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A construction of a fuzzy topology from a strong fuzzy metric

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Abstract

After the inception of the concept of a fuzzy metric by I. Kramosil and J. Michalek, and especially after its revision by A. George and G. Veeramani, the attention of many researches was attracted to the topology induced by a fuzzy metric. In most of the works devoted to this subject the resulting topology is an ordinary, that is a crisp one. Recently some researchers showed interest in the fuzzy-type topologies induced by fuzzy metrics. In particular, in the paper (J. J. Miñana, A. Šostak, Fuzzifying topology induced by a strong fuzzy metric, Fuzzy Sets and Systems 300 (2016), 24–39) a fuzzifying topology $\mathcal{T} : 2^X \to [0,1]$ *induced by a fuzzy metric* $m : X \times X \times [0,\infty)$ *was constructed. In this paper we extend this construction to get the fuzzy topology* $\mathcal{T} : [0,1]^X \to [0,1]$ *and study some properties of this fuzzy topology.*

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1. Introduction

After the concept of a fuzzy metric was defined by I. Kramosil and J. Michalek in 1975 [16] and later redefined in a slightly revised form in 1994 by A. George and P. Veeramani [4], many researches became interested in the topological structure of a fuzzy metric space. In particular different properties

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of the topologies induced by fuzzy metrics and operations with such topologies were studied by A. George and P. Veeramani, V. Gregori, S. Romaguera, A. Sapena, D. Mihet, J.J. Miñana, S. Morillas et al., see e.g. [4], [5], [11], [10], [7], [8], [25], et al.

In most of the works devoted to this subject the resulting topology is an ordinary, that is a crisp one. Recently some researchers showed interest in the fuzzy-type topologies induced by fuzzy metrics. In particular in the papers [37], [22], [26] fuzzy pseudometrics $m : X \times X \times (0, \infty)$ were applied to induce fuzzifying topologies $\mathcal{T} : 2^X \to [0, 1]$. However, as far as we know, there was still no research in the field of "fullbodied" fuzzy topologies $\mathcal{T} : [0, 1]^X \to [0, 1]$ induced by fuzzy metrics. It is the principal aim of this paper to develop such construction. Specifically, we consider here an LM-fuzzy topology $T^m : L^X \to M$ generated by a fuzzy metric m, where L and M are complete sublattices of the unit interval $[0, 1]$. Our approach here is based on the construction of a fuzzifying topology presented in [26] which is extended to an LM-fuzzy topology by applying the Lowen ω-functor.

The structure of the paper is as follows. In the next section we recall basic notions and results used in the sequel. In particular, the concepts of a fuzzy metric, a fuzzy topology (specifically, an LM-fuzzy topology) are recalled here. We recall here also the standard construction of a fuzzy topology from a decreasing family of ordinary topologies and modify it by applying the Lowen ω-functor.

The next, 3^{rd} section is the main one in this work. Here we realize the general construction of a fuzzy topology from a family of topologies in case when these topologies are induced by a strong fuzzy metric and consider some properties of this construction. The main result here is Theorem 3.8 showing, in a certain sense, "the good behaviour" of this construction. The next, Corollary 3.9 presents the "categorical version" of this theorem. In the last, 4^{th} Section some perspectives for the future research are sketched.

2. Preliminaries

2.1. **Fuzzy metrics.** Basing on the concept of a statistical metric introduced by K. Menger [24], see also [29], I. Kramosil and J. Michalek in [16] defined the notion of a fuzzy metric. A. George and P. Veeramani [4] slightly modified the original concept of a fuzzy metric. At present in most cases research involving fuzzy metrics is done in the context of George-Veeramani definition. This approach is accepted also in our paper.

Definition 2.1 ([4])**.** A fuzzy metric on a set X is a fuzzy set $m : X \times X \times \mathbb{R}^+ \to [0, 1]$, where $\mathbb{R}^+ = (0, +\infty)$, such that:

(1GV) $m(x, y, t) > 0$ for all $x, y \in X$, and all $t \in \mathbb{R}^+$;
(2GV) $m(x, y, t) = 1$ if and only if $x = y$;
(3GV) $m(x, y, t) = m(y, x, t)$ for all $x, y \in X$ and all $t \in \mathbb{R}^+$;
(4GV) $m(x, z, t + s) \geq m(x, y, t) * m(y, z, s)$ for all $x, y, z \in X$, $t, s \in \mathbb{R}^+$;
(5GV) $m(x, y, -) : \mathbb{R}^+ \to [0, 1]$ is continuous for all $x, y \in X$.

The pair (X, m) is called a fuzzy metric space.

If it is important to specify the t-norm in the definition of a fuzzy metric we use notations $(m, *)$ and $(X, m, *)$ for a fuzzy metric and the fuzzy metric space respectively.

In case when axiom (2GV) is replaced by a weaker axiom

(2′GV) if $x = y$, then $m(x, y, t) = 1$

we get definitions of a fuzzy pseudo-metric, and the corresponding fuzzy pseudo-metric space.

Note that axiom (4GV) combined with axiom (2′GV) implies that a fuzzy metric $m(x, y, t)$ is non-decreasing on the third argument.

Definition 2.2. A fuzzy metric $m : X \times X \times (0, \infty) \to [0, 1]$ is called strong if, in addition to the properties (1GV) - (5GV), the following stronger versions of axioms (4GV) and (5GV) are satisfied

(4^sGV) $m(x, z, t) \geq m(x, y, t) * m(y, z, t)$ for all $x, y, z \in X$ and for all $t > 0$.

(5^sGV) $m(x, y, -) : \mathbb{R}^+ \to [0, 1]$ for all $x, y \in X$ is continuous and non-decreasing.

Remark 2.3. In the original definition of a strong fuzzy metric it was defined as a mapping $m : X \times X \times \mathbb{R}^+ \to [0, 1]$ satisfying axioms (1GV), (2GV), (3GV), (4^sGV) and (5GV). However, as it was noticed, such combination of axioms does not imply axiom (4GV) and hence a strong fuzzy metric need not be a fuzzy metric: the corresponding example can be found in [6]. Therefore in our definition of a strong fuzzy metric we replace axiom (5GV) by axiom (5^sGV) by assuming additionally that m is non-decreasing. This condition, on one hand, can be obtained "gratis" from the system of axioms (1GV), (2GV), (3GV), (4GV) and (5GV) and, on the other hand, it allows to obtain the condition (4GV). Thus, under the present definition every strong fuzzy metric is a fuzzy metric.

Definition 2.4 ([11])**.** A fuzzy metric m on a set X is said to be *stationary*, if m does not depend on t, i.e. if for each $x, y \in X$, the function $m_{x,y}(t) = m(x, y, t)$ is constant. In this case we can write $m(x, y)$ instead of $m(x, y, t)$.

The next concept implicitly appears in [10]:

Definition 2.5. Given two fuzzy metric spaces $(X, m, *_m)$ and $(Y, n, *_n)$ a mapping $f : X \to Y$ is called continuous if for every $\varepsilon \in (0, 1)$, every $x \in X$ and every $t \in (0, \infty)$ there exist $\delta \in (0, 1)$ and $s \in (0, \infty)$ such that $n(f(x), f(y), t) > 1 - \varepsilon$ whenever $m(x, y, s) > 1 - \delta$. In symbols:

$$\forall \varepsilon \in (0, 1), \forall x \in X, \forall t \in (0, \infty) \ \exists \ \delta \in (0, 1), \exists s \in (0, \infty) \text{ such that}$$

$$m(x, y, s) > 1 - \delta \Longrightarrow n(f(x), f(y), t) > 1 - \varepsilon$$

Fuzzy metric spaces as objects and continuous mappings between them as morphisms form a category which we denote **FuzMS**.

In a fuzzy metric space $(X, m, *)$ a (crisp) topology on X is introduced as follows [4], [5]:

Given a point $x \in X$, a number $\varepsilon \in (0,1)$ and $t > 0$, we define a ball at a level t with center x and radius $\varepsilon > 0$ as the set $B_\varepsilon(x,t) = \{y \in X \mid \ m(x,y,t) > 1-\varepsilon\}$. Obviously

$$t \leq s \Longrightarrow B_\varepsilon(x,t) \subseteq B_\varepsilon(x,s) \ \text{ and } \ \varepsilon \leq \delta \Longrightarrow B_\varepsilon(x,t) \subseteq B_\delta(x,t).$$

It is shown in [4], see also [5], that the family $\{B_\varepsilon(x,t) \mid x \in X, t \in (0,\infty), \varepsilon \in (0,1)\}$ is a base for some topology T^m on X. Besides one can easily verify the following proposition:

Proposition 2.6 ([4])**.** *Given two fuzzy metric spaces $(X, m, *_m)$ and $(Y, n, *_n)$ a mapping $f : (X, m, *_m) \to (Y, n, *_n)$ is continuous if and only if the mapping of the induced topological spaces $f : (X, T^m) \to (Y, T^n)$ is continuous.*

Hence, by assigning to a fuzzy metric space $(X, m, *_m)$ the induced topological space (X, T^m) and assigning to a continuous mapping $f : (X, m, *_m) \to (Y, n, *_n)$ the mapping $f : (X, T^m) \to (Y, T^n)$, we get a functor Φ : **FuzMS** $\to$ **TOP** where **TOP** is the category of topological spaces.

2.2. **Fuzzy topologies.** The first approach to the study of topological-type structures in the context of fuzzy sets was undertaken in 1968 by C.L. Chang [1]; soon later this approach was essentially developed and extended by J.A. Goguen [3]. According to this approach a (Chang-Goguen) L-fuzzy topology on a set X, where L is a complete infinitely distributive lattice, is a subfamily of the family L^X of L-fuzzy subsets of X satisfying certain counterparts of the usual topological axioms. An alternative approach to the fuzzification of the subject of topology was undertaken in 1980 by U. Höhle [12]. According to this approach, an (L-)fuzzy topology on a set X is defined as a mapping $\mathcal{T} : 2^X \to L$ satisfying certain functional versions of topological axioms. Later, in 1991, the same concept was rediscovered by Mingsheng Ying [33], by making deep analysis of topological axioms and properties of topological spaces by means of fuzzy logic. Mingsheng Ying called such structures by *fuzzifying topologies* and just this term is used now by most authors when speaking about such structures. Finally, in 1985 in [17] and [30] (independently) a general view on the concept of a topology in the context of fuzzy sets and fuzzy structures was proposed; later, in [18], [19], this approach led to the concept of an LM-fuzzy topology where L and M are complete infinitely distributive lattices. According to this approach an LM-fuzzy topology is a certain mapping $\mathcal{T} : L^X \to M$, see Definition 2.7. It is the aim of this work to present a construction of an LM-fuzzy topology on a fuzzy metric space. Besides, since we will deal with fuzzy topological-type structures generated by fuzzy metrics $m : X \times X \times (0,\infty) \to [0,1]$, we will restrict here with the case when L and M are complete sublattices *of the unit interval* $[0,1]$ containing 0 and 1.

Definition 2.7 ([30, 17, 18])**.** Given a set X, a mapping $\mathcal{T} : L^X \to M$ is called an LM-fuzzy topology on X if it satisfies the following axioms:

(1) $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1$ here given a constant $a \in [0,1]$ by a_X we denote the constant function taking value a for all $x \in X$, that is $a_X : X \to \{a\} \subseteq [0,1]$;
(2) $\mathcal{T}(A \wedge B) \geq \mathcal{T}(A) \wedge \mathcal{T}(B)$ $\forall A, B \in L^X$;
(3) $\mathcal{T}(\bigvee_i A_i) \geq \bigwedge_i \mathcal{T}(A_i)$ $\forall \{A_i : i \in I\} \subseteq L^X$.

A pair $(X, \mathcal{T})$ is called an LM-fuzzy topological space.

In case when $L = [0,1]$ and when we do not specify the range M we will call $[0,1], [0,1]$-fuzzy topologies just as fuzzy topologies.

Remark 2.8. The intuitive meaning of the value $\mathcal{T}(A)$ is the degree to which a fuzzy set $A \in L^X$ is open.

Remark 2.9. Chang-Goguen L-fuzzy topological spaces can be characterized now as $L2$-fuzzy topological spaces where $2 = \{0,1\}$ is the two-element lattice, and L-fuzzifying topological spaces are just $2L$-fuzzy topological spaces.

Definition 2.10. Given two LM-fuzzy topological spaces $(X, \mathcal{T}^X)$ and $(Y, \mathcal{T}^Y)$ a mapping $f : (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ is called continuous if

$$\mathcal{T}^X\left(f^{-1}(B)\right) \geq \mathcal{T}^Y(B) \quad \forall B \in L^Y.$$

Given $\alpha \in M$ and a fuzzy topology $\mathcal{T} : L^X \to M$ let $\mathcal{T}_\alpha = \{A \in L^X : \mathcal{T}(A) \geq \alpha\}$.

The following theorem is well-known and easy to prove:

Theorem 2.11. *A mapping $f : (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ is continuous if and only if the mapping $f : (X, \mathcal{T}^X_\alpha) \to (Y, \mathcal{T}^Y_\alpha)$ is continuous for each $\alpha \in [0,1]$, that is $f^{-1}(B) \in \mathcal{T}^X_\alpha$ whenever $B \in \mathcal{T}^Y_\alpha$.*

Since the composition $g \circ f : (X, \mathcal{T}^X) \to (Z, \mathcal{T}^Z)$ of two continuous mappings $f : (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ and $g : (Y, \mathcal{T}^Y) \to (Z, \mathcal{T}^Z)$ is obviously continuous and since the identity mapping id : $(X, \mathcal{T}^X) \to (X, \mathcal{T}^X)$ is continuous, we come to the category **FuzTop(**LM**)** of LM-fuzzy topological spaces as objects and their continuous mappings as morphisms.

2.3. **Construction of LM-fuzzy topologies from families of crisp topologies.** In this section we describe a scheme allowing to construct LM-fuzzy topologies from decreasing families of ordinary topologies.

Let K be a sup-dense subset of M, that is for every $\alpha \in M, \alpha \neq 0$ there exists a subset K^α of K such that $\alpha = \sup K^\alpha$. Further, let a non-increasing family of topologies $\{T_\alpha : \alpha \in K\}$ on a set X be given, that is

$$\alpha < \beta, \alpha, \beta \in K \implies T_\alpha \supseteq T_\beta$$

and $T_0 = L^X$ whenever $0 \in K$.

Further, let $\omega(T_\alpha)$ be the family of all lower semi-continuous functions $A : (X, T_\alpha) \to [0,1]$. It is well known see [20], [21] (and easy to verify) that $\omega(T_\alpha)$ satisfies the axioms of a Chang-Goguen fuzzy topology, (actually even a stratified Chang-Goguen fuzzy topology) that is

(1) $\omega(T_\alpha)$ contains all constant functions $c_X : (X, T_\alpha) \to [0,1], c \in [0,1]$,

(2) $\omega(T_\alpha)$ is closed under finite meets
(3) $\omega(T_\alpha)$ is closed under arbitrary joins.

Besides $\mathcal{T}_0 = [0,1]^X$

We shall need also the following well-known fact, see e.g. [20], [21]:

Proposition 2.12. *Given two topological spaces (X, T^X) and (Y, T^Y) a mapping $f : (X, T^X) \to (Y, T^Y)$ is continuous if and only if the mapping of the corresponding Chang-Goguen fuzzy topological spaces $f : (X, \omega(T^X)) \to (Y, \omega(T^Y))$ is continuous.*

Let $int_\alpha A$ be the interior of a fuzzy set $A \in [0,1]^X$ in the Chang-Goguen fuzzy topology $\omega(T_\alpha)$.

Theorem 2.13. *By setting $\mathcal{T}(A) = \sup\{\alpha : int_\alpha A = A\}$ an LM-fuzzy topology $\mathcal{T} : L^X \to M$ is defined.*

Proof. Notice first that

$$\alpha \le \beta \Longrightarrow int_\beta A \le int_\alpha A \le A, \forall A \in L^X \text{ and } \forall \alpha, \beta \in M,$$

and hence the definition of the mapping $\mathcal{T} : L^X \to M$ is correct.

(1) Since constant functions $c_X : (X, T_\alpha) \to K \subseteq [0,1],\ \ c \in K$ are continuous for every $\alpha \in K$, we conclude that $int_\alpha c_X = c_X$, and hence $\mathcal{T}(c_X) = 1$.

(2) Let $A, B \in L^X$ and let $\mathcal{T}(A) = \alpha$, $\mathcal{T}(B) = \beta$. Without loss of generality assume that $\alpha \le \beta$. Then for every $\varepsilon > 0$ such that $\alpha - \varepsilon \in K$ we have $A = int_{\alpha-\varepsilon} A$, and $B = int_{\beta-\varepsilon} B = int_{\alpha-\varepsilon} B$. Hence

$$A \wedge B = (int_{\alpha-\varepsilon} A) \wedge (int_{\alpha-\varepsilon} B) = int_{\alpha-\varepsilon}(A \wedge B).$$

Thus $\mathcal{T}(A \wedge B) \ge \alpha$ and hence $\mathcal{T}(A \wedge B) \ge \mathcal{T}(A) \wedge \mathcal{T}(B)$.

(3) Let $\{A_i : i \in I\} \subseteq L^X$ be a family of fuzzy subsets of X and let $\alpha = \bigwedge_{i\in I} \mathcal{T}(A_i)$. Then for every $\varepsilon > 0$ such that $\alpha - \varepsilon \in K$ and every $i \in I$ it holds $A_i = int_{\alpha-\varepsilon} A_i$. Hence

$$\bigvee_{i\in I} A_i = \bigvee_{i\in I} int_{\alpha-\varepsilon} A_i \le int_{\alpha-\varepsilon} \bigvee_{i\in I} A_i.$$

Since the opposite inequality is obvious, we have

$$\bigvee\nolimits_{i\in I} A_i = int_{\alpha-\varepsilon} \bigvee\nolimits_{i\in I} A_i,$$

and hence $\mathcal{T}(\bigvee_{i\in I} A_i) \ge \bigwedge_{i\in I} \mathcal{T}(A_i)$. □

Remark 2.14. By making restrictions of the construction in an LM-fuzzy topology on the range L of fuzzy sets or on the range M of the fuzzy topology we come to the following special cases:

(1) Let $L = 2$ is the two-element lattice, that is $2 = \{0,1\}$ and $M = [0,1]$. In this case our construction reduces to the construction described in [26] and gives an $2M$-fuzzy topology, that is a fuzzifying topology.

(2) Let $L = [0,1]$ and $M = 2$ be the two element lattice. In this case $K = \{0,1\}$ or $K = \{1\}$ and our construction gives a fuzzy topology $\mathcal{T} : L^X \to 2$ such that $\mathcal{T}_0 = L^X$ and $\mathcal{T}_1 = \omega(T_1)$ is the given topology on the set X.

(3) Let $L = M = 2$. Then our construction gives $\mathcal{T} : 2^X \to 2$ such that $\mathcal{T}_0 = 2^X$ is the discrete topology and $\mathcal{T}_1 = T_1$ is the given topology.

Let K be a sup-dense subset of the unit interval and let $\{T_\alpha^X : \alpha \in K\}$, $\{T_\alpha^Y : \alpha \in K\}$ be non-increasing families of topologies on the sets X and Y respectively. Further, let $\{\omega(T_\alpha^X) : \alpha \in K\}$, $\{\omega(T_\alpha^Y) : \alpha \in K\}$ be the corresponding Chang-Goguen fuzzy topologies, and let $\mathcal{T}^X : L^X \to M$ and $\mathcal{T}^Y : L^Y \to M$ be the LM-fuzzy topologies constructed from families $\{\omega(T_\alpha^Y) : \alpha \in K\}$ and $\{\omega(T_\alpha^Y) : \alpha \in K\}$, respectively.

Theorem 2.15. *A function $f : (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ is continuous if and only if the function $f : (X, \omega(T_\alpha^X)) \to (Y, \omega(T_\alpha^Y))$ is continuous for every $\alpha \in K$,*

Proof. Assume first that $f : (X, \omega(T_\alpha^X)) \to (Y, \omega(T_\alpha^Y))$ is continuous for every $\alpha \in K$ and, given $B \in L^Y$, let $\mathcal{T}^Y(B) = \beta$. We have to show that $\mathcal{T}^X(f^{-1}(B)) \geq \beta$. In case $\beta = 0$ the statement is obvious. Therefore we assume that $\beta > 0$. Then for every $\varepsilon > 0$ such that $\beta - \varepsilon \in K$it holds $int_{\beta-\varepsilon} B = B$ and hence $B \in \omega(T_{\beta-\varepsilon}^Y)$ (here without loss of generality we assume that $\beta - \varepsilon \in K$). From the continuity of all mappings $f : (X, \omega(T_\alpha^X)) \to (Y, \omega(T_\alpha^Y))$ it follows that $f^{-1}(B) \in \omega(T_{\beta-\varepsilon}^X)$. Hence for every $\delta > 0$ such that $\beta - \varepsilon - \delta \in K$ it holds $f^{-1}(B) = int_{\beta-\varepsilon-\delta} f^{-1}(B)$. From here we easily get the required inequality $\mathcal{T}^X(f^{-1}(B)) \geq \beta = \mathcal{T}^X(B)$.

Conversely, assume that $f : (X, \omega(T_\alpha^X)) \to (Y, \omega(T_\alpha^Y))$ is not continuous for some α. Then there exists $\varepsilon > 0$ such that $\alpha - \varepsilon \in K$ and $V \in \omega(T_{\alpha-\varepsilon}^Y)$ but $f^{-1}(V) \notin \omega(T_{\alpha-\varepsilon}^X)$. However this means that $\mathcal{T}^X(f^{-1}(V)) \leq \alpha - \varepsilon < \mathcal{T}^Y(V)$, and hence the function $f : (X, \mathcal{T}^X) \to (Y, \mathcal{T}^Y)$ is not continuous. □

3. LM-fuzzy topology induced by a strong fuzzy metric

3.1. **Construction of an LM-fuzzy topology on a strong fuzzy metric space.** Let $(X, m, *)$ be a strong fuzzy metric space. In order to define a relation between properties of a fuzzy metric for a fixed parameter $t \in \mathbb{R}^+$ and α-levels of the LM-fuzzy topology, that we are going to construct, we take a strictly increasing continuous bijection $\varphi : (0, \infty) \to (0, 1)$.

We fix $\alpha \in (0,1)$ and consider the family

$$B_\alpha = \{B_m(x, r, t) : x \in X, r \in (0,1)\}, \text{ where } t = \varphi^{-1}(\alpha).$$

Then, B_α is a base of a topology T_α^m on the set X. Indeed, it is easy to verify (see e.g. [10]) that $m_t : X \times X \to [0,1]$ defined by $m_t(x,y) = m(x,y,t)$ for $x, y \in X$, is a stationary fuzzy metric on X which has as a base the family $\{B_{m_t}(x, r) : x \in X, r \in (0,1)\}$. This topology is characterized in the next theorem.

Theorem 3.1 ([26]). *Let $\alpha \in (0,1)$ and let $U \in 2^X$. Then $U \in T^m_\alpha$ if and only if for each $x \in U$ there exists $\delta \in (0,1)$ such that $B_m(x,\delta,t) \subseteq U$, where $t = \varphi^{-1}(\alpha)$.*

From this theorem and taking into account that for each $x \in X$, for each $\delta \in (0,1)$, and for every $t > 0$ the inclusion $B_m(x,\delta,s) \subseteq B_m(x,\delta,t)$ holds whenever $0 < s < t$, we obtain the next corollary.

Corollary 3.2 ([26]). *If $U \in T^m_\alpha$, then $U \in T^m_\beta$ whenever $\beta < \alpha$, and hence the family $\{T^m_\alpha : \alpha \in (0,1)\}$ is non-increasing.*

Now referring to Subsection 2.3 from Corollary 3.2 we get the following

Theorem 3.3. *Let (X,m) be a strong fuzzy metric space. By setting $\mathcal{T}^m(A) = \bigvee\{\alpha : A \in \omega(T^m_\alpha)\}$ for every $A \in L^X$ we get an LM-fuzzy topology $\mathcal{T}^m : L^X \to M$.*

In the sequel we refer to the fuzzy topology $\mathcal{T}^m$ constructed in the previous theorem as a fuzzy topology induced by the fuzzy metric m.

3.2. **Case of a principal fuzzy metric.**

Definition 3.4. ([8]) $(X,m,*)$ is called *principal* (or just, m is principal) if $\{B(x,\varepsilon,t) : r \in (0,1)\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$.

By definition of a principal fuzzy metric and Corollary 3.2, it is easy to verify that if $(X,m,*)$ is a strong principal fuzzy metric space, then $T_\alpha = T_\beta$ for all $\alpha,\beta \in (0,1)$. Hence also $\omega(T_\alpha) = \omega(T_\beta)$ for all $\alpha,\beta \in (0,1]$ and therefore the resulting fuzzy topology is a Chang-Goguen type topology, or $L2$-fuzzy topology in our notations. In particular, as one can expect, fuzzy topology generated by the standard fuzzy metric, that is by fuzzy metric m_d

$$m_d(x,y,t) = \frac{t}{t + d(x,y)}$$

where $d : X \times X \to [0,\infty)$ is an ordinary metric on the set X, is a Chang-Goguen fuzzy topology. We reformulate this fact as follows:

Example 3.5. Let (X,d) be a metric space and let m_d be the corresponding standard metric. Further, let $\mathcal{T}^{m_d}$ be the fuzzy topology induced by m_d. Then for every $U \in [0,1]^X$

$$\mathcal{T}^{m_d}(U) = \begin{cases} 1 & \text{if } U \text{ is lower semicontinuous} \\ 0 & \text{otherwise} \end{cases}$$

3.3. **Continuity of mappings of LM-fuzzy topological spaces versus continuity of mappings of strong fuzzy metric spaces.** As different from the concordant situation in case of fuzzy metrics and the induced topologies, (see Proposition 2.6), the concept of continuity of mappings of fuzzy metric spaces (Definition 2.10) is not coherent with the concept of continuity of the mappings of the induced LM-fuzzy topological spaces (Definition 2.10). This fact was known already in case of fuzzyfying topologies induced by fuzzy metrics. [26]. Therefore, in order to describe the relations between continuity of mappings between fuzzy metric spaces and the continuity of mappings between the induced fuzzy topological spaces, we need to consider the following stronger version of continuity for fuzzy metric spaces introduced in [7]:

Definition 3.6. A mapping $f : (X, m, *_m) \to (Y, n, *_n)$ is called strongly continuous at a point $x \in X$ if given $\varepsilon \in (0,1)$ and $t > 0$ there exists $\delta \in (0,1)$ such that $m(x, y, t) > 1 - \delta$ implies $n(f(x), f(y), t) > 1 - \varepsilon$. We say that $f : (X, m, *_m) \to (Y, n, *_n)$ is strongly continuous (on X) if it is strongly continuous at each point $x \in X$.

Remark 3.7. In paper [7] this property of a mapping $f : (X, m, *_m) \to (Y, n, *_n)$ was called t-continuity. Here, we think it is reasonable (following also [26]) to recall this property as *strong continuity* first because it is well related with the concept of a strong fuzzy metric which is fundamental for this paper, and second, because the prefix t in front of the adjective "continuous" seems to be misleading in this context.

Theorem 3.8. *A mapping $f : (X, \mathcal{T}^m) \to (Y, \mathcal{T}^n)$ of LM-fuzzy topological spaces induced by fuzzy metrics $m : X \times X \times (0, \infty) \to [0,1]$ and $n : Y \times Y \times (0, \infty) \to [0,1]$, respectively, is continuous if and only if the mapping $f : (X, m, *_m) \to (Y, n, *_n)$ is strongly continuous.*

Proof. Suppose that a mapping $f : (X, \mathcal{T}^m) \to (Y, \mathcal{T}^n)$ is continuous. Then for every $\alpha \in [0,1]$ the mapping $f : (X, \mathcal{T}^m_\alpha) \to (Y, \mathcal{T}^n_\alpha)$ is continuous (Theorem 2.11). Let $x \in X$ and take any $B_n(f(x), \varepsilon, t) \in \mathcal{T}^n_\alpha$, where $\alpha = \varphi(t)$. Since $B_n(f(x), \varepsilon, t) \in \mathcal{T}^n_\alpha$, where $\alpha = \varphi(t)$ and f is continuous, we have that $f^{-1}(B_n(f(x), \varepsilon, t)) \in \mathcal{T}^m_\alpha$. Therefore, for each $x' \in f^{-1}(B_n(f(x), \varepsilon, t))$ we can find $\delta \in (0,1)$ such that $B_m(x', \delta, t) \subseteq f^{-1}(B_n(f(x), \varepsilon, t))$. In particular, since $x \in f^{-1}(B_n(f(x), \varepsilon, t))$ there exists $\delta \in (0,1)$ such that $B_m(x, \delta, t) \subseteq f^{-1}(B_n(f(x), \varepsilon, t)$. However, this means that if $x' \in B_m(x, \delta, t)$, that is if $m(x, x', t) > 1 - \delta$, then $x' \in f^{-1}(B_n(f(x), \varepsilon, t))$, that is $n(f(x), f(y), t) > 1 - \varepsilon$. Therefore, the mapping $f : (X, m, *_m) \to (Y, n, *_n)$ is strongly continuous at a point x, and, since $x \in X$ is arbitrary, the mapping $f : (X, m, *_m) \to (Y, n, *_n)$ is strongly continuous.

Conversely, suppose that a mapping $f : (X, m, *_m) \to (Y, n, *_n)$ is strongly continuous, but $f : (X, \mathcal{T}^m) \to (Y, \mathcal{T}^n)$ is not continuous. Then there, applying Theorem 2.15 we conclude that there exists $\alpha \in [0,1]$ such that $f : (X, \omega(T^m_\alpha)) \to (Y, (T^n_\alpha))$ is not continuous. Then, we can find $V \in L^Y$ such that $V \in T^n_\alpha$, but $f^{-1}(V) \notin T^n_\alpha$. Referring to Proposition 2.12 without loss of generality we may assume that $V \in 2^X$.

The inequality $f^{-1}(V) \notin T^m_\alpha$ means that there exists $x_0 \in f^{-1}(V)$ such that $A \not\subseteq f^{-1}(V)$ for each $A \in T^m_\alpha$ containing point x_0, and, in particular, $B_m(x_0, \delta, t) \not\subseteq f^{-1}(V)$ for each $\delta \in (0,1)$, where $t = \varphi^{-1}(\alpha)$.

On the other hand, since $f(x_0) \in V \in \mathcal{T}^n_\alpha$, we can find $\varepsilon \in (0,1)$ such that $B_n(f(x_0), \varepsilon, t) \subseteq V$. Therefore, we have found $x_0 \in X$ and $\varepsilon \in (0,1)$ such that for each $\delta \in (0,1)$ it holds $B_m(x_0, \delta, t) \not\subseteq f^{-1}(B_n(f(x_0), \varepsilon_0, t))$ where $t = \varphi^{-1}(\alpha)$. However, this means that for each $\delta \in (0,1)$ there exists a point

$x \in X$ such that $m(x_0, x, t) > 1 - \delta$, but $n(f(x_0), f(x), t) \leq 1 - \varepsilon$ and hence the mapping $f : (X, m, *_m) \to (Y, n, *_n)$ is not strongly continuous at the point x_0. The obtained contradiction completes the proof. □

Let **FuzSM** denote the the subcategory of the category **FuzM** whose objects are strong metric spaces and whose morphisms are strongly continuous mappings of strong fuzzy metric spaces. From the previous theorem we get the following statement:

Corollary 3.9. *By assigning to every strong fuzzy metric space* $(X, m, *)$ *the* LM*-fuzzy topological space* $\mathfrak{F}(X, m, *) = (X, \mathcal{T}^m)$ *and assigning to every strongly continuous mapping* $f : (X, m, *_m) \to (Y, n, *_n)$ *the mapping* $\mathfrak{F}(f) = f : (X, \mathcal{T}^m) \to (Y, \mathcal{T}^n)$ *we obtain the functor* $\mathfrak{F}$: **FuzSMet** → **FuzTop**(LM).

4. Conclusion

We presented here a method allowing to construct for a given strong fuzzy metric space $(X, m, *)$ an LM-fuzzy topological space $(X, \mathcal{T}^m)$ where L, M are complete sublattices of the unit interval $[0, 1]$. In case $L = \{0, 1\}$ this construction comes to the construction of a fuzzifying topology developed in [26]. Although we restricted ourselves by the case of strong fuzzy metric spaces, it is clear that all concepts considered here and all results obtained in an obvious way can be reformulated for the case of strong fuzzy *pseudometric* spaces.

At the end of the last section a functor $\mathfrak{F}$: **FuzSMet** → **FuzTop**(L, M) was introduced. An actual problems is to study properties of this functor. In particular, we plan to study preservation of such operations as products, co-products, quotients, etc., by this functor. Another challenge is to study categorical properties of the subcategory $\mathfrak{F}$**FuzSMet** in the category **FuzTop**(L, M) as well as categorical properties of the subcategory **FuzSMet** in the category **FuzMet**

It is important to consider concrete examples of strong fuzzy metrics and the induced fuzzy topologies. As it was said above, in case of a principal fuzzy metric our construction brings forward to a Chang-Goguen fuzzy topology. In particular, starting with the standard fuzzy metric, we come to a Chang-Goguen fuzzy topology. Therefore, to get a general, say LM-fuzzy topology for $L = M = [0, 1]$ we have to start with a strong fuzzy metric which is not principal. Some examples of such fuzzy metrics can be found in [6] and [26].

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