0(v) &= \frac{1}{4}F(v+2) + \frac{1}{4}F(v-2).\end{aligned} \tag{3.18}$$

Now we turn to the inverse volume operator which appears in the matter field part. As such, we first define the quantity $|p|^{-1/2}$ in the following way:

$$|p|^{-1/2} = \mathrm{sgn}(p)\frac{4}{\kappa\gamma\bar{\mu}}\mathrm{Tr}\left(\sum_{IJ}\tau^{IJ}h_{IJ}\left\{h_{IJ}^{-1}, V^{\frac{n-1}{2n}}\right\}\right). \tag{3.19}$$

Note that under the replacement $\{,\} \to \frac{1}{i\hbar}[,]$, we have

$$\begin{aligned}&\mathrm{Tr}\left(\sum_{IJ}\tau^{IJ}h_{IJ}\left[h_{IJ}^{-1}, V^{\frac{n-1}{2n}}\right]\right) \\ &= \frac{n}{2}\left(\sin\left(\frac{\bar{\mu}c}{2}\right)V^{\frac{n-1}{2n}}\cos\left(\frac{\bar{\mu}c}{2}\right) - \cos\left(\frac{\bar{\mu}c}{2}\right)V^{\frac{n-1}{2n}}\sin\left(\frac{\bar{\mu}c}{2}\right)\right).\end{aligned} \tag{3.20}$$

Since in the classical level we have $V^{-1} = |p|^{-\frac{n}{n-1}}$, thus the action of the inverse volume operator on a quantum state $\Psi(v)$ is just a suitable power of Eq. (3.19), as follows:

$$\begin{aligned}\widehat{V^{-1}}\Psi(v) &= \left(\frac{2n}{\kappa\gamma\hbar(\Delta_n)^{\frac{1}{n-1}}}\right)^{\frac{2n}{n-1}}\left(\frac{\kappa\gamma\hbar(\Delta_n)^{\frac{1}{n-1}}}{2(n-1)}\right)^{\frac{n+1}{n-1}}v^{\frac{2}{n-1}} \\ &\quad\times\left||v+1|^{\frac{n-1}{2n}} - |v-1|^{\frac{n-1}{2n}}\right|^{\frac{2n}{n-1}}\Psi(v) \\ &= \frac{2(n-1)}{\kappa\gamma\hbar(\Delta_n)^{\frac{1}{n-1}}}\left(\frac{n}{n-1}\right)^{\frac{2n}{n-1}}v^{\frac{2}{n-1}} \\ &\quad\times\left||v+1|^{\frac{n-1}{2n}} - |v-1|^{\frac{n-1}{2n}}\right|^{\frac{2n}{n-1}}\Psi(v) \\ &:= B(v)\Psi(v).\end{aligned} \tag{3.21}$$

In the semiclassical region, namely in the large v region, the eigenvalue of the inverse volume operator $\widehat{V^{-1}}$ approaches its classical value and turns out to be

$$\left(\frac{2(n-1)}{\kappa\gamma\hbar(\Delta_n)^{\frac{1}{n-1}}}\right)\frac{1}{|v|}. \tag{3.22}$$

Collecting all the above ingredients, and noting that $\hat{p}_\phi\Psi(v,\phi) = -i\hbar\frac{\partial\Psi(v,\phi)}{\partial\phi}$, we finally obtain the full quantum Hamiltonian constraint

$$\hbar^2\frac{B(v)}{2}\frac{\partial^2\Psi(v,\phi)}{\partial^2\phi} = \hat{H}_{\rm gr}\Psi(v). \tag{3.23}$$

3.2 Odd dimensional sector

For the case of the spacetime dimensions $n+1$ being odd, we let $s = \frac{(n-2)}{2}$, and note that we have the following classical identity [22]:

$$\frac{\pi^{[a|IK}\pi_K^{b]J}}{\sqrt{h}} = \frac{1}{2(n-2)!}\epsilon^{aba_1b_1\ldots a_sb_s}\epsilon^{IJKI_1J_1\cdots I_sJ_s} = n_K\pi_{a_1I_1K_1}\pi^{K_1}_{b_1J_1}\cdots\pi_{a_sI_sK_s}\pi^{K_s}_{b_sJ_s}\sqrt{h}^{n-2} \quad (3.24)$$

where the n^I can be written in terms of π_{aIJ} as

$$n^I = \frac{1}{n!}\epsilon^{a_1b_1\ldots a_{s+1}b_{s+1}}\epsilon^{II_1J_1\ldots I_{s+1}J_{s+1}} = \pi_{a_1I_1K_1}\pi^{K_1}_{b_1J_1}\ldots\pi_{a_{s+1}I_{s+1}K_{s+1}}\pi^{K_{s+1}}_{b_{s+1}J_{s+1}}\sqrt{h}^{n-1}. \quad (3.25)$$

Thus we can rewrite the Hamiltonian constraint as follows:

$$\begin{aligned} H_{gr} &= -\frac{1}{2\kappa\gamma^2}\int \mathrm{d}\Sigma F_{abIJ}\frac{\pi^{[a|IK}\pi_K^{b]J}}{\sqrt{h}} \\ &= -\frac{1}{4(n-2)!\kappa^{n-1}\gamma^n} \\ &\quad\times\epsilon^{aba_1b_1\cdots a_sb_s}\epsilon^{IJKI_1J_1\cdots I_sJ_s}(n-1)^{n-2}\frac{1}{2^{n-2}} \\ &\quad\times\int \mathrm{d}\Sigma\Big(F_{abIJ}n_K\{A_{a_1I_1K_1},V\} \\ &\quad\times\{A^{K_1}_{b_1J_1},V\}\cdots\{A_{a_sI_sK_s},V\}\{A^{K_s}_{b_sJ_s},V\}\Big). \end{aligned} \quad (3.26)$$

Following the recipe prescribed in the last subsection, by replacing the connection by a holonomy and the Poisson bracket by the commutator we obtain the quantum Hamiltonian constraint operator,

$$\begin{aligned} \hat{H}_{\rm gr} &= \frac{n(n-1)^{4n-3}2^{2n-3}}{(2n-3)^{2n-2}(i\hbar)^{2n-2}\kappa^{2n-1}\gamma^{2n}\bar{\mu}^{2n}}\sin^2(\bar{\mu}c) \\ &\quad\times\left(\sin\left(\frac{\bar{\mu}c}{2}\right)V^{\frac{2n-3}{2n-2}}\cos\left(\frac{\bar{\mu}c}{2}\right)\right. \\ &\quad\left.-\cos\left(\frac{\bar{\mu}c}{2}\right)V^{\frac{2n-3}{2n-2}}\sin\left(\frac{\bar{\mu}c}{2}\right)\right)^{2n-2} \\ &= \sin(\bar{\mu}c)\hat{F}\sin(\bar{\mu}c) \end{aligned} \quad (3.27)$$

where the action of $\hat{F}$ on a quantum state $\Psi(v)$ is defined by

$$\begin{aligned} \hat{F}\Psi(v) &= -\frac{n\hbar(n-1)^{2n-2}}{4(2n-3)^{2n-2}\gamma(\Delta_n)^{\frac{1}{n-1}}}|v|^2 \\ &\quad\times\left(|v-1|^{\frac{2n-3}{2n-2}}-|v+1|^{\frac{2n-3}{2n-2}}\right)^{2n-2}\Psi(v) \\ &\equiv F(v)\Psi(v). \end{aligned} \quad (3.28)$$

This operator acts on a quantum state $\Psi(v)\in\mathcal{H}_{\rm kin}$, giving a difference equation

$$\hat{H}_{\rm gr}\Psi(v) = f_+(v)\Psi(v+4)+f_0(v)\Psi(v) + f_-(v)\Psi(v-4) \quad (3.29)$$

where

$$\begin{aligned} f_+(v) &= -\frac{1}{4}F(v+2) \\ &= \frac{n\hbar(n-1)^{2n-2}}{16(2n-3)^{2n-2}\gamma(\Delta_n)^{\frac{1}{n-1}}}|v+2|^2 \\ &\quad\times\left(|v+1|^{\frac{2n-3}{2n-2}}-|v+3|^{\frac{2n-3}{2n-2}}\right)^{2n-2} \\ f_-(v) &= -\frac{1}{4}F(v-2) \\ f_0(v) &= \frac{1}{4}F(v+2)+\frac{1}{4}F(v-2). \end{aligned} \quad (3.30)$$

The action of the inverse volume operator keeps the same form as in the even dimensional case. Therefore, the full quantum Hamiltonian constraint also turns out to be the following:

$$\hbar^2\frac{B(v)}{2}\frac{\partial^2\Psi(v,\phi)}{\partial^2\phi} = \hat{H}_{\rm gr}\Psi(v). \quad (3.31)$$

4 Singularity resolution

Now we come to deal with the issue of the singularity resolution. In order to proceed, we take the same strategy as adopted in [18]. To be more specific, we first make some reasonable simplifications on our quantum Hamiltonian constraint equation such that the whole dynamical system becomes simpler and exactly solvable. Then the discussion of the issue of singularity resolution will be made within this exactly solvable formalism [18]. We make the following replacements as in [18]:

$$B(v)\longmapsto\left(\frac{2(n-1)}{\kappa\gamma\hbar(\Delta_n)^{\frac{1}{n-1}}}\right)\frac{1}{|v|}$$

and

$$F(v)\longmapsto -\frac{n\hbar}{4\gamma(\Delta_n)^{\frac{1}{n-1}}}|v|.$$

The validity of the first replacement amounts to assuming $\mathcal{O}(\frac{1}{|v|})\ll 1$, which also in turn implies the second replacement.

In the corresponding quantum theory, the Hamiltonian constraint equation now reduces to

$$\begin{aligned} \frac{\partial^2\Psi(v)}{\partial\phi^2} &= -\hat{\Theta}\Psi(v) \\ &= \frac{n\kappa}{4(n-1)}v\sin(b)v\sin(b)\Psi(v) \\ &= \frac{n\kappa}{16(n-1)}v\left[(v+2)\Psi(v+4)-2v\Psi(v)\right. \\ &\quad\left.+(v-2)\Psi(v-4)\right] \end{aligned} \quad (4.1)$$

where we denote quantum state $\Psi(v) \equiv \Psi(v, \phi)$ for short. Equation (4.1) gives rise to a Klein–Gordon type equation. The physical state for the quantum dynamics of the $n+1$ dimensional LQC thus is given by the "positive frequency" square root of Eq. (4.1) as

$$\frac{\partial \Psi(v)}{\partial \phi} = i\sqrt{\Theta}\Psi(v). \tag{4.2}$$

Note that here exists a superselection ambiguity, namely, for any real number $\epsilon \in [0, 4)$ the states $\Psi(v)$ supported on points $v = 4k + \epsilon$ with k being an integer lead to the same dynamics. Thus as in [19] we just fix $\epsilon = 0$. Moreover, note that because the state $|0\rangle$ has zero norm, it is excluded out of the physical Hilbert space. The physical inner product between the two states reads

$$\langle \Psi_1, \Psi_2\rangle_{\text{phy}} := \frac{1}{\pi}\sum_{v=4k}\frac{1}{|v|}\bar{\Psi}_1(v)\Psi_2(v). \tag{4.3}$$

Note that (b, v) forms a canonical conjugate pair, thus the Fourier transform $\Psi(b)$ has a support on the interval $(0, \pi)$. Therefore the Fourier transformation and its corresponding inverse transformation are defined, respectively, as

$$\Psi(b) := \sum_{v=4k} e^{\frac{i}{2}vb}\Psi(v),$$
$$\Psi(v) = \frac{1}{\pi}\int_0^{\pi} e^{-\frac{i}{2}vb}\Psi(b)\mathrm{d}b. \tag{4.4}$$

Now we set $\chi(v) = \frac{1}{\pi v}\Psi(v)$, then the constraint equation (4.1) becomes a second-order differential equation

$$\frac{\partial^2\chi(b)}{\partial^2\phi} = \frac{n\kappa}{n-1}\sin^2(b)\frac{\partial^2\chi(b)}{\partial^2 b} \tag{4.5}$$

We define the following new variable x to make this equation simpler,

$$x = \sqrt{\frac{n-1}{n\kappa}}\ln\left(\tan\left(\frac{b}{2}\right)\right) \tag{4.6}$$

Then the constraint equation (4.5) becomes the standard Klein–Gordon type equation,

$$\frac{\partial^2\chi(b)}{\partial^2\phi} = \frac{\partial^2\chi(b)}{\partial^2 x}. \tag{4.7}$$

The physical Hilbert space is the span of positive frequency solutions to Eq. (4.7). This equation can be further simplified if we decompose the solution into left and right moving sectors as

$$\chi(x) = \chi_L(x_+) + \chi_R(x_-), \tag{4.8}$$

here $x_\pm = \phi \pm x$. In addition, $\chi(x)$ has the following symmetry:

$$\chi(-x) = -\chi(x). \tag{4.9}$$

This feature enables us to make a further decomposition

$$\chi(x) = \frac{1}{\sqrt{2}}\left(F(x_+) - F(x_-)\right), \tag{4.10}$$

where $F(x_\mp)$ by definition are negative/positive frequency solutions to Eq. (4.7). The physical inner product (4.3) now reads

$$\langle \chi_1, \chi_2\rangle_{\text{phy}} := i\int_{-\infty}^{\infty}\mathrm{d}x\left[\left(\frac{\partial \bar{F}_1(x_+)}{\partial x}\right)F_2(x_+) - \left(\frac{\partial \bar{F}_1(x_-)}{\partial x}\right)F_2(x_-)\right]. \tag{4.11}$$

Now the expectation value of the volume operator can be calculated as follows:

$$\begin{aligned}\langle \hat{V}\rangle|_\phi &= (\chi, \hat{V}|_\phi\chi)_{\text{phy}} = \frac{\hbar\kappa\gamma(\Delta_n)^{\frac{1}{n-1}}}{2(n-1)}(\chi, |\hat{v}|\chi)_{\text{phy}} \\ &= i\frac{\hbar\kappa\gamma(\Delta_n)^{\frac{1}{n-1}}}{2(n-1)}\int_{-\infty}^{\infty}\mathrm{d}x\left[\left(\frac{\partial\bar{F}(x_+)}{\partial x}\right)(\hat{v}F(x_+)) - \left(\frac{\partial\bar{F}(x_-)}{\partial x}\right)(-\hat{v}F(x_-))\right] \\ &= \frac{\hbar\kappa\gamma(\Delta_n)^{\frac{1}{n-1}}}{(n-1)\sqrt{\beta}}\int_{-\infty}^{\infty}\mathrm{d}x\left|\frac{\partial F}{\partial x}\right|^2\cosh(\sqrt{\beta}(x-\phi)) \\ &= V_+e^{\sqrt{\beta}\phi} + V_-e^{-\sqrt{\beta}\phi}\end{aligned} \tag{4.12}$$

where $\beta = \frac{n\kappa}{n-1}$ and

$$V_\pm = \frac{\hbar\kappa\gamma(\Delta_n)^{\frac{1}{n-1}}}{(n-1)\sqrt{\beta}}\int_{-\infty}^{\infty}\left|\frac{\partial F}{\partial x}\right|^2 e^{\mp\sqrt{\beta}x}\mathrm{d}x. \tag{4.13}$$

From Eq. (4.12), it is clear that the expectation value of $\hat{V}$ admits a nonzero minimum $V_{\min} = 2\sqrt{V_+V_-}$. This implies that all states undergo a big bounce rather than experience a singularity which has zero expectation of the volume operator. To justify this conclusion, let us turn to the matter density $\rho = \langle\rho|_{\phi_0}\rangle$. If our picture is right, this important physical observable should have an upper bound. The classical definition of the matter density reads $\rho = \frac{p_\phi^2}{2V^2}$ and we can see that the matter density will to infinity at the singularity point as the volume will go to zero. Thus as a comparison we calculate the expectation value of ρ, if the singularity is really resolved, the expectation value of matter density must have an upper bound. To this aim, we first need to know the matrix elements of the observable $\hat{p}_\phi$, which read

$$\frac{1}{\hbar}\langle F_1, \hat{p}_\phi F_2\rangle_{\text{phy}} = \int_{-\infty}^{\infty}\left(\frac{\partial\bar{F}_1(x)}{\partial x}\right)\frac{\partial F_2(x)}{\partial x}\mathrm{d}x. \tag{4.14}$$

Now we use a fixed state $\chi(x) = \frac{1}{\sqrt{2}}(F(x_+) - F(x_-))$ to calculate the expectation value of matter density at the moment of ϕ_0,

$$\langle\rho|_{\phi_0}\rangle = \frac{(\langle\hat{p}_\phi\rangle)^2}{2(\langle\hat{V}\rangle)^2} = \frac{(n-1)^2\beta\hbar^2}{2\hbar^2\kappa^2\gamma^2(\Delta_n)^{\frac{2}{n-1}}} \frac{\left[\int_{-\infty}^{\infty}\mathrm{d}x\left|\frac{\partial F}{\partial x}\right|^2\right]^2}{\left[\int_{-\infty}^{\infty}\mathrm{d}x\left|\frac{\partial F}{\partial x}\right|^2\cosh(\sqrt{\beta}x)\right]^2} \leq \frac{n(n-1)}{2\kappa\gamma^2(\Delta_n)^{\frac{2}{n-1}}} = \rho_c \tag{4.15}$$

where we use the fact $\cosh(\sqrt{\beta}x) \geq 1$ in the second line. An interesting fact is that in Sect. 5 we will find that this upper bound of the expectation value of matter density coincides with the critical matter density which comes from the effective Friedmann equation.

5 Effective Hamiltonian

One of the most delicate and valuable issues is the effective description of LQC, since it predicts the possible quantum gravity effects to low-energy physics. Both the canonical [24–27] and the path integral perspective [28–34] of the effective Hamiltonian of LQC has been studied.

With the Hamiltonian constraint equation (4.1) in hand, we now derive the effective Hamiltonian within the $n+1$ dimensional timeless path integral formalism. In the timeless path integral formalism, the dynamics is encoded into the transition amplitude, which equals the physical inner product [28–31], i.e.,

$$A(v_f,\phi_f;\ v_i,\phi_i) = \langle v_f,\phi_f|v_i,\phi_i\rangle_{\text{phy}} = \lim_{\alpha_o\to\infty}\int_{-\alpha_o}^{\alpha_o}\mathrm{d}\alpha\langle v_f,\phi_f|e^{i\alpha\hat{C}}|v_i,\phi_i\rangle, \tag{5.1}$$

where the subscripts i and f represent the initial and final states, and $\hat{C} \equiv \hat{\Theta}+\hat{p}_\phi^2/\hbar^2$. As shown in [31,32], by inserting some suitable complete basis and do multiple group averaging, Eq. (5.1) is equivalent to the calculation of

$$\langle v_f,\phi_f|e^{i\sum_{m=1}^{I}\epsilon\alpha_m\hat{C}}|v_i,\phi_i\rangle = \sum_{v_{I-1},\ldots v_1}\int\mathrm{d}\phi_{I-1}\cdots\mathrm{d}\phi_1 \times\prod_{m=1}^{I}\langle\phi_m|\langle v_m|e^{i\epsilon\alpha_m\hat{C}}|v_{m-1}\rangle|\phi_{m-1}\rangle. \tag{5.2}$$

Note that the action of the constraint equation is of the Klein–Gordon type, and thus its action on the gravitational part and the scalar field part can be calculated separately. So we first calculate the matter part and get

$$\langle\phi_m|e^{i\epsilon\alpha_m\frac{\hat{p}_\phi^2}{\hbar^2}}|\phi_{m-1}\rangle = \int\mathrm{d}p_{\phi_m}\langle\phi_m|p_{\phi_m}\rangle\langle p_{\phi_m}|e^{i\epsilon\alpha_m\frac{\hat{p}_\phi^2}{\hbar^2}}|\phi_{m-1}\rangle = \frac{1}{2\pi\hbar}\int\mathrm{d}p_{\phi_m}e^{i\epsilon(\frac{p_{\phi_m}}{\hbar}\frac{\phi_m-\varphi_{m-1}}{\epsilon}+\alpha_m\frac{p_{\phi_m}^2}{\hbar^2})}. \tag{5.3}$$

For the gravity part, we expand the exponential and neglect the higher-order terms thus getting

$$\int\mathrm{d}\phi_m\langle\phi_m|\langle v_m|e^{-i\epsilon\alpha_m\hat{\Theta}}|v_{m-1}\rangle|\phi_{m-1}\rangle = \delta_{v_m,v_{m-1}} -i\epsilon\alpha_m\int\mathrm{d}\phi_m\langle\phi_m|\langle v_m|\hat{\Theta}|v_{m-1}\rangle|\phi_{m-1}\rangle + \mathcal{O}(\epsilon^2). \tag{5.4}$$

By using Eq. (4.1), the matrix elements of $\langle\phi_m|\langle v_m|\hat{\Theta}|v_{m-1}\rangle|\phi_{m-1}\rangle$ can be evaluated as follows:

$$2\pi\hbar\int\mathrm{d}\phi_m\langle\phi_m|\langle v_m|\hat{\Theta}|v_{m-1}\rangle|\phi_{m-1}\rangle = \int\mathrm{d}\phi_m\mathrm{d}p_{\phi_m}e^{i\epsilon(\frac{p_{\phi_m}}{\hbar}\frac{\phi_m-\phi_{m-1}}{\epsilon})}\frac{n\kappa}{16(n-1)}v_{n-1} \times\frac{v_m+v_{m-1}}{2}(\delta_{v_m,v_{m-1}+4}-2\delta_{v_mn,v_{m-1}}+\delta_{v_m,v_{m-1}-4}).$$

By using the following identity:

$$\delta_{v_m,v_{m-1}+4}-2\delta_{v_m,v_{m-1}}+\delta_{v_m,v_{m-1}-4} = \frac{4}{\pi}\int_0^\pi\mathrm{d}b_m e^{-ib_m(v_m-v_{m-1})/2}\sin^2(b_m),$$

Eq. (5.4) can be rewritten in a compact form,

$$2\pi\hbar\int\mathrm{d}\phi_m\langle\phi_m|\langle v_m|e^{-i\epsilon\alpha_m\hat{\Theta}}|v_{m-1}\rangle|\phi_{m-1}\rangle = \int\mathrm{d}\phi_m\mathrm{d}p_{\phi_m}e^{i\epsilon(\frac{p_{\phi_m}}{\hbar}\frac{\phi_m-\phi_{m-1}}{\epsilon})}\frac{1}{\pi}\int_0^\pi\mathrm{d}b_n e^{-ib_m(v_m-v_{m-1})/2} \times\left[1-i\alpha_m\epsilon\frac{n\kappa}{16(n-1)}v_{m-1}\frac{v_m+v_{m-1}}{2}4\sin^2 b_m\right].$$

Combining all the above ingredients, the physical transition amplitude can be written as follows:

$$A(v_f,\phi_f;\ v_i,\phi_i) = \lim_{I\to\infty}\lim_{\alpha_{Io},\ldots,\alpha_{1o}\to\infty}\left(\epsilon\prod_{m=2}^{I}\frac{1}{2\alpha_{mo}}\right)\int_{-\alpha_{Io}}^{\alpha_{Io}}\mathrm{d}\alpha_I\cdots\int_{-\alpha_{1o}}^{\alpha_{1o}}\mathrm{d}\alpha_1 \times\int_{-\infty}^{\infty}\mathrm{d}\phi_{I-1}\cdots\mathrm{d}\phi_1\left(\frac{1}{2\pi\hbar}\right)^I\int_{-\infty}^{\infty}\mathrm{d}p_{\phi_I}\cdots\mathrm{d}p_{\phi_1} \times\sum_{v_{I-1},\ldots,v_1}\left(\frac{1}{\pi}\right)^I\int_0^\pi\mathrm{d}b_I\cdots\mathrm{d}b_1 \times\prod_{m=1}^{I}\exp i\epsilon\left[\frac{p_{\phi_m}}{\hbar}\frac{\phi_m-\phi_{m-1}}{\epsilon}-\frac{b_m}{2}\frac{v_m-v_{m-1}}{\epsilon} +\alpha_m\left(\frac{p_{\phi_m}^2}{\hbar^2}-\frac{n\kappa}{16(n-1)}v_{m-1}\frac{v_m+v_{m-1}}{2}4\sin^2 b_m\right)\right].$$

In the 'continuum limit', the final result of the transition amplitude reads

$$\begin{aligned}&A(v_f,\phi_f;\ v_i,\phi_i)\\&=c\int\mathcal{D}\alpha\int\mathcal{D}\phi\int\mathcal{D}p_\phi\int\mathcal{D}v\int\mathcal{D}b\\&\times\exp\left(\frac{i}{\hbar}\int_0^1 \mathrm{d}\tau\left[p_\phi\dot\phi-\frac{\hbar b}{2}\dot v\right.\right.\\&\left.\left.+\hbar\alpha\left(\frac{p_\phi^2}{\hbar^2}-\frac{n\kappa}{4(n-1)}v^2\sin b\right)\right]\right),\end{aligned}$$

where c is an overall constant which does not affect the dynamics. Hence, the effective Hamiltonian constraint in our $n+1$ dimensional LQC model can be simply read off as

$$\begin{aligned}C_{\mathrm{eff}}&=\frac{p_\phi^2}{\hbar^2}-\frac{n\kappa}{4(n-1)}v^2\sin^2 b\\&=\frac{p_\phi^2}{\hbar^2}-\frac{n(n-1)\Delta_n^2}{\hbar^2\kappa\gamma^2\bar\mu^{2n}}\sin^2(\bar\mu c).\end{aligned}\tag{5.5}$$

When we take the large scale limit which by definition is $\sin b\to b$ (or $\sin(\bar\mu c)\to\bar\mu c$ in (c,p) representation), we observed that the classical Hamiltonian constraint (3.5) is recovered from Eq. (5.5) up to a inverse volume factor $\frac{1}{|V|}$. The reason for this is simply that the Hamiltonian constraint in the previous sections describes the evolution in the proper time of isotropic observers. To consider this point, the factor $\frac{1}{|V|}$ has to be multiplied by C_{eff} to obtain the correct result. As a consequence, we finally find the physical effective Hamiltonian,

$$\begin{aligned}H_F&=\frac{1}{|v|}C_{\mathrm{eff}}=-\frac{n\hbar}{4\gamma(\Delta_n)^{\frac{1}{n-1}}}|v|\sin^2 b\\&+\frac{\hbar\kappa\gamma(\Delta_n)^{\frac{1}{n-1}}}{2(n-1)}|v|\rho,\end{aligned}\tag{5.6}$$

where the $n+1$ dimensional matter density by definition is $\rho=\frac{2(n-1)^2p_\phi^2}{v^2\hbar^2\kappa^2\gamma^2(\Delta_n)^{\frac{2}{n-1}}}$.

6 Effective equation

Now we are ready to derive the physical evolution equation of the $n+1$ dimensional Universe, and most important is of course the modified Friedmann equation. To this aim, we combine the effective Hamiltonian constraint H_F (5.6) with the symplectic structure of $n+1$ dimensional loop quantum cosmology. One can easily obtain the equations of motion for the volume v and the scalar field ϕ, respectively:

$$\dot v=\{v,H_F\}=\frac{n}{\gamma(\Delta_n)^{\frac{1}{n-1}}}|v|\sin(b)\cos(b),\tag{6.1}$$

$$\dot\phi=\{\phi,H_F\}=\frac{4(n-1)p_\phi}{\hbar\kappa\gamma(\Delta_n)^{\frac{1}{n-1}}|v|}.\tag{6.2}$$

By using Eq. (6.1), it is easy to see that

$$H^2=\left(\frac{\dot a}{a}\right)^2=\left(\frac{\dot v}{nv}\right)^2=\frac{1}{\gamma^2(\Delta_n)^{\frac{2}{n-1}}}\sin^2(b)\cos^2(b).\tag{6.3}$$

On the other hand, the effective Hamiltonian constraint $H_F=0$ can be rewritten in the following compact form:

$$\sin^2 b=\frac{2\kappa\gamma^2(\Delta_n)^{\frac{2}{n-1}}\rho}{n(n-1)}=\frac{\rho}{\rho_c}.\tag{6.4}$$

Here we define $\rho_c=\frac{n(n-1)}{2\kappa\gamma^2(\Delta_n)^{\frac{2}{n-1}}}$ as the $n+1$ dimensional critical matter density. As shown in Sect. 4, this ρ_c in fact is the upper bound of the matter density. With the help of this equation, the modified Friedmann equation reads

$$H^2=\left(\frac{\dot a}{a}\right)^2=\frac{2\kappa}{n(n-1)}\rho\left(1-\frac{\rho}{\rho_c}\right)\tag{6.5}$$

From Eq. (6.5), it is easy to see that when $\rho=\rho_c$, we have $\dot v=0$, which implies the existence of a quantum bounce at that point. To justify this, we calculate the second derivative of v at the point of $\rho=\rho_c$,

$$\ddot v|_{\rho=\rho_c}=\{\dot v,H_F\}|_{\rho=\rho_c}=\frac{n^2}{\gamma^2(\Delta_n)^{\frac{2}{n-1}}}|v|\neq 0.\tag{6.6}$$

Obviously, this implies a quantum bounce occurs at that point. Moreover, we combine the continuity equation (which is nothing but the equation of motion for scalar field ϕ) in $n+1$ dimensions,

$$\dot\rho+nH(\rho+p)=0,\tag{6.7}$$

with Eq. (6.5), we can easily obtain another dynamical equation of the $n+1$ dimensional Universe which is the so-called Raychaudhuri equation with loop quantum correction,

$$\frac{\ddot a}{a}=\frac{2\kappa}{n(n-1)}\rho\left(1-\frac{\rho}{\rho_c}\right)-\frac{\kappa}{n-1}(\rho+p)\left(1-\frac{2\rho}{\rho_c}\right).\tag{6.8}$$

7 Conclusion

In this paper, a detailed construction of the $n+1$ dimensional LQC is presented. We start from the classical connection dynamics of $n+1$ dimensional general relativity together with symmetry reduction procedures, and then, using the nonperturbative loop quantization method, we find that the dynamical evolution of the $n+1$ dimensional Universe is fully determined by a difference equation. Interestingly, in the quantum theory, the even dimensional sector and the odd dimensional sector exhibit qualitative different features. In order to obtain the effective equations of $n+1$ dimensional LQC which contain quantum corrections to the classical equations, we then generalize the timeless path integral formalism of LQC to the $n+1$ dimensional case and use it to derive the modified effective Hamiltonian of $n+1$ dimensional LQC. Based on this effective Hamiltonian, the Friedmann equation as well as the Raychaudhuri equation with loop quantum corrections are obtained. Our results indicate that the classical singularity is resolved by a quantum bounce in arbitrary spacetime dimensions. In addition, we find that the heuristic replacement $c \rightarrow \frac{\sin(\bar{\mu}c)}{\bar{\mu}}$ with $\bar{\mu} = (\frac{\Delta_n}{p})^{\frac{1}{n-1}}$ works not only for the $3+1$ dimensional case, but also for the more general dimensional case.

Our work offers possibilities to explore the issues of LQC with a number spacetime dimensions higher than 4. In particular, nowadays, higher dimensional cosmology has become a rather popular and active field. For example, by using the dimensional reduction method, the cosmic acceleration can be naturally explained by some 5 dimensional models [7]. Hence it is also very interesting to study these issues within our higher dimensional LQC formalism. Moreover, the results we developed in this paper lay a foundation for future phenomenological investigations of possible quantum gravity effects in higher dimensional quantum cosmology.

Another interesting topic is to derive the LQC directly from the LQG. In the $3+1$ dimensional case, some interesting efforts have been made toward this important direction [35–37]. It is quite interesting to discuss this topic in our $n+1$ dimensional LQC setting, and we leave all these interesting and delicate topics for future study.

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