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A QUICK TEST FOR NONISOMORPHISM OF ONE-RELATOR GROUPS

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ABSTRACT. A sequence of integers is read off from the presentation of a finitely generated torsion-free one-relator group with nontrivial second integral homology, without recourse to group-theoretic manipulations. This test sequence is derived from the cup coproduct on the coring of the integral homology module of the group, and reflects information about the group's second lower central factor group.

Test sequences differ only if the corresponding groups are nonisomorphic. The test process can be generalised to any one-relator group with nontrivial second integral homology.

1. Introduction. It is generally difficult to determine when two one-relator groups are isomorphic (see [1, p. 75] for example), though S. J. Pride [7] has shown that the isomorphism problem for two-generator one-relator groups with torsion is solvable. This paper describes a test for nonisomorphism of one-relator groups with isomorphic integral homology modules. The test is directly applicable only to groups which are torsion-free, finitely generated and have nontrivial second integral homology, but may be extended to any one-relator group with nontrivial second integral homology.

The test's main advantages are that it is quick, that it involves arithmetic calculations only, rather than manipulation of group elements, and that the required test integers can be read off from any presentation of the group.

In §2 the test is described and in §3, proof that it measures nonisomorphism is given. In §4 it is shown that the test may be extended to any one-relator group by comparison with a suitable group to which the test directly applies.

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2. The test. This technique isolates a sequence of integers from a presentation of a torsion-free finitely generated one-relator group with nontrivial second integral homology (hereinafter referred to as a *testable* group). Sequences determined from two testable groups differ only if the groups themselves differ, in which case a specific homomorphism defined between their respective second lower central factor groups is *not* an isomorphism.

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Two steps are involved: the first reads off a skew-symmetric integral matrix from a presentation of a testable group and the second derives from this matrix a test sequence of integers which is independent of isomorphisms of the group and hence is presentation-free.

In all that follows, let $G = \langle x_1, \ldots, x_n : r \rangle$ be a presentation of a testable group, let F be the free group generated by $\{x_1, \ldots, x_n\}$, let

$$1 \to R \to F \xrightarrow{\pi} G \to 1$$

be the corresponding free presentation of G, and let ε: $\mathbf{Z}F \to \mathbf{Z}$ be the augmentation map of the group ring $\mathbf{Z}F$ over the integers $\mathbf{Z}$. Recall that $\partial w/\partial z$ is the *Fox derivative* [2] in $\mathbf{Z}F$ of the element w of $\mathbf{Z}F$ with respect to the generator z of F.

(2.1) Definition. For each word w and pair of generators (y, z) in F, define the integer $\langle w; y, z \rangle$ to be

$$\langle w; y, z \rangle = \varepsilon(\partial^2 w/\partial y \partial z).$$

That is, for $y \neq z$, $\langle w; y, z \rangle$ is the exponent sum in w of occurrences of y preceding each occurrence of z^{+1}, minus the exponent sum in w of occurrences of y preceding each occurrence of z^{-1}. Thus $\langle yz^{-2}y^2z; y, z \rangle = (-1) + (-1) + (3) = 1$.

Recall further the definition of the invariant factors of an integral matrix. If V is an $n \times n$ matrix with integer entries the *ith determinantal divisor* $d_i(V)$ of V for $0 \leqslant i \leqslant n$ is defined to be $d_0(V) = 1$,

$$d_i(V) = \gcd\{\det v : v \text{ is an } i \times i \text{ submatrix of } V\}, \qquad 1 \leqslant i \leqslant n.$$

(2.2) Definition. The *ith invariant factor* $s_i = s_i(V)$ of V is

$$s_i(V) = \begin{cases} d_i(V)/d_{i-1}(V), & 1 \leqslant i \leqslant \operatorname{rank} V, \\ 0, & \operatorname{rank} V < i \leqslant n. \end{cases}$$

If V is skew-symmetric with rank $V = 2l$ it is true (see [5, Theorems IV.2, IV.3] for example) that $s_{2k}(V) = s_{2k-1}(V)$ $1 \leqslant k \leqslant l$.

These two definitions are all that is required to implement the test.

Test Step 1. (i) For each pair of generators (x_i, x_j) with $i < j$ in G, calculate $\langle r; x_i, x_j \rangle$.

(ii) Form the skew-symmetric matrix $M = M(G)$ with entries

$$m_{kl} = \begin{cases} \langle r; x_k, x_l \rangle, & k < l, \\ 0, & k = l, \\ -\langle r; x_l, x_k \rangle, & k > l. \end{cases}$$

Test Step 2. Calculate the invariant factors $s_i(M(G))$, $1 \leqslant i \leqslant n$. The sequence of invariant factors $TS(G) = (s_1, s_2, \ldots, s_n)$ is the required *test sequence for G*. If the test sequences for two presentations differ at any co-ordinate then they are presentations of nonisomorphic groups.

Computation in Test Step 1(i) may be simplified if a representation $r \equiv r^*$ mod F_3 is known, where

$$r^* = \prod_{j=2}^{n} \prod_{i<j} [x_i, x_j]^{m_{ij}}.$$

If $\langle r^*, [x_i, x_j]\rangle$ denotes the exponent sum in r^* of commutator $[x_i, x_j]$ then the identity

$$\langle r; x_i, x_j\rangle = \langle r^*, [x_i, x_j]\rangle, \qquad 1 \leqslant i < j \leqslant n,$$

may be used; that is, $\langle r; x_i, x_j\rangle = m_{ij}$, $1 \leqslant i < j \leqslant n$. Note this implies that *any* skew-symmetric $n \times n$ integral matrix V appears as $V = M(G)$ for at least one testable group G.

In illustration, consider $G = \langle x_1, x_2, x_3, x_4: r\rangle$ where

$$r = [x_3, x_4][x_1, x_2][x_1, x_4^2][x_3, x_2][x_2, x_4][x_3, x_4]^3$$

and $H = \langle x_1, x_2, x_3, x_4: s\rangle$ where

$$s = r[x_4, x_3].$$

Then

$$M(G) = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 4 \\ -2 & -1 & -4 & 0 \end{bmatrix} \quad \text{and} \quad M(H) = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 3 \\ -2 & -1 & -3 & 0 \end{bmatrix}.$$

Since $d_1(M(G)) = 1$ it follows that $s_1(M(G)) = s_2(M(G)) = 1$, and as $d_4(M(G)) = [d_3(M(G))]^2 = 4$, it follows that $s_3(M(G)) = s_4(M(G)) = 2$ and $TS(G) = (1, 1, 2, 2)$. Similarly, $TS(H) = (1, 1, 1, 1)$, hence $G \not\cong H$.

(2.3) Example. Let $G = \langle x_1, x_2: r\rangle$ be a testable two-generator group. Then $r \equiv [x_1, x_2]^p \bmod F_3$ for a unique integer p, so that $TS(G) = (|p|, |p|)$, thus there are infinitely many distinct isomorphism classes of such groups.

(2.4) Example. Let $G = \langle x_1, x_2, x_3: r\rangle$ be a testable three-generator group. Then $r \equiv [x_1, x_2]^k[x_1, x_3]^l[x_2, x_3]^m \bmod F_3$ and $TS(G) = (g, g, 0)$ where $g = (k, l, m)$. Thus there are infinitely many distinct isomorphism classes of such groups.

(Analysis similar to that of the prior two examples clearly holds for groups with any number of generators; for instance there is a doubly-countable number of distinct isomorphism classes of testable five-generator groups.)

3. The homomorphism $\mathfrak{D}(G)$. We prove that the test sequence $TS(G)$ measures (in a way to be made precise) the second term G_2/G_3 of the lower central factors of G. Recall that for a testable group G, $H_2(G; \mathbf{Z}) \cong \mathbf{Z}$ and $H_1(G; \mathbf{Z})$ is free abelian.

Let $\eta: R \cap F_2/[R, F] \to H_2(G; \mathbf{Z})$ be the Hopf isomorphism, $\mu: F_2/F_3[R, F] \to \bigwedge_2 H_1(G; \mathbf{Z})$ be the isomorphism given by $[f, f^*]F_3[R, F] \to \pi(f)G_2 \wedge \pi(f^*)G_2$, $[\ ,\ ]: \bigwedge_2 H_1(G; \mathbf{Z}) \to G_2/G_3$ be the homomorphism given by $gG_2 \wedge g^*G_2 \to [g, g^*]G_3$, and $\iota: R \cap F_2/[R, F] \to F_2/F_3[R, F]$ be the homomorphism induced by inclusion.

(3.1) Definition. The homomorphism $\mathfrak{D}(G)$: $H_2(G; \mathbf{Z}) \to \bigwedge_2 H_1(G; \mathbf{Z})$ is defined to be $\mathfrak{D}(G) = \mu \circ \iota \circ \eta^{-1}$.

In [3] it is shown firstly that the sequence

$$H_2(G; \mathbf{Z}) \overset{\mathfrak{D}(G)}{\to} \bigwedge_2 H_1(G; \mathbf{Z}) \overset{[\,,\,]}{\to} G_2/G_3 \to 1$$

is exact, secondly that

$$\mathfrak{D}(G) \circ \eta(r[R, F]) = \sum_{j=2}^{n} \sum_{i=1}^{j-1} \langle r; x_i, x_j \rangle (x_i G_2 \wedge x_j G_2),$$

and thirdly that $\mathfrak{D}(G) = \phi \circ \Omega(G)$ where $\Omega(G)$ is the *diagonal comultiplication* (cup coproduct)

$$\Omega(G): H_2(G; \mathbf{Z}) \to H_1(G; \mathbf{Z}) \nabla H_1(G; \mathbf{Z}) \subseteq H_1(G; \mathbf{Z}) \otimes H_1(G; \mathbf{Z})$$

and ϕ is the isomorphism defined by

$$(x_i G_2 \otimes x_j G_2 - x_j G_2 \otimes x_i G_2) \to x_i G_2 \wedge x_j G_2, \qquad 1 \leqslant i < j \leqslant n.$$

Suppose G and H are testable groups with isomorphic integral homology modules. If α: $H_2(G; \mathbf{Z}) \to H_2(H; \mathbf{Z})$ is an isomorphism (so $\alpha = \pm 1$) and β: $H_1(G; \mathbf{Z}) \to H_1(H; \mathbf{Z})$ is any isomorphism, it is easy to deduce that

$$(\beta \otimes \beta) \circ \Omega(G) = \Omega(H) \circ \alpha \Leftrightarrow (\beta \wedge \beta) \circ \mathfrak{D}(G) = \mathfrak{D}(H) \circ \alpha.$$

When isomorphisms α and β exist satisfying these conditions, we write $\Omega(G) \sim \Omega(H)$ via (α, β), or, equivalently, $\mathfrak{D}(G) \sim \mathfrak{D}(H)$ via (α, β), and $\sim$ is an equivalence relation on the set of such homomorphisms.

Since $\Omega(G) \sim \Omega(H)$ via (α, β) if and only if $M(H) = \pm B^{\top} M(G) B$, where B is the change-of-basis matrix corresponding to β, we conclude that $\Omega(G) \sim \Omega(H)$ if and only if $M(G)$ and $M(H)$ have the same invariant factors (for instance, see **[5**, Theorem IV.3]). This specifies the connection between the test sequence $TS(G)$ and the homomorphism $\mathfrak{D}(G)$:

$$TS(G) = TS(H) \Leftrightarrow \mathfrak{D}(G) \sim \mathfrak{D}(H).$$

A preliminary definition is needed before the relationship between $\mathfrak{D}(G)$ and G_2/G_3 may be described. Any isomorphism β: $H_1(G; \mathbf{Z}) \to H_1(H; \mathbf{Z})$ induces a set map β': $G_2/G_3 \to H_2/H_3$ given by

$$\beta'\left(\prod_{i=1}^{n} [g_i, g_i']^{s_i} G_3 \right) = \prod_{i=1}^{n} [h(g_i), h(g_i')]^{s_i} H_3$$

where $h(g)$ is a coset representative of $\beta(gG_2)$. It follows that

$$[\ ,\ ] \circ (\beta \wedge \beta) = \beta' \circ [\ ,\ ].$$

(3.2) THEOREM. *Let G and H be testable groups and let β: $H_1(G; \mathbf{Z}) \to H_1(H; \mathbf{Z})$ be an isomorphism. Then $\mathfrak{D}(G) \sim \mathfrak{D}(H)$ via (α, β) if and only if β': $G_2/G_3 \to H_2/H_3$ is an isomorphism.*

PROOF. Consider the commuting diagram

$$\begin{array}{ccccccc}
H_2(G; \mathbf{Z}) & \overset{\mathfrak{D}(G)}{\to} & \wedge_2 H_1(G; \mathbf{Z}) & \overset{[\ ,\ ]}{\to} & G_2/G_3 & \to & 1 \\
 & & \beta \wedge \beta \downarrow & \circ & \downarrow \beta' & & \\
H_2(H; \mathbf{Z}) & \underset{\mathfrak{D}(H)}{\to} & \wedge_2 H_1(H; \mathbf{Z}) & \underset{[\ ,\ ]}{\to} & H_2/H_3 & \to & 1
\end{array}$$

If $\mathfrak{D}(G) \sim \mathfrak{D}(H)$ via (α, β) then β' is an isomorphism immediately. Conversely, if β' is an isomorphism, then since (in either row) Ker[,] is unique up to isomorphism, there exists an isomorphism α: $H_2(G; \mathbf{Z}) \to H_2(H; \mathbf{Z})$ such that $(\beta \wedge \beta) \circ \mathfrak{D}(G) = \mathfrak{D}(H) \circ \alpha$. $\square$

(3.3) COROLLARY. *If G and H are testable groups with isomorphic integral homology then $TS(G) = TS(H)$ if and only if there exists an isomorphism β: $H_1(G; \mathbf{Z}) \to H_1(H; \mathbf{Z})$ inducing an isomorphism β': $G_2/G_3 \to H_2/H_3$.* $\square$

Finally, suppose γ: $G \to H$ is an isomorphism of testable groups. Then γ induces isomorphisms γ_{ab}: $H_1(G; \mathbf{Z}) \to H_1(H; \mathbf{Z})$ and γ_2: $G_2/G_3 \to H_2/H_3$ with $\gamma_2 = \gamma'_{ab}$ necessarily. Hence

$$TS(G) \neq TS(H) \Rightarrow G \ncong H.$$

4. General use of the test.

(4.1) *Groups with torsion.* (See [**6**, Corollary 4.13.1], for example.) If G and H are torsion-free one-relator groups with $G = \langle x_1, \ldots, x_n: r\rangle$ and $H = \langle x_1, \ldots, x_n: s\rangle$ and if $G(k) = \langle x_1, \ldots, x_n: r^k\rangle$ and $H(k) = \langle x_1, \ldots, x_n: s^k\rangle$ then

$$G \ncong H \Rightarrow G(k) \ncong H(k) \quad \text{for all } k \in \mathbf{Z}.$$

(4.2) *Infinitely generated groups.* If G^* is an infinitely generated one-relator group it can be decomposed as a free product $G^* = G * G'$ where G is a finitely-generated one-relator group on a minimum number of generators and G' is free on the set $X(G)$. If $H^* = H * H'$ is another such decomposition, the Kurosh decomposition theorem [**4**, §35] implies

$$G^* \cong H^* \Leftrightarrow G \cong H \text{ and card } X(G) = \text{card } X(H),$$

so in particular $G \ncong H \Rightarrow G^* \ncong H^*$.

This analysis also applies to finitely generated groups not presented on a minimal generating set.

(4.3) *Groups with trivial second integral homology.* If G^* is a one-relator group with $H_2(G^*; \mathbf{Z}) = 0$ it may be embedded in a one-relator group G with $H_2(G; \mathbf{Z}) \neq 0$ in such a fashion that if H^* is another such group then in certain cases $G \ncong H$ implies $G^* \ncong H^*$.

Let $G^* = \langle x_1, \ldots, x_n: r^*\rangle$ where r^* is cyclically reduced of length > 1 and $n > 2$. If F^* is the free group on $\{x_1, \ldots, x_n\}$ then $r^* \notin F_2^*$ and thus some generator of F^* has nonzero exponent sum in r^*. By [**6**, Theorem 3.5] it may be assumed that only one generator (x_1 say) has nonzero exponent sum (α say) in r^*. Then G^* can be embedded in a one-relator group $G = \langle a, b, x_2, \ldots, x_n: r\rangle$ in which every generator has zero exponent sum in r. In fact $G \cong G^* *_A Y$ where Y is the free group on $\{a, b\}$ and the (free) subgroup A of G^* generated by $\{x_1\}$ is amalgamated by the isomorphism $x_1 \to [a, b]$. If G^* is torsion-free then so is G and $H_2(G; \mathbf{Z}) \cong \mathbf{Z}$. Finally, suppose H^* is another such group with isomorphic integral homology. Necessarily H^* has a presentation $\langle x_1, \ldots, x_n: s^*\rangle$ where x_1 is the only generator with exponent sum nonzero (in fact equal to α). If $G \ncong H$, then one of two conclusions may be drawn: either $G^* \ncong H^*$, or else $G^* \cong H^*$ but no isomorphism ϕ: $G^* \to H^*$ has $\phi(x_1) = x_1^{\pm 1}$.

5. Concluding remarks. Since Ker $\iota = R \cap F_3[R, F]/[R, F]$ for a testable group G, it follows that $\mathfrak{D}(G) \sim 0$ if and only if $r \in F_3$. Equivalently, if $H = \langle x_1, \ldots, x_n: s\rangle$ is another testable group, then $\mathfrak{D}(G) = \mathfrak{D}(H)$ if and only if $r \equiv s \bmod F_3$, or, in other words, $TS(G) \neq TS(H)$ only if $r \not\equiv s \bmod F_3$. Thus the test sequence is not sufficient to distinguish between one-relator groups. For instance, if $G = \langle x_1, x_2: [x_1, x_2^4]\rangle$ and $H = \langle x_1, x_2: [x_1^2, x_2^2]\rangle$, then $TS(G) = TS(H)$, but $G \not\cong H$ since G has infinite cyclic centre and H has trivial centre.

However, the test sequence for G does generally carry more information than is given by G_2/G_3, by virtue of Corollary (3.3). In fact, if the relator r of the testable group G is written in the form $r \equiv r^* \bmod F_3$ specified earlier, then the relation matrix of the abelian group G_2/G_3 is the single column matrix $(\langle r^*, [x_i, x_j]\rangle)$ with $1 \leqslant i < j \leqslant n$. Hence $G_2/G_3 \cong \mathbf{Z}/\langle g\rangle \oplus \mathbf{Z}^{m-1}$, where $m = \frac{1}{2}n(n-1)$ and $g = \gcd\{\langle r^*, [x_i, x_j]\rangle, 1 \leqslant i < j \leqslant n\}$. Because $TS(G) = (g, g, s_3, \ldots, s_n)$ it follows that for another testable group H, $G_2/G_3 \cong H_2/H_3$ if and only if $TS(G)$ and $TS(H)$ agree on their *first two* coordinates.

References

1. G. Baumslag, *Some problems on one-relator groups* (Proc. Second Internat. Conf. Theory of Groups, Canberra, 1973), Lecture Notes in Math., Vol. 372, Springer-Verlag, Berlin, 1974, pp. 75–81.

2. R. H. Fox, *Free differential calculus*. I, Ann. of Math. (2) **57** (1953), 547–560.

3. K. J. Horadam, *The diagonal comultiplication on homology*, J. Pure Appl. Algebra (to appear).

4. A. G. Kurosh, *The theory of groups*, Vol. II, Chelsea, New York, 1956.

5. M. Newman, *Integral matrices*, Academic Press, New York, 1972.

6. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*, 2nd rev. ed., Dover, New York, 1976.

7. S. J. Pride, *The isomorphism problem for two-generator one-relator groups with torsion is solvable*, Trans. Amer. Math. Soc. **227** (1977), 109–139.

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