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Entropy spectrum of (1 + 1) dimensional stringy black holes

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Abstract We explore the entropy spectrum of (1 + 1) dimensional dilatonic stringy black holes via the adiabatic invariant integral method known as Jiang and Han's method (Phys Lett B 718:584, 2012) and the Bohr–Sommerfeld quantization rule. It is found that the corresponding spectrum depends on black hole parameters like charge, ADM mass, and, more interestingly, on the dilatonic field. We calculate the entropy of the present black hole system via the Euclidean treatment of quantum gravity and study the thermodynamics of the black hole and find that the system does not undergo any phase transition.

1 Introduction

The fundamental difficulties in merging quantum theory with gravitation theory are well known. We know that the beauty of black hole thermodynamics lies in the powerful way it speaks of the unity of physics. To find answers to the difficulties, plenty of studies are going on by applying the known laws of quantum mechanics to general relativity. As a result of these studies, it is found that it is very difficult to study the systems in four dimensions. So extensive studies have been performed on lower dimensional gravity theories. String theory is one approach to quantum gravity. The developments in string theory have provided a good framework to consider the quantum properties of the black holes. The low energy string theory has several black hole solutions [1–3]. The study of lower dimensional black hole systems will help us to address many problems that arise in higher dimensional quantum gravity models. Hence several studies have been done in black hole thermodynamics of two dimensional gravity models. In addition, from these understandings as well as using the black hole/string correspondence principle one can understand the conceptual issues regarding the microscopic origin of black hole entropy.

In recent years the dilatonic black holes have received much attention, because it is widely believed that the investigations on this black hole solution will lead to an exact explanation of microscopic origin of black hole entropy (Bekenstein–Hawking entropy) [4–6]. Also there exist some conceptual problems regarding the end point of black hole evaporation through thermal radiation [8–10]. The two dimensional Einstein–Hilbert action is just a Gauss–Bonnet term, a topological invariant in the two dimensional space time. So the two dimensional models are locally trivial. Hence it is necessary to introduce some extra fields to this model. The best candidate for this is the dilaton field, which arises in the compactifications from higher dimensional models or from string theory. This theory also possesses different black hole solutions. These solutions play an important role in our understanding of quantum gravity. Now, as pointed out in [11], higher dimensional black holes in string theory can be related to two dimensional solutions through U-duality. Hence the higher dimensional black holes in string theory [2,12–15] are related to many two dimensional solutions including the two dimensional charged black holes of McGuigan et al. [3]. In the present study the two dimensional dilatonic black hole is considered which is analogous to the above mentioned two dimensional string black hole.

Bekenstein [16] proposed that the horizon area of a non-extremal black hole is an adiabatic invariant, quantities which vary very slowly compared to the variation of the external perturbation of the system, and the horizon area of a black hole is quantized in units of the Planck length. As a result of this, much attention has been given to the entropy spectrum quantization. Many methods have been put forward to calculate this entropy spectrum spacing. A first method in this direction was introduced by Hod [17,18]. This method relies on the quasinormal modes and the corresponding frequency. Hod employed Bohr's correspondence principle to quantize the entropy and found that in the asymptotic limit,

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it is related to the real part of quasinormal frequencies. Later, Kunstatter [19] derived an equally spaced entropy spectrum for Schwarzschild black hole. For this derivation, the quantity $I=\int \frac{dE}{\omega_R}$ was taken as the adiabatic invariant, where E is the energy and ω_R is the real part of the quasinormal frequency. Maggiore in 2007 [20] refined the idea of Hod by proving that the physical frequency of the quasinormal modes is determined by its real and imaginary parts. Recently Majhi and Vagenas [21] proposed a new method to quantize the entropy without using quasinormal modes. They used the idea of connecting an adiabatic invariant quantity to the Hamiltonian of the black hole, and they obtained an equally spaced entropy spectrum. Jiang and Han [22] modified this idea by pointing out the fact that the adiabatic invariant quantity $\int p_i\, dq_i$ used in Majhi and Vagenas's method is not canonically invariant. Therefore they made a modification by using the adiabatic invariant quantity of the covariant form $I=\oint p_i\, dq_i$. We adopt Jiang and Han's method to quantize the entropy of the two dimensional stringy black hole.

The organization of this paper is as follows. In Sect. 2, we introduce the two dimensional charged dilatonic black hole solution and discuss the thermodynamics of the corresponding black hole. In Sect. 3, we obtain the entropy of the black hole via a Euclidean treatment of quantum gravity and also the entropy spectrum and the corresponding spacing is studied. Our paper concludes with a short discussion of the results in Sect. 4.

2 Two dimensional stringy black hole and its thermodynamics

The action corresponding to Maxwell gravity coupled to a dilatonic field (Φ) can be described by [3]

$$S=\frac{1}{2\pi}\int d^2x\sqrt{-g}e^{-2\Phi}\left(R+4(\nabla\Phi)^2-\lambda-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{1}$$

in which R is the Ricci scalar, λ is the effective central charge, and $F_{\mu\nu}$ is the electromagnetic field tensor. The equations of motion corresponding to the metric, gauge, and dilaton fields are, respectively, given by

$$\begin{aligned}&R_{\mu\nu}-2\nabla_\mu\nabla_\nu\Phi-\frac{1}{2}F_{\mu\sigma}F^\sigma_\nu=0,\\&\nabla_\nu(e^{-2\Phi}F^{\mu\nu})=0,\\&R-4\nabla_\mu\nabla^\mu\Phi+4\nabla_\mu\Phi\nabla^\mu\Phi-\lambda-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}=0.\end{aligned} \tag{2}$$

The solution to the equation of motion leads to the two dimensional dilatonic black hole, whose metric is given by

$$ds^2=-f(r)dt^2+\frac{dr^2}{f(r)}, \tag{3}$$

with the metric function

$$f(r)=1-2me^{-Qr}+q^2e^{-2Qr}, \tag{4}$$

the dilaton field

$$\Phi=\Phi_0-\frac{Q}{2}r, \tag{5}$$

and the electromagnetic field tensor

$$F_{tr}=\sqrt{2}Qqe^{-Qr}. \tag{6}$$

The condition of asymptotic flatness for the spacetime requires $\lambda=-Q^2$. In these equations m and Q are proportional to black hole mass and black hole charge, respectively. Then the horizons are located at

$$r_\pm=\frac{1}{Q}\ln\left(m\pm\sqrt{m^2-q^2}\right). \tag{7}$$

From the above expression it is evident that, in order to have an event horizon at $r_\pm$, the condition $m^2-q^2\geq 0$ has to be satisfied. The solution given by (4) is analogous to the string theoretic black hole [1–3]. So we call this particular black hole solution a two dimensional charged "stringy" black hole.

Now we will investigate the thermodynamic aspects of a charged stringy black hole. From (4), and using the condition $f(r)=0$, at the horizon, we can deduce the mass of the black hole:

$$m=\frac{e^{Qr}+q^2e^{-Qr}}{2}. \tag{8}$$

The temperature of the two dimensional charged dilatonic black hole or stringy black hole can be derived using the relation

$$T_H=\frac{\kappa}{2\pi}, \tag{9}$$

where the surface gravity κ is defined by

$$\kappa=\frac{1}{2}\frac{\partial f(r)}{\partial r}\bigg|_{r=r_+}. \tag{10}$$

This yields the Hawking temperature

$$T_H=\frac{\lambda\sqrt{m^2-q^2}}{\pi(m+\sqrt{m^2-q^2})}, \tag{11}$$

and when the charge of the string becomes zero, the temperature will reduce to

$$T=\frac{\lambda}{2\pi}. \tag{12}$$

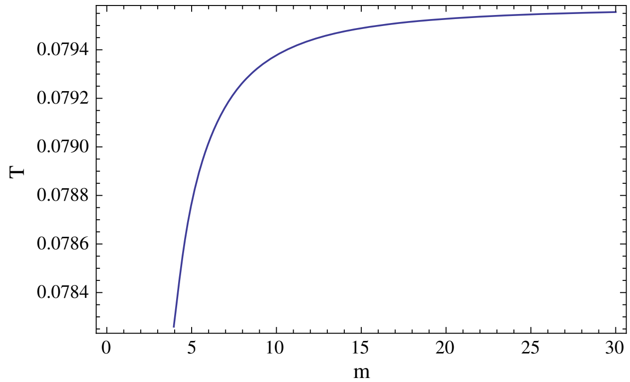

Fig. 1 Variation of temperature of the black hole with respect to the mass of the black hole in Planck units

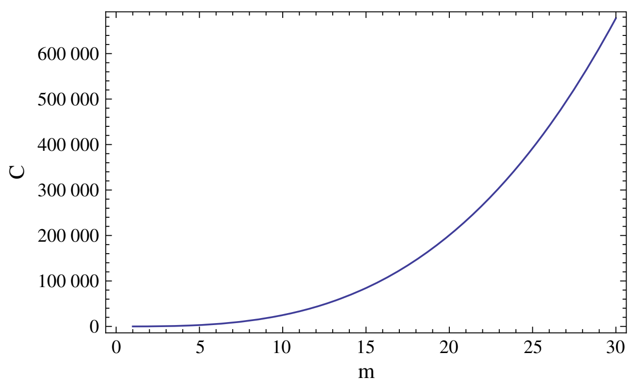

Fig. 2 Variation of the specific heat of the black hole with respect to the mass of the black hole in Planck units

The variation of the Hawking temperature with respect to the mass of the black hole is depicted in Fig. 1. From the thermodynamic relation $C = \frac{\partial m}{\partial T}$, one can arrive at the specific heat of the black hole:

$$C = \frac{2\pi(m\sqrt{m^2-q^2}+m^2-q^2)}{Q(m-\sqrt{m^2-q^2})}, \tag{13}$$

and the corresponding variation with respect to mass of the black hole is plotted in Fig. 2. From these two figures it is evident that the two dimensional dilatonic stringy black hole does not undergo any kind of phase transition. Ising solved the 1D Ising model in 1925 and found that there is no phase transition in 1D systems. We can see that these results match with the $(1+1)$ dimensional dilatonic stringy black hole system. This system also shows no phase transition. So this black hole behaves like a 1D Ising model system as far as thermodynamic behaviors are considered.

3 Entropy and entropy spectrum

As we mentioned in the introduction, the best way to study the thermodynamics of gravitational field is via the Euclidean treatment of quantum gravity. The partition function that explains the black hole thermodynamics is evaluated as the Euclidean path integral over the space of all field configurations with the saddle point approximation around the black hole solution [23]. It is found that the entropy of the black hole under consideration originates from the above mentioned Euclidean path integral calculations. By following similar arguments, one can perform an analysis to obtain the entropy of a two dimensional dilatonic stringy black hole. This analysis can begin by writing the partition function as

$$Z = \mathrm{e}^{I_{\mathrm{E}}} = \mathrm{e}^{\beta F}, \tag{14}$$

where I is the action evaluated for the Euclideanized gravitational field, β is the inverse temperature, and F is the Helmholtz free energy, $F = M - TS$. So this equation implies that the Euclidean action evaluated on the black hole system can be identified as the inverse temperature times the free energy of the system. Here the Euclidean action is given by

$$I_{\mathrm{E}} = \beta M - \frac{S}{\kappa_B} + \beta \sum_i \mu_i Q_i, \tag{15}$$

where μ_i's are the chemical potentials which correspond to Q_i's, which are the charges. From the Wick-rotated form of (1), the Euclidean minisuperspace action can be constructed. From this construction the corresponding black hole entropy can be calculated by comparing it with the partition function and it is given by

$$S = 4\pi \mathrm{e}^{-2\Phi_0}(m + \sqrt{m^2-q^2}). \tag{16}$$

Now it might also be possible to write down the ADM mass of the black hole as described in [1]; thus

$$M = 2mQ\mathrm{e}^{-2\Phi_0}. \tag{17}$$

This mass relation can also be derived by adopting the derivation of Arnowitt et al. [24]. It is also found that this ADM mass agrees with the thermodynamic evolution of energy. Now one arrives at an expression, where S is a function of the ADM mass (M) and the electric charge (Q_{el}) as follows:

$$S = \frac{2\pi}{Q}\left(M + \sqrt{M^2 - 2Q_{\mathrm{el}}^2}\right), \tag{18}$$

where the electric charge is given by

$$Q_{\mathrm{el}} = \sqrt{2}qQ\mathrm{e}^{-2\Phi_0} = \frac{q}{\sqrt{2}m}M. \tag{19}$$

Obviously one can expect a matching between this result and that of near extremal AdS_2 type black holes as considered in [3]. The AdS_2 type black hole solution is given by

$$g(r) = 1 - (m/\lambda)\mathrm{e}^{-2\lambda r} + (q^2/4\lambda^2)\mathrm{e}^{-4\lambda r}. \tag{20}$$

Now let us consider the near horizon limit of extremal black holes in the above equation using the condition $m = q$. In this case we can find the black hole entropy:

$$S = \frac{\sqrt{2}\pi q}{\lambda}. \tag{21}$$

Now it can be shown from (18) that the entropy of the flat dilatonic black hole agrees with the entropy (21) of the near extremal limit of AdS_2 type black hole solution [3,6,7].

Now we will quantize the entropy of the two dimensional dilatonic stringy black hole via the adiabatic invariance and Bohr–Sommerfeld quantization rule. According to the tunneling picture, when a particle tunnels in or out, the black hole horizon will oscillate due to the gain or loss of the black hole mass [25]. Such an oscillating horizon can be studied using the adiabatic invariant quantity,

$$I = \oint p_i \,\mathrm{d}q_i = \int_{q_i^{\mathrm{in}}}^{q_i^{\mathrm{out}}} p_i^{\mathrm{out}} \,\mathrm{d}q_i + \int_{q_i^{\mathrm{out}}}^{q_i^{\mathrm{in}}} p_i^{\mathrm{in}} \,\mathrm{d}q_i, \tag{22}$$

where p_i^{in} or p_i^{out} is the conjugate momentum corresponding to the coordinate q_i^{in} or q_i^{out}, respectively, and $i = 0, 1, 2$. For the horizon of a black hole $q_1^{\mathrm{in}} = r_{\mathrm{h}}^{\mathrm{in}} (q_1^{\mathrm{out}} = r_{\mathrm{h}}^{\mathrm{out}})$ and $q_0^{\mathrm{in}} (q_0^{\mathrm{out}}) = \tau$, where τ is the Euclidean time and r_{h} is the horizon radius. By implementing the Hamilton equation $\dot{q}_i = \frac{\mathrm{d}H}{\mathrm{d}p_i}$, where H is the Hamilton of the system, the integral can be rewritten

$$\begin{aligned} \int_{q_i^{\mathrm{out}}}^{q_i^{\mathrm{in}}} p_i^{\mathrm{in}} \,\mathrm{d}q_i &= \int_{\tau_{\mathrm{out}}}^{\tau_{\mathrm{in}}} \int_0^H \mathrm{d}H' \,\mathrm{d}\tau + \int_{r_{\mathrm{h}}^{\mathrm{out}}}^{r_{\mathrm{h}}^{\mathrm{in}}} \int_0^H \frac{\mathrm{d}H'}{\dot{r}_{\mathrm{h}}} \,\mathrm{d}r_{\mathrm{h}} \\ &= 2 \int_{r_{\mathrm{h}}^{\mathrm{out}}}^{r_{\mathrm{h}}^{\mathrm{in}}} \int_0^H \frac{\mathrm{d}H'}{\dot{r}_{\mathrm{h}}} \,\mathrm{d}r_{\mathrm{h}}, \end{aligned} \tag{23}$$

where $\dot{r}_h = \frac{\mathrm{d}r_h}{\mathrm{d}\tau}$ is the oscillating velocity of the black hole horizon. We know that the tunneling and oscillation take place at the same time. Hence we can write the relation connecting the black hole horizon oscillating velocity and velocity of the tunneling particle as [26]

$$\dot{r}_{\mathrm{h}} = -\dot{r}. \tag{24}$$

The metric given by (3) is Euclideanized by introducing the transformation $t \to -i\tau$. Let a photon travel across the black hole horizon, then the radial geodesic is given by

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = \pm i f(r), \tag{25}$$

where the $+$ and $-$ signs correspond to the outgoing and ingoing paths, respectively. Now the action given in (22) can be written

$$\oint p_i \,\mathrm{d}q_i = -4i \int_{r^{\mathrm{out}}}^{r^{\mathrm{in}}} \int_0^H \frac{\mathrm{d}H'}{f(r)} \,\mathrm{d}r. \tag{26}$$

Using the near horizon approximation, $f(r)$ can be Taylor expanded as

$$f(r) = f(r)_{\ r_{\mathrm{h}}} + (r - r_{\mathrm{h}}) \frac{\mathrm{d}f(r)}{\mathrm{d}r}_{\ r_{\mathrm{h}}} + \cdots. \tag{27}$$

We also know that at the horizon $r = r_{\mathrm{h}}$ there is a pole. To avoid this we consider a contour integral in such a way that the half loop is going above the pole from right to left. Evaluating the adiabatic invariant integral (26), using the Cauchy integral theorem, we arrive at

$$\oint p_i \,\mathrm{d}q_i = 4\pi \int_0^H \frac{\mathrm{d}H'}{\kappa} = 2\hbar \int_0^H \frac{\mathrm{d}H'}{T}, \tag{28}$$

where κ is the surface gravity of the black hole; it is related to the Hawking temperature by $T = \frac{\hbar\kappa}{2\pi}$.

We can write the Smarr formula of the two dimensional dilatonic stringy black hole as

$$\mathrm{d}M = \mathrm{d}H = T\,\mathrm{d}S - \phi\,\mathrm{d}Q, \tag{29}$$

where ϕ is the electrostatic potential. Then (28) becomes

$$\oint p_i \,\mathrm{d}q_i = 2\hbar \left(S - \frac{2\pi m}{Q} (1 - \cos\alpha) \right), \tag{30}$$

where $\alpha = \sin^{-1}(q/m)$. The Bohr–Sommerfeld quantization rule is given by

$$\oint p_i \,\mathrm{d}q_i = 2\pi n\hbar, \quad n = 1, 2, 3, \ldots. \tag{31}$$

Comparing the above two Eqs. (30) and (31), one can write the entropy spectrum

$$S = n\pi + \frac{2\pi m}{Q} (1 - \cos\alpha). \tag{32}$$

Using (17), we can rewrite the entropy spectrum in terms of the ADM mass of the black hole

$$S = n\pi + \frac{\pi}{Q^2} M \mathrm{e}^{2\Phi_0} (1 - \cos\alpha). \tag{33}$$

It is interesting to note that the entropy of the two dimensional dilatonic stringy black holes is quantized. From the above relation (33), it is evident that the entropy spectrum depends on the value of the dilatonic field at the horizon, as $S \propto \mathrm{e}^{2\Phi_0}$. Hence we may conclude that there is a background entropy due to this dilatonic field.

4 Results and discussion

In this work we have calculated the entropy of a $(1+1)$ dimensional dilatonic stringy black hole via the Euclidean treatment of quantum gravity. We have studied the thermodynamics of the black hole. By calculating the temperature and heat capacity of the black hole we conclude that the present black hole system does not undergo any kind of phase transition. This behavior matches with the one dimensional Ising model system in statistical studies [27]. We have investigated the quantization of the entropy of $(1+1)$ dimensional dilatonic stringy black holes via an adiabatic invariant integral method put forward by Majhi and Vagenas, as well as the Bohr–Sommerfeld quantization rule. We have found that the entropy spectrum is quantized and also that the entropy spectrum depends on the black hole parameters: the electric charge, the ADM mass, and the dilatonic field. This is supported by the area law in higher dimensional gravity theory. By considering the two dimensional dilaton theory as the dimensional reduction from higher dimensional theories, one can conclude that the dilaton field is associated with the radius of the compactified coordinates. At this point, it is interesting enough to recall the ideas of inflationary cosmology [28–30], which imply the presence of a scalar field that drives the inflation. Here we can match this scalar field with the dilaton field. The dependence of the entropy spectrum on dilatonic field points toward the microscopic origin of the black hole entropy and hence toward quantum gravity. This dilatonic field can also be considered as the source which drives the inflation in the context of inflationary cosmology. So further studies on these black hole solutions can unfold the mysteries regarding the inflationary stage of the universe and also the microscopic origin of the Bekenstein–Hawking entropy.

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