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Spaces of Whitney Functions with Basis

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Abstract. We construct a basis in the spaces of Whitney functions $\mathcal{E}(K)$ for two model cases, where $K \subset \mathbb{R}$ is a sequence of closed intervals tending to a point. In the proof we use a convolution property for the coefficients of scaling Chebyshev polynomials.

0. Introduction

The problem of the existence of bases in concrete spaces of functions is one of the most important parts of the structure theory of Fréchet spaces. It became more exciting after the Grothendieck problem was solved in the negative in [13], [2], [1], [8], [11]. Still there is no example of a concrete functional nuclear F – space without a basis. For a long time the space of C^∞ – functions on a sharp cusp has been considered as a candidate for this role ([2], see also [12]).

Here we give a construction of a basis in the space of Whitney functions $\mathcal{E}(K)$ for two model cases, where the compact $K \subset \mathbb{R}$ is a sequence of intervals tending to a point. The proof is based on a convolution property for the coefficients of scaling Chebyshev polynomials (Sect. 3). The method can be applied for the construction of a special basis in the space $C^\infty[0,1]$ and subsequently for the space of C^∞ – functions on a graduated sharp cusp ([5]). As a tool we use the Dynin – Mityagin criterion for the property of being a basis in a nuclear Fréchet space (T.1.1 below).

1. Preliminaries

Let $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$, where $I_k = [a_k, b_k] = [x_k - \delta_k, x_k + \delta_k]$. Let $h_k = a_k - b_{k+1}$, $b_1 \le 1$. Suppose that $a_k \downarrow 0$, $h_k \downarrow 0$ and $\delta_k \downarrow 0$.

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The topology in the space $\mathcal{E}(K)$ of Whitney functions is defined by the norms

$$\|f\|_p = |f|_p + \sup\left\{\frac{\left|(R_y^p f)^{(i)}(x)\right|}{|x-y|^{p-i}} : x, y \in K, x \neq y, i = 0, 1, \ldots, p\right\},$$

$p \in \mathbb{N}_0$, where $|f|_p = \sup\{\left|f^{(i)}(x)\right| : x \in K, i \leq p\}$ and $R_y^p f(x) = f(x) - T_y^p f(x)$ is the Taylor remainder. Let $\mathcal{E}_0(K)$ denote the subspace of $\mathcal{E}(K)$ consisting of functions which vanish at zero together with all their derivatives.

We will use the Chebyshev polynomials

$$T_N(x) = \cos(N \cdot \arccos x) = \sum_{s=0}^{N} A_s^{(N)} \cdot x^s,$$

where for $s = N - 2j$

$$A_s^{(N)} = (-1)^j N\, 2^{N-2j-1} \frac{(N-j-1)!}{j!(N-2j)!}, \quad j = 0, 1, \ldots, [N/2]$$

(see for instance [14], 6.10.6) and $A_s^{(N)} = 0$ if one of the numbers N, s is even and the other one is odd.

On the other hand, $\cos^p t = \sum_{n=0}^{p} B_n^{(p)} \cos nt$, where $0 \leq B_n^{(p)} \leq 1$.

For $|\Delta| \geq 1$, and $0 < \varepsilon \leq 1$ we have that

$$T_N(\Delta + \varepsilon \cos t) = \sum_{s=0}^{N} A_s^{(N)} \sum_{p=0}^{s} \binom{s}{p} \Delta^{s-p} \varepsilon^p \cos^p t = \sum_{n=0}^{N} \beta_n^{(N)}(\Delta, \varepsilon) \cos nt,$$

where

$$\beta_n^{(N)}(\Delta, \varepsilon) = \sum_{s=n}^{N} A_s^{(N)} \sum_{p=n}^{s} B_n^{(p)} \binom{s}{p} \Delta^{s-p} \varepsilon^p.$$

Since for $x \geq 1$

$$\sum_{s=0}^{N} \left|A_s^{(N)}\right| x^s = \sum_{j=0}^{[N/2]} (2x)^{N-2j} \frac{N}{2} \cdot \frac{(N-j-1)!}{j!(N-2j)!}$$

and $N/2 \leq N - j$, we have

$$\sum_{s=n}^{N} \left|A_s^{(N)}\right| x^s \leq \sum_{j=0}^{[N/2]} \binom{N-j}{j} (2x)^{N-2j} \leq \sum_{i=0}^{N} \binom{N}{i} (2x)^{N-i} = (2x+1)^N.$$

Therefore

$$\left|\beta_n^{(N)}(\Delta, \varepsilon)\right| \leq \varepsilon^n \sum_{s=n}^{N} \left|A_s^{(N)}\right| \sum_{p=0}^{s} \binom{s}{p} \left|\Delta^{s-p}\right| \leq \varepsilon^n (2|\Delta| + 3)^N. \tag{1.1}$$

As Mityagin proved in [7] the Chebyshev polynomials give a basis in the space $C^\infty[-1,1]$ and this space is isomorphic to the space s of rapidly decreasing sequences.

Let T_{nk} denote the restriction to I_k of the scaling Chebyshev polynomial, that is $T_{nk}(x) = T_n\left(\frac{x - x_k}{\delta_k}\right)$, $x \in I_k$ and $T_{nk} = 0$ for $x \in K \setminus I_k$.

Let $\xi_{0k}(f) = \frac{1}{\pi}\int_0^\pi f(x_k+\delta_k\cos t)\,dt,\ \xi_{nk}(f) = \frac{2}{\pi}\int_0^\pi f(x_k+\delta_k\cos t)\cos nt\,dt,\ n\in\mathbb{N}$. The functionals (ξ_{nk}) are, clearly, biorthogonal to (T_{nk}).

We will use the convention that $\sum_{i=m}^n = 0$ for $m > n$ and $0^0 = 1$.

Since the space $\mathcal{E}(K)$ is nuclear, we can use the following criterion ([7], T.9).

Theorem 1.1. (Dynin−Mityagin.) *Let E be a nuclear Fréchet space and $\{e_n \in E,\ \eta_n \in E',\ n \in \mathbb{N}\}$ be a biorthogonal system such that the set of functionals $(\eta_n)_{n=1}^\infty$ is total over E. Let for every p there exist q and C such that for all n*

$$\|e_n\|_p \cdot |\eta_n|_{-q} \ \le\ C\,. \tag{1.2}$$

Then the system $\{e_n, \eta_n\}$ is an absolute basis in E.

Here and subsequently, $|\cdot|_{-q}$ denotes the dual norm: for $\eta\in E'$ let $|\eta|_{-q} = \sup\{|\eta(f)|,\ \|f\|_q \le 1\}$.

2. Basis in the space $\mathcal{E}_0(K)$

This section contains a slightly modified version of the joint result [3]. For the convenience of the reader we repeat it here, thus making the exposition self−contained. In an analogous way a basis was constructed in [6] for the subspace of the space of C^∞−functions defined on a stepped sharp cusp, consisting of all the functions vanishing at the cusp.

For $f\in\mathcal{E}(K)$ let f_k be equal to f on I_k and zero on $K\setminus I_k$, $X_k = \{f\in\mathcal{E}_0(K) : \mathrm{supp} f\subset I_k\}$. Using Taylor expansion at zero of the corresponding extensions of the functions it is easy to obtain the following characterization for elements of the subspace $\mathcal{E}_0(K)$.

Lemma 2.1. *The function f from $\mathcal{E}(K)$ belongs to $\mathcal{E}_0(K)$ iff for every r and for every N there exists C_0 such that $|f_k|_r \le C_0\, b_k^N\, \|f\|_{r+N}$ for any k.*

Theorem 2.2. *Let K be a compact set as in the Preliminaries. If there exists M such that $h_k \ge b_k^M$ for any k, then the system $\{T_{nk},\xi_{nk}\}_{n=0,\,k=1}^{\infty,\,\infty}$ is a basis in the space $\mathcal{E}_0(K)$.*

Proof. Clearly, the system of functionals ξ_{nk} is total; thus we only need to check the condition (1.2). Fix $p\in\mathbb{N}_0$. Since $\left|T_n^{(j)}(x)\right| \le n^{2j}$ for $|x|\le 1,\ j\le n$, it follows that $|T_{nk}|_p \le (n^2/\delta_k)^u$, where $u=\min\{n,p\}$. On the other hand, for any $f_k\in X_k$ we get $\|f_k\|_p \le 4\,|f_k|_p\, h_k^{-p}$, as is easy to check. Thus $\|T_{nk}\|_p \le 4n^{2u}\delta_k^{-u}h_k^{-p}$.

Let us evaluate the dual norms of the ξ_{nk} as functionals on $C^\infty(I_k)$. Fix $r\in\mathbb{N}_0$, $f\in C^\infty(I_k)$. If $0<n\le r$ then using the Taylor expansion of f at x_k, we get

$$\xi_{nk}(f) \ =\ \frac{2}{\pi}\int_0^\pi f^{(n)}(\theta)\,\frac{\delta_k^n}{n!}\,\cos^n t\cdot\cos nt\,dt$$

with $\theta = \theta(t) \in I_k$. Hence

$$|\xi_{nk}(f)| \;\le\; 2\,\frac{\delta_k^n}{n!}\,|f|_r \;\le\; C'_r(\delta_k/n)^n\,|f|_r\,.$$

For $n = 0$ we replace the middle term in the last product by 1 and the bound is valid as well.

If $r < n$, then we can take the polynomial Q_{n-1} of best approximation to f on I_k in the norm $|\cdot|_0$. Then by the Jackson theorem

$$|\xi_{nk}(f)| \;\le\; \frac{2}{\pi}\int_0^\pi |f - Q_{n-1}|\,dt \;\le\; 2\,|f - Q_{n-1}|_0 \;\le\; C''_r\delta_k^r n^{-r}\,|f|_r$$

(see for instance [10], 5.1.5). Thus with $C_r = \max\{C'_r,\, C''_r\},\ v = \min\{n,\, r\}$ we have

$$|\xi_{nk}(f)| \;\le\; C_r(\delta_k/n)^v\,|f|_r\,. \tag{2.1}$$

Let now $r = 2p,\ q = (M+2)p,\ f \in \mathcal{E}_0(K)$. By the lemma,

$$|\xi_{nk}(f)| \;=\; |\xi_{nk}(f_k)| \;\le\; C_r(\delta_k/n)^v C_0\, b_k^{Mp}\,\|f\|_q\,.$$

Thus

$$|\xi_{nk}|_{-q} \;\le\; C_r C_0(\delta_k/n)^v h_k^p$$

and

$$\|T_{nk}\|_p\,|\xi_{nk}|_{-q} \;\le\; 4C_rC_0 n^{2u-v} \;\le\; 4C_rC_0(2p)^p\,.$$

This proves the theorem. □

Corollary 2.3. $\mathcal{E}_0(K) = \left(\bigoplus X_k\right)_s$.

Here and in the sequel $X = \left(\bigoplus X_k\right)_s$ means that every $f \in X$ has a unique representation $f = \sum_{k=1}^\infty f_k$ with $f_k \in X_k$ and moreover for any $p \in \mathbb{N}$ the sequence $(\|f_k\|_p)_{k=1}^\infty$ is rapidly decreasing.

Proof. For any $p,\, Q \in \mathbb{N}$ let $N = M(Q+p),\ q = p + N$. As before,

$$\|f_k\|_p \;\le\; 4\,|f_k|_p\, h_k^{-p} \;\le\; C_q\,\|f\|_q\, b_k^N h_k^{-p} \;\le\; C_q\,\|f\|_q\, h_k^Q\,.$$

Since $h_k \downarrow$ and $\sum h_k < \infty$, there exists a constant C such that $h_k \le Ck^{-1}$. Thus $\|f_k\|_p \le C_qC^Q\,\|f\|_q k^{-Q}$ and $(\|f_k\|_p) \in s$. □

3. Convolution property and a new biorthogonal system

Fix in an arbitrary way three (maybe overlapping) finite intervals $I_1,\, I_2,\, I_3 \subset \mathbb{R}$. Let $\widetilde{T_{ni}},\ i = 1,2,3,\ n \in \mathbb{N}_0$, be the corresponding scaling Chebyshev polynomials considered on $\mathbb{R},\ p,\, r \in \mathbb{N}_0,\ p \le r$. Then

$$\sum_{q=p}^{r} \xi_{p3}\big(\widetilde{T_{q2}}\big)\,\xi_{q2}\big(\widetilde{T_{r1}}\big) \;=\; \xi_{p3}\big(\widetilde{T_{r1}}\big)\,.$$

This property is a corollary of the following fact from elementary linear algebra.

Lemma 3.1. *Let* $(e_{i1})_{i=1}^n,\ (e_{i2})_{i=1}^n,\ (e_{i3})_{i=1}^n$ *be bases in an* n *– dimensional vector space,* $\alpha_{ik}(e_{jl})$ *be the* i *– th coefficient in the expansion of* e_{jl} *in the* k *– th basis. Then*

$$\sum_{q=1}^{n} \alpha_{p3}(e_{q2})\alpha_{q2}(e_{r1}) \;=\; \alpha_{p3}(e_{r1})\,.$$

Proof. The numbers $\alpha_{j3}(e_{q2})$ form the transition matrix $M_{3\leftarrow 2}$ from the second basis to the third. Here j is the index of the row, q is the index of the column. Analogously, $M_{2\leftarrow 1} = [\alpha_{i2}(e_{r1})]_{i,\,r=1}^{n,\,n}$. Thus in the sum above we see a product of the $p-$th row of $M_{3\leftarrow 2}$ with the $r-$th column of $M_{2\leftarrow 1}$, that is the $(p,r)-$th element of $M_{3\leftarrow 2}M_{2\leftarrow 1} = M_{3\leftarrow 1}$. □

Now if we apply this lemma to the Chebyshev bases (or arbitrary other scaling polynomials of increasing degree), then the terms with $q < p$ and $q > r$ vanish due to the orthogonality of ξ_{qk} to all polynomials of degree less than q.

Fix a compact set K as in the Preliminaries. For $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we will denote by P_{nk} the function equal to $\widetilde{T_{nk}}$ on $[0, b_k] \cap K$ and zero otherwise on K. Let $l : \mathbb{N} \to \mathbb{N}_0$ be a nondecreasing function. The concrete form of this function will depend on the compact set K considered. We introduce a new biorthogonal system which will be a basis in $\mathcal{E}(K)$ for two model cases.

Fix a natural number k. If $n \geq l(k)$, then let $e_{nk} = T_{nk},\ \eta_{nk} = \xi_{nk}$; for $n < l(k)$ let $e_{nk} = P_{nk}$,

$$\eta_{nk} \;=\; \xi_{nk} - \sum_{i=n}^{l(k-1)-1} \xi_{nk}(P_{i\,k-1})\,\xi_{i\,k-1}\,. \tag{3.1}$$

Lemma 3.2. *The system of functionals* $(\eta_{nk})_{n=0,\,k=1}^{\infty,\,\infty}$ *is total on* $\mathcal{E}(K)$ *and biorthogonal to* $(e_{nk})_{n=0,\,k=1}^{\infty,\,\infty}$.

Proof. Biorthogonality of $\{e, \eta\}$ can be easily checked from that of $\{T, \xi\}$ and from the convolution property.

Suppose that for some $f \in \mathcal{E}(K)$ we have $\eta_{nk}(f) = 0$ for all n, k. Since $\eta_{nk}(f) = \xi_{nk}(f) = \xi_{nk}(f_k) = 0$ for $n \geq l(k)$, we see that f_k is a polynomial on I_k of degree at most $l(k)$. Now let us take $k_0 = \min\{k : l(k) > 0\}$. Then $\eta_{nk_0} = \xi_{nk_0}$ and $f_{k_0} \equiv 0$. Next for $k = k_0 + 1,\ n < l(k)$ we obtain $\xi_{nk}(f_k) = \eta_{nk}(f) = 0$. Thus $f_k \equiv 0$. Continuing in this way we see that $f_k \equiv 0$ for any k and $f \equiv 0$. □

4. Estimation of norms

Let us first deduce some bounds for the norms of the elements and of the biorthogonal functionals for an arbitrary compact set K of the above–mentioned type (we can omit here the condition of monotonicity of $(h_k),\ (\delta_k)$).

Lemma 4.1. *Let* d *equal* $b_{k-1} - x_k$ *for* $n < q < l(k)$ *and* $d = \delta_{k-1}$ *for* $q \leq n < l(k)$. *Then for* $n < l(k)$

$$|\eta_{nk}|_{-q} \ \leq \ 4 \left[\delta_k^q + d^q \sum_{i=n}^{l(k-1)-1} |\xi_{nk}(P_{i\,k-1})| \right].$$

Proof. Let us remark that

$$\eta_{nk}(f) \ = \ \frac{2}{\pi} \int_0^\pi \left[f(x_k + \delta_k \cos t) \cos nt - f(x_{k-1} + \delta_{k-1} \cos t) \sum_{i=n}^{l-1} \xi(P) \cos it \right] dt\,.$$

Here and in the sequel we use the notation

$$\sum_{i=n}^{l-1} \xi(P) \ := \ \sum_{i=n}^{l(k-1)-1} \xi_{nk}(P_{i\,k-1})\,.$$

Suppose that $0 < n < q < l(k)$. The case $n = 0$ can be considered in the same manner with a change of the coefficient before the integral. Expanding both functions at x_k up to the $q-$th degree, we represent the expression in the square brackets in the following form

$$\begin{aligned}
[\cdots] \ = \ & \sum_{j=0}^{q-1} \frac{1}{j!}\, f^{(j)}(x_k) \delta_k^j \cos^j t \cos nt \\
& - \sum_{j=0}^{q-1} \frac{1}{j!}\, f^{(j)}(x_k)(x_{k-1} - x_k + \delta_{k-1} \cos t)^j \sum_{i=n}^{l-1} \xi(P) \cos it + \text{Remainder}\,,
\end{aligned}$$

which is equal to

$$\begin{aligned}
& \frac{1}{q!}\, f^{(q)}(\theta)\, \delta_k^q \cos^q t\, \cos nt - \big[1/q!\; f^{(q)}(x_k)(x_{k-1} - x_k + \delta_{k-1} \cos t)^q \\
& \qquad\qquad + R^q_{x_k} f(x_{k-1} + \delta_{k-1} \cos t)\big] \sum_{i=n}^{l-1} \xi(P) \cos it
\end{aligned}$$

with $\theta \in I_k$.

Let us show that the main part of the expansion will vanish after integration. By orthogonality we only need to consider the case $j \geq n$. We will compare the coefficients of $f^{(j)}(x_k)/j!$ in both sums after integration. Since

$$\frac{2}{\pi} \int_0^\pi \cos^j t \cos nt\, dt \ = \ B_n^{(j)}\,,$$

the coefficient in the first sum equals $\delta_k^j\, B_n^{(j)}$.

For the second sum, due to the orthogonality, the corresponding coefficient is

$$\sum_{m=n}^{j} \binom{j}{m} (x_{k-1} - x_k)^{j-m} \delta_{k-1}^m \left[\cos^m t \sum_{i=n}^{m} \xi(P) \cos it \right].$$

The expression in the square brackets after integration gives

$$\sum_{i=n}^{m} \xi(P)\, B_i^{(m)}\,. \tag{4.1}$$

Thus it remains to prove that

$$\delta_k^j\, B_n^{(j)} \;=\; \sum_{m=n}^{j} \binom{j}{m} (x_{k-1}-x_k)^{j-m} \delta_{k-1}^m \sum_{i=n}^{m} \xi(P) B_i^{(m)}\,. \tag{4.2}$$

Let us consider the sum (4.1) separately. Clearly,

$$\xi_{nk}(P_{i\,k-1}) \;=\; \beta_n^{(i)}(\Delta,\varepsilon) \;=\; \sum_{s=n}^{i} A_s^{(i)} \sum_{p=n}^{s} B_n^{(p)} \binom{s}{p} \Delta^{s-p} \varepsilon^p$$

with $\Delta = \frac{x_k - x_{k-1}}{\delta_{k-1}}$, $\varepsilon = \frac{\delta_k}{\delta_{k-1}}$. Changing the order of summation, we represent (4.1) as

$$\sum_{s=n}^{m} \sum_{p=n}^{s} B_n^{(p)} \binom{s}{p} \Delta^{s-p} \varepsilon^p \sum_{i=s}^{m} B_i^{(m)} A_s^{(i)}\,.$$

But the last sum here is the coefficient of $\cos^s t$ in the expansion of $\cos^m t$ in powers of $\cos t$, that is $\sum_{i=s}^m B_i^{(m)} A_s^{(i)} = 1$ if $s = m$ and it is zero for $s < m$.

Hence

$$\sum_{i=n}^{m} \xi(P) B_i^{(m)} \;=\; \sum_{p=n}^{m} B_n^{(p)} \binom{m}{p} \Delta^{m-p} \varepsilon^p\,.$$

Therefore the right−hand side of (4.2) can be written as

$$\sum_{m=n}^{j} \binom{j}{m} (-\Delta\delta_{k-1})^{j-m} \delta_{k-1}^m \sum_{p=n}^{m} B_n^{(p)} \binom{m}{p} \Delta^{m-p} \varepsilon^p$$

$$= \sum_{p=n}^{j} B_n^{(p)} \varepsilon^p \delta_{k-1}^j \Delta^{j-p} \sum_{m=p}^{j} \binom{j}{m} \binom{m}{p} (-1)^{j-m}\,.$$

Since $\binom{j}{m}\binom{m}{p} = \binom{j}{p}\binom{j-p}{m-p}$ the last sum here is

$$\binom{j}{p} \sum_{m=p}^{j} \binom{j-p}{m-p} (-1)^{j-m} 1^{m-p} \;=\; 0$$

for $p < j$. Thus we have only the case $p = j$ and the total sum is

$$B_n^{(j)} \varepsilon^j \delta_{k-1}^j \;=\; B_n^{(j)} \delta_k^j$$

and (4.2) is proved. Therefore,

$$|\eta_{nk}(f)| \;\le\; \frac{2}{\pi} \int_0^\pi |\text{Remainder}|\, dt$$

$$\le\; \frac{2}{q!}\, |f|_q \delta_k^q + (b_{k-1} - x_k)^q \left(\frac{2}{q!}\, |f|_q + 2\|f\|_q \right) \sum_{i=n}^{l-1} |\xi(P)|\,.$$

This establishes the formula for the first case.

Let now $q \le n < l(k)$. Then expanding up to degree q the first function at x_k and the second at x_{k-1}, we immediately have

$$\eta_{nk}(f) = \frac{2}{\pi}\int_0^\pi f^{(q)}(\theta_0)\delta_k^q \cos^q t \cos nt\, dt - \sum_{i=n}^{l-1} \xi(P)\cdot \frac{2}{\pi}\int_0^\pi f^{(q)}(\theta_1)\delta_{k-1}^q \cos^q t \cos it\, dt$$

with $\theta_j \in I_{k-j},\ j = 0, 1$. This proves the lemma. □

We now turn to the elements P_{nk}. To simplify notation we write $R(f, p)$ instead of

$$\sup\left\{\left|(R_y^p f)^{(i)}(x)\right| \cdot |x-y|^{i-p},\ i \le p,\ x, y \in K,\ x \ne y\right\}.$$

Let as before denote $u = \min\{n, p\}$.

Lemma 4.2. *The following estimates hold*

$$|P_{nk}|_p \ \le\ 2^{n-1}\frac{n!}{(n-u)!}\,\delta_k^{-n}\, b_k^{n-u}\,, \qquad R(P_{nk}, p)\ \le\ e\cdot h_{k-1}^{-p}\,|P_{nk}|_u\,.$$

Proof. Let us write the function P_{nk} in the form

$$P_{nk}(x) \ =\ 2^{n-1}\delta_k^{-n}\prod_{i=1}^n (x-\theta_i)\,,$$

where $\theta_i \in I_k,\ x \in [0, b_k] \cap K$. For $j \le n$ the $j-$th derivative of $\prod_{i=1}^n (x-\theta_i)$ is a sum of $\frac{n!}{(n-j)!}$ terms of the type $(x-\theta_{i_1})\cdots(x-\theta_{i_{n-j}})$. Since $|x-\theta_i| < b_k$ for all i, we obtain the first bound of the lemma. Now if $x,\ y \in [0, b_k] \cap K$ and $p < n$, then $R(P_{nk}, p) \le 2\,|P_{nk}|_p$ by the Lagrangian form for Taylor's remainder. If $p \ge n$, then it is zero. Suppose that x and y lie on different sides of the hole h_{k-1}. Let for instance $y \le b_k$. Then $|x-y| \ge h_{k-1}$ and

$$R(P_{nk}, p)\ \le\ \sup_i \sum_{j=i}^u \frac{1}{(j-i)!}\, h_{k-1}^{j-p} \sup_y \left|P_{nk}^{(j)}(y)\right|$$

and the second bound of the lemma is clear. □

5. Basis in $\mathcal{E}(K)$ for $\mathcal{E}(K) \not\simeq s$

Here we consider a concrete compact set K. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an arbitrary increasing function such that $\varphi(t) \ge t$, let $\delta_{k+1} = 1/\varphi\big(\delta_k^{-1}\big)$ and $I_k = [(b-2)\cdot\delta_k, b\cdot\delta_k]$.

If $t^N/\varphi(t) \to 0$ for all N as $t \to \infty$, then the spaces $\mathcal{E}(K)$ and s are not isomorphic (see T.3 in [4]). Moreover, one can easily construct a family, having the cardinality of the continuum, of pairwise nonisomorphic spaces $\mathcal{E}(K_\alpha)$ by choosing suitable scales of functions φ_α (see [4], T.1 for more details).

Theorem 5.1. *Suppose $b \geq 6$. Let the sequence $(\delta_k) \downarrow 0$ be such that $b\delta_1 \leq 1$, $3\delta_{k+1} \leq \delta_k$ for all k, and $K = \{0\} \cup \bigcup_{k=1}^{\infty}[(b-2)\delta_k, b\delta_k]$. Then the system $\{e_{nk}, \eta_{nk}\}_{n=0,\, k=1}^{\infty,\, \infty}$ is a basis in the space $\mathcal{E}(K)$.*

Proof. Here $h_k = (b-2)\delta_k - b\delta_{k+1}$. We check at once that $h_k \geq h_{k+1}$ and therefore the compact set K satisfies the conditions of the Preliminaries. Moreover,

$$h_k \;\geq\; 2\delta_k\,, \quad h_k \;\geq\; b_{k+1}$$

and we can apply [4] for the isomorphic classification of the space $\mathcal{E}(K)$.

Let us fix $l = l(k)$ such that

$$(5.1) \qquad (2b+1)^{l(k)} \cdot \delta_{k-1} \;\leq\; 1\,.$$

In order to get this, one can take $l = [(k-2)\nu]$ with $\nu = \frac{\ln 3}{\ln(2b+1)}$, because

$$(2b+1)^l \delta_{k-1} \;\leq\; (2b+1)^{\nu(k-2)} 3^{-k+2}\delta_1 \;=\; \delta_1 \;<\; 1\,.$$

In addition for this l we can take k_0 such that

$$(5.2) \qquad l(k) \;\leq\; \delta_{k-1}^{-1}\,, \quad k \;>\; k_0\,.$$

Since $\xi_{nk}(P_{i\,k-1}) = \beta_n^{(i)}(\Delta, \varepsilon)$ with $|\Delta| = \left|\frac{x_k - x_{k-1}}{\delta_{k-1}}\right| < b-1,\ \varepsilon = \frac{\delta_k}{\delta_{k-1}}$ we conclude from (1.1) and (5.1) that

$$(5.3) \qquad \sum_{i=n}^{l-1} |\xi(P)| \;\leq\; \sum_{i=n}^{l(k-1)-1} \varepsilon^n (2b+1)^i \;\leq\; \varepsilon^n (2b+1)^{l(k)} \;\leq\; \delta_k^n \delta_{k-1}^{-n-1}\,.$$

Fix $p \in \mathbb{N},\ q = 3p+2$ and k_q with $k_q \geq k_0,\ l(k_q) \geq q$. Let $C_0 = \max \|e_{nk}\|_p \cdot |\eta_{nk}|_{-q}$ for $0 \leq n \leq q,\ 1 \leq k \leq k_q$.

If $k \leq k_q,\ n > q$ or $k > k_q$ and $n \geq l(k)$, then $e_{nk} = T_{nk},\ \eta_{nk} = \xi_{nk}$ due to the choice of $l(k)$ and k_q. As in Theorem 2.2 we have the bound

$$|T_{nk}|_p \;\leq\; n^{2p} \cdot \delta_k^{-p}\,.$$

Arguing as in Lemma 4.2, we get for $y \in I_k,\ x \in K \setminus I_k$

$$\sum_{j=i}^{p} \frac{1}{(j-i)!} \left|T_{nk}^{(j)}(y)\right| |x-y|^{j-p} \;\leq\; \sum_{j=i}^{p} \frac{1}{(j-i)!}\, n^{2j} \delta_k^{-j} h_k^{j-p} \;\leq\; e \cdot n^{2p} \delta_k^{-p}\,,$$

as $\delta_k \leq h_k$. Thus $\|T_{nk}\|_p \leq 4 \cdot n^{2p} \delta_k^{-p}$.

For biorthogonal functionals it is enough in this case to use (2.1) with q instead of r

$$|\xi_{nk}|_{-q} \;\leq\; C_q (\delta_k/n)^q\,,$$

as $n \geq q$. Therefore, $\|T_{nk}\|_p \cdot |\xi_{nk}|_{-q} \leq 4C_q$.

It remains to analyze the case $k > k_q,\ 0 \leq n < l(k)$. Here $e_{nk} = P_{nk},\ \eta_{nk}$ is defined by (3.1). Fix k.

If $0 \le n < p$, then by Lemma 4.2

$$\|P_{nk}\|_p \;\le\; C_1' \delta_k^{-n}\, \delta_{k-1}^{-p}\,,$$

where C_1' does not depend on k and n.

On the other hand, by Lemma 4.1 and (5.3)

$$|\eta_{nk}|_{-q} \;\le\; 4\big[\delta_k^q + (b\cdot\delta_{k-1})^q \cdot \delta_k^n \delta_{k-1}^{-n-1}\big]\,.$$

Therefore,

$$\|P_{nk}\|_p\,|\eta_{nk}|_{-q} \;\le\; 4\,C_1'\big[\delta_k^{q-n}\delta_{k-1}^{-p} + b^q\cdot \delta_{k-1}^{q-n-p-1}\big] \;\le\; 4C_1'(1+b^q) \;=\; C_1\,.$$

If $p \le n < q$, then with the same bound for $|\eta_{nk}|_{-q}$ we have by Lemma 4.2

$$|P_{nk}|_p \;\le\; 2^{n-1} n^p \delta_k^{-n} (b\delta_k)^{n-p}\,, \quad R(P_{nk},p) \;\le\; e(2\delta_{k-1})^{-p}\,|P_{nk}|_p\,.$$

Thus $\|P_{nk}\|_p \le C_2' \cdot \delta_k^{-p}\delta_{k-1}^{-p}$ and

$$\|P_{nk}\|_p\,|\eta_{nk}|_{-q} \;\le\; 4C_2'\big[\delta_k^{q-p}\delta_{k-1}^{-p} + b^q(\delta_k/\delta_{k-1})^{n-p}\delta_{k-1}^{q-2p-1}\big] \;\le\; 4C_2'(1+b^q) \;=\; C_2\,.$$

Suppose $q \le n < l(k)$. Then

$$|P_{nk}|_p \;\le\; \frac{1}{2}\,(2b)^n n^p \delta_k^{-p}\,, \quad R(P_{nk},p) \;\le\; e(2\delta_{k-1})^{-p}\,|P_{nk}|_p\,.$$

Therefore,

$$\|P_{nk}\|_p \;\le\; 2(2b)^l l^p \delta_k^{-p}\delta_{k-1}^{-p} \;\le\; 2\delta_k^{-p}\delta_{k-1}^{-2p-1}\,,$$

by (5.1) and (5.2). Also, by Lemma 4.1 and (5.3)

$$|\eta_{nk}|_{-q} \;\le\; 4\big[\delta_k^q + \delta_{k-1}^q \cdot \delta_k^n \delta_{k-1}^{-n-1}\big]$$

and

$$\|P_{nk}\|_p\,|\eta_{nk}|_{-q} \;\le\; 8\big[\delta_k^{q-p}\delta_{k-1}^{-2p-1} + (\delta_k/\delta_{k-1})^{n-p}\delta_{k-1}^{q-3p-2}\big] \;\le\; 8 \;=\; C_3\,,$$

due to the choice of q. The constant $C = \max_{i\le 3} C_i$ does not depend on n, k, hence the theorem follows from Theorem 1.1 and Lemma 3.2. □

Now let us introduce the projections

$$S_k \;=\; \sum_{j=1}^{k-1}\sum_{n=0}^{l(j)-1} \eta_{nj}(\,\cdot\,)\,P_{nj}\,, \quad Q_k \;=\; \sum_{n=0}^{\infty} \eta_{nk}(\,\cdot\,)e_{nk}$$

in the space $\mathcal{E}(K)$. Clearly, $Q_k(f) = f - S_k(f)$ on I_k, $Q_k(f)$ is a polynomial of degree $l(k)-1$ on $[0,b_{k+1}]\cap K$ and $Q_k(f)=0$ otherwise on K. Let $X_k = Q_k(\mathcal{E}(K))$.

Corollary 5.2. $\mathcal{E}(K) = \big(\bigoplus X_k\big)_s$.

Proof. In fact, it is enough to show that for all p and for all M there exist q and C such that

$$\sum_{k=1}^{\infty} \|Q_k(f)\|_p \cdot k^M \ \le\ C\,\|f\|_q\,.$$

We can repeat all the arguments of the theorem with $q = 3(p+1)+M$ and show in this way that the double series

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty} |\eta_{nk}(f)|\cdot\|e_{nk}\|_p k^M$$

is convergent. □

6. Basis in $\mathcal{E}(K)$ for $\mathcal{E}(K) \simeq s$

Let now $b_k = 2^{-k+1} = 2b_{k+1},\ \delta_k = 2^{-k-2} = 2\delta_{k+1}$ for $k \in \mathbb{N}$. Clearly, $a_k = 6\delta_k,\ h_k = 2\delta_k$. From [9] and [4] it follows that the spaces $\mathcal{E}(K)$ and s are isomorphic. Let us give an explicit form of the basis in $\mathcal{E}(K)$ which can be applied for the construction of a special basis in the space $C^\infty[0,1]$.

Theorem 6.1. *Let* $K = \{0\} \cup \bigcup_{k=1}^{\infty}\left[3\cdot 2^{-k-1}, 2^{-k+1}\right]$. *Then the system* $\{e_{nk}, \eta_{nk}\}_{n=0,\,k=1}^{\infty,\,\infty}$ *is a basis in the space* $\mathcal{E}(K)$.

Proof. Let us take $l(k) = [k/4]$. Since for our case $\Delta = -7/2,\ \varepsilon < 1$, we replace (5.3) by

$$\sum_{i=n}^{l-1} |\xi(P)| \ \le \sum_{i=n}^{l(k-1)-1} 10^i \ <\ 10^{l(k-1)}\ <\ \delta_k^{-1}\,. \tag{6.1}$$

Fix $p \in \mathbb{N}$, take $q = 3p+2$ and $k_q = 4q$.

Let C_0 be the same as in Theorem 5.1. The estimates of $\|e_{nk}\|_p \cdot |\eta_{nk}|_{-q}$ for $n \ge l(k)$ or $n < l(k),\ k \le k_q$ are the same as above. Similarly, for fixed $k > k_q$ if $n < p$ then

$$|P_{nk}|_p \ \le\ 2^{n-1} n!\,\delta_k^{-n}\,, \quad R(P_{nk}, p)\ \le\ e\cdot(4\delta_k)^{-p}\,|P_{nk}|_n\,.$$

Therefore

$$\|P_{nk}\|_p \ \le\ p!\,\delta_k^{-2p}\,.$$

By Lemma 4.1 and (6.1)

$$|\eta_{nk}|_{-q} \ \le\ 4\big[\delta_k^q + (b_{k-1}-x_k)^q\cdot\delta_k^{-1}\big]\,.$$

Here $b_{k-1} - x_k = 8\delta_{k-1} - 7\delta_k = 9\delta_k$. Hence, $|\eta_{nk}|_{-q} \le C\delta_k^{q-1}$, where C does not depend on n, k, and the product $\|P_{nk}\|_p \cdot |\eta_{nk}|_{-q}$ is uniformly bounded.

If $p \le n < q$, then $\|P_{nk}\|_p \le 2^{4q} q^p \delta_k^{-2p}$, as is easy to check. For $|\eta_{nk}|_{-q}$ we use the previous bound and obtain the desired conclusion.

Suppose now that $q \le n$. Then by Lemma 4.2

$$|P_{nk}|_p \ \le\ 2^{n-1} n^p \delta_k^{-n} (8\delta_k)^{n-p}$$

and

$$\|P_{nk}\|_p \ \le\ (1+e)(4\delta_k)^{-p}\, |P_{nk}|_p \ \le\ 2^{4n+1} n^p \delta_k^{-2p}\,.$$

Since $n < l(k) \le k/4$ and $k < \delta_k^{-1}$, we have

$$2^{4n+1} n^p \ \le\ 2 \cdot 2^k k^p \ <\ \delta_k^{-p-1}\,.$$

Thus $\|P_{nk}\|_p \le \delta_k^{-3p-1}$.

From Lemma 4.1 with $\delta_{k-1} = 2\delta_k$ and (6.1) it follws that

$$|\eta_{nk}|_{-q} \ \le\ 4\big[\delta_k^q + (2\delta_k)^q \cdot \delta_k^{-1}\big] \ \le\ 2^{q+3}\delta_k^{q-1}\,.$$

Therefore $\|P_{nk}\|_p \cdot |\eta_{nk}|_{-q} \le 2^{q+3}$ due to the choice of q.

Thus for the system $\{e_{nk}, \eta_{nk}\}_{n=0,\, k=1}^{\infty,\, \infty}$ we have the Dynin–Mityagin estimate (1.2) and in view of Lemma 3.2 the proof is complete. □

In the same manner as above we can obtain the following

Corollary 6.2. $\mathcal{E}(K) = \big(\bigoplus X_k\big)_s$.

Remark 6.3. The basis in $\mathcal{E}(K)$ cannot be constructed as an extension of the basis in the subspace of the functions vanishing at zero. In fact, $\mathcal{E}_0(K)$ is not complemented in $\mathcal{E}(K)$ because the quotient space $\mathcal{E}(K)/\mathcal{E}_0(K)$ is isomorphic to the space $\omega = \mathbb{R}^{\mathbb{N}}$ and does not have a continuous norm.

In turn if we take the basis projection

$$Q_0 \ =\ \sum_{k=1}^{\infty} \sum_{n=l(k)}^{\infty} \xi_{nk}(\,\cdot\,) T_{nk}\,,$$

on the "vanishing part" $X_0 = Q_0(\mathcal{E}(K))$ of the space $\mathcal{E}(K)$ with $X_1 = (I - Q_0)(\mathcal{E}(K))$, then $X_0 \subset \mathcal{E}_0(K)$ as a proper subspace and the exact sequence

$$0 \ \longrightarrow\ X_0 \ \longrightarrow\ \mathcal{E}(K) \ \longrightarrow\ X_1 \ \longrightarrow\ 0$$

splits.

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