Defective Coloring Revisited

Lenore Cowen
DEPARTMENT OF MATHEMATICAL SCIENCES
AND DEPARTMENT OF COMPUTER SCIENCE
JOHNS HOPKINS UNIVERSITY
BALTIMORE, MARYLAND, USA

Wayne Goddard
DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF NATAL
DURBAN, SOUTH AFRICA

C. Esther Jesurum
DEPARTMENT OF APPLIED MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS, USA

ABSTRACT

A graph is (k,d)-colorable if one can color the vertices with k colors such that no vertex is adjacent to more than d vertices of its same color. In this paper we investigate the existence of such colorings in surfaces and the complexity of coloring problems. It is shown that a toroidal graph is (3, 2)- and (5, 1)-colorable, and that a graph of genus γ is $(\chi_\gamma/(d+1)+4,d)$-colorable, where χ_γ is the maximum chromatic number of a graph embeddable on the surface of genus γ. It is shown that the $(2,k)$-coloring, for $k \geq 1$, and the (3, 1)-coloring problems are NP-complete even for planar graphs. In general graphs (k,d)-coloring is NP-complete for $k \geq 3, d \geq 0$. The tightness is considered. Also, generalizations to defects of several algorithms for approximate (proper) coloring are presented. © 1997 John Wiley & Sons, Inc.

1. INTRODUCTION

We define a (k,d)-coloring of a graph as a coloring of the vertices with k colors such that each vertex has at most d neighbors of its same color. For a graph G we define $\chi_d(G)$ as the minimum k such that there is a (k,d)-coloring of G. So a $(k,0)$-coloring is a proper coloring, and $\chi_0(G)$ is the usual vertex chromatic number of the graph.

Journal of Graph Theory Vol. 24, No. 3, 205—219 (1997)

CCC 0364-9024/97/030205-15

The parameter $\chi_d(G)$, also called the *defective chromatic number* of G, has been well-studied for planar graphs (see, for example, [10, 13, 15, 19, 26, 32]). In this paper, we extend the study of $\chi_d(G)$ to graphs embeddable on the torus, and surfaces of higher genera. We also consider algorithmic issues for the construction of defective colorings in general graphs, planar graphs, and graphs of bounded degree.

1.1. Previous Work

Defective coloring was introduced almost simultaneously by Burr and Jacobson (see [1]), Harary and Jones [18] and Cowen, Cowen and Woodall [10], and has continued to be an active area of research. Surveys of this and related colorings are given in [13] and [32]. Cowen, Cowen and Woodall [10] focussed on graphs embedded on surfaces and gave a complete characterization of all k and d such that every outerplanar graph is (k, d)-colorable, and a complete characterization of all k and d such that every planar graph is (k, d)-colorable. Namely, every outerplanar graph is 3-colorable, every outerplanar graph is (2, 2)-colorable, and there exist outerplanar graphs which are not (2, 1)-colorable. There does not exist a d such that every planar graph is $(1, d)$- or $(2, d)$-colorable; there exist planar graphs which are not (3, 1)-colorable, but every planar graph is (3, 2)-colorable. Together with the (4, 0)-coloring implied by the 4-Color Theorem, this solves defective chromatic number for the plane. Recently, Poh [26] and Goddard [15] showed that any planar graph has a special (3, 2)-coloring in which each color class is a linear forest (thus each color class is the disjoint union of paths), though this can in fact be read out of a more general result of Woodall [32, Theorem 2.2]. The interested reader is also referred to the excellent recent book of Jensen and Toft [19, cf. pages 40, 63.] for related problems and references.

For general surfaces, it was shown in [10] that for each genus $g \geq 0$, there exists a $k = k(g)$ such that every graph on the surface of genus g is $(4, k)$-colorable. This was improved to $(3, k)$-colorable by Archdeacon [2].

For general graphs, a result of Lovász from the 1960s [22], which has been rediscovered many times since (cf. [4, 8, 21, 32]), provides an upper bound on the defective chromatic number of a graph.

Theorem 1.1 (Lovász). For any k, any graph of maximum degree Δ can be $(k, \lfloor \Delta/k \rfloor)$-colored.

Proof. Take a k-coloring that minimizes the number of monochromatic edges. This must be the desired coloring, because if not, consider some vertex v with more than $\lfloor \Delta/k \rfloor$ of its neighbors self-colored. Since in any k-coloring of the vertices of G, there is always one color class with at most $\lfloor \Delta/k \rfloor$ members in the neighbor set of v, we can flip v's color to this color, thereby decreasing the number of monochromatic edges and contradicting the minimality of the original coloring. ∎

Corollary.

$$\chi_d(G) \leq \left\lceil \frac{\Delta+1}{d+1} \right\rceil = \left\lfloor \frac{\Delta}{d+1} \right\rfloor + 1.$$

The papers [1, 13, 14] provide other bounds on the defective chromatic number in terms of other parameters and in terms of other defective chromatic numbers.

The complexity of constructing defective colorings is less well-studied. However, R. Cowen [11] showed that (2, 1)-coloring is NP-Complete for general graphs. We also remark that the proof that any planar graph is (3, 2)-colorable in [10] is constructive, and gives a simple quadratic-time algorithm for (3, 2) coloring planar graphs (just as the proof of the 5-color theorem is constructive, and immediately implies an n^2 algorithm—improved by [9, 12, 24, 31] to a linear-time algorithm).

1.2. This Paper

In this paper, we first extend the results on the plane to the torus. First, we show that any graph embeddable on the torus is (3, 2)-colorable. Then we show that any such graph is also (5, 1)-colorable. In both cases, there exist quadratic time algorithms which construct the coloring. The question of whether or not every toroidal graph is (4, 1)-colorable remains an open question.

Second, we consider defective colorings of graphs on arbitrary surfaces. For genus γ, let $\chi_d(\gamma)$ be the maximum d-defect coloring number of all graphs embeddable on the surface S_γ. We show that $\chi_d(\gamma) \leq \chi_0(\gamma)/(d+1) + 4$. Also, Archdeacon [2] showed that every graph embeddable on the surface S_γ is $(3, 3\gamma + O(1))$-colorable. We show that this is improveable to $(3, c\sqrt{\gamma})$-colorable, which shows that the maximum defect needed for 3-colorability is within a constant factor of that needed for the maximum clique on that surface.

Finally, we consider complexity results and approximation algorithms for defective coloring. We show, perhaps surprisingly, that determining if a graph is (2, 1)-colorable is NP-complete for planar graphs, and this generalizes to $(2, d)$-coloring for $d \geq 1$. We show that determining if a planar graph is (3, 1)-colorable is also NP-Complete. And in general graphs we show that (k, d)-coloring is NP-Complete for all $k \geq 3$, and all $d \geq 0$, as expected. A simple reduction from proper coloring and the result of [23] shows that for any constant defect d, there exists an $\varepsilon > 0$ such that χ_d cannot be approximated within a factor of n^ε unless P $=$ NP.

These impossibility results for general graphs do not, of course, rule out good algorithms for defective coloring of bounded-degree graphs. A simple greedy algorithm produces the Lovász coloring cited above. We give polynomial-time approximation algorithms for defective coloring, in the spirit of Wigderson [30], and others who improved his bounds [5, 17, 20]. We show how to generalize both Wigderson's original algorithm, and the recent algorithms of Karger, Motwani and Sudan [20] to defects, and achieve a tradeoff between the defect and number of colors used.

The paper concludes with some open problems.

2. DEFECTIVE COLORING ON THE TORUS

Since every planar graph embeds on the torus there does not exist a d such that every toroidal graph is $(2, d)$-colorable. For 3 colors we need a result that is slightly stronger than, but whose proof is strongly similar to the proofs of, Theorem 5 in [10] and Theorem 1 in [15].

Theorem 2.2. Every planar graph can be (3, 2)-colored such that any two specified vertices v_1 and v_2 receive specified colors and such that for $i = 1, 2 v_i$ has no neighbor with the same color (except for possibly v_{3-i}).

Proof. We prove this by induction on the number of vertices.

First case: v_1 and v_2 are adjacent, and are required to be the same color. Then we contract them to a single vertex, choose a second vertex arbitrarily, and then use the induction hypothesis to (3, 2)-color the resultant graph so that the vertex v_1v_2 is given the specified color. When we uncontract, v_1 and v_2 are properly colored in the resulting (3, 2)-coloring except for the edge v_1v_2, as required.

Second Case: v_1 and v_2 are not adjacent. We may assume G is a maximal planar graph. So there must be a cycle that separates v_1 and v_2 in G: let W be such a cycle of minimum length

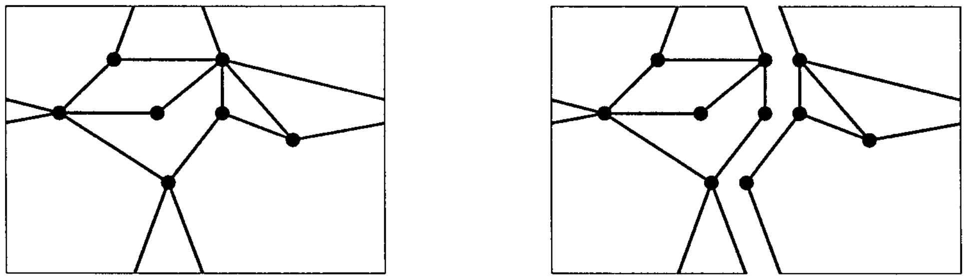

FIGURE 1. Cutting along minimum noncontractible cycle C making C_1 and C_2 yields a planar graph; C_1 and C_2 are then contracted to single vertices.

(so W is chord-free). Let $G_1(G_2)$ consist of $v_1(v_2)$ and all the other vertices and edges inside (outside) W. Let $G'_1(G'_2)$ be obtained from $G_1(G_2)$ by contracting W into a single new vertex w. These are both planar graphs with fewer vertices than G. Now by induction, color $G'_1(G'_2)$ with the requisite specified color for $v_1(v_2)$ and w specified to get a color distinct from either the color specified for v_1 or v_2 (possible, since there are three colors), and the vertices v_1, v_2 and w each without defect. We now transfer these colors back to G, giving all the vertices of W the color assigned to w.

Third case: v_1 and v_2 are adjacent and required to be different colors. Then we insert a new vertex on the edge between them, and without violating planarity add edges if necessary, to make G again a maximal planar graph. We then proceed as in the second case. ∎

Theorem 2.3. Every toroidal graph can be (3, 2)-colored.

Proof. Without loss of generality we may assume that G is a maximal toroidal graph. Let C be a minimal noncontractible cycle of G. Then cut down the middle of C: split every vertex and every edge of C into two parts yielding G' with two copies C_1 and C_2 of the cycle. For each edge linking a vertex v in C to a vertex w outside C that edge remains linking w and one of the copies of v as indicated. See Figure 1. At the same time this cut turns the torus into a sphere with the graph G' embedded on the sphere such that C_1 and C_2 are the boundaries of regions.

Form graph G'' from G' by contracting C_1 and C_2 each to a single vertex v_1 and v_2. Since G'' is planar, by the above theorem we can 3-color the vertices of G'' such that each color class has maximum degree two, and v_1 and v_2 both receive color 1 while none of their neighbors have color 1.

This yields immediately a coloring of G where all the vertices of C receive color 1. This is the desired (3, 2)-coloring. ∎

Note that this coloring can be found in quadratic time. In using Theorem 2.2, a total of at most a linear number of cycles are found, and a suitable cycle can be found in linear time. Concerning the complexity of transforming the toroidal graph to the plane, a combinatorial embedding of the graph on the torus can be found in linear time by the work of Mohar [25]. One way to then find a noncontractible cycle is to consider a breath-first search tree, and all the elementary cycles that contain an edge outside the tree. One of these cycles must be noncontractible (as every cycle in the graph is a combination of these elementary cycles). There are a linear number of elementary cycles. To test whether a cycle is suitable, one may use the embedding of the original graph, and then determine the genus of the two subgraphs by counting the vertices, edges, and faces and using Euler's formula.

We turn next to colorings with defect 1.

Lemma 2.1. If H is planar and u and v are vertices of H, then H can be (5, 1)-colored such that the induced graph $\langle N(u) \cup N(v) - \{u, v\}\rangle$ is 3-colored.

Proof. Find a maximal collection $\mathcal{P}$ of internally disjoint (u, v)-paths of length 3. Construct H' by, for each path ua_ib_iv in $\mathcal{P}$, contracting the edge a_ib_i. The resultant graph H' is planar.

So by the 4-color theorem, we can 4-color the graph H'. If we uncontract to get H we have a (4, 1)-coloring of H, since all vertices are properly colored, save those pairs of vertices that were contracted, which are adjacent with the same color and so have defect **1.** If u and v receive the same color in this coloring, we are done.

Otherwise, suppose u receives color **1** and v receives color **2.** Then any common neighbor is colored **3** or **4.** In particular, all the internal vertices of the paths in $\mathcal{P}$ receive color **3** or **4,** and color classes **1** and **2** are both independent sets. Now re-color every vertex in $\langle N(u) \cup N(v) - \{u, v\}\rangle$ that has color **1** or **2** with a new color, color **5.** Trivially $\langle N(u) \cup N(v) - \{u, v\}\rangle$ is 3-colored.

To prove the theorem we need to show that the new coloring is a (5, 1)-coloring. For this it is sufficient to show that the vertices with color **5** form an independent set. But suppose that there are vertices a and b which are adjacent and colored **5.** Then in the original coloring of H they must have had different colors. Say vertex a had color **2** and vertex b had color **1.** Then vertex a cannot be adjacent to vertex v and so must be adjacent to vertex u. Similarly, vertex b must be adjacent to vertex v. This yields a path $uabv$ of length **3** in H. Since neither a nor b received color **3** or **4,** this is a path internally disjoint from the ones in $\mathcal{P}$–-a contradiction. ∎

The time to algorithmically construct the coloring of Lemma 2.3 is dominated by the time to 4-color the graph, which is quadratic by the forbidden minors algorithm of Robertson, Sanders, Seymour and Thomas [28]. The next theorem shows how to transform the toroidal graph to the required planar graph, and this can also be done in quadratic time, by the method discussed above for finding a minimal non-contractable cycle.

Theorem 2.4 . One can (5, 1)-color any graph embedded in the torus.

Proof. Let G be embedded in the torus. Then there exists a minimal noncontractible cycle C that is an induced cycle. Construct a planar graph H by cutting along the edges of C, to form two copies of C, and contracting the two cycles to two single vertices u and v. By the above lemma we can (5, 1)-color H so that the neighbors of u and v are 3-colored. This translates to a (5, 1)-coloring of $G - C$ where the neighbors of C are 3-colored. Since there are two colors which we may use for C, we obtain the desired conclusion. ∎

Actually we obtain the conclusion with at least one of the color classes being an independent set. But we are unable to resolve the following question.

Question 2.5. Is every graph on the torus (4, 1)-colorable?

3. GENERAL GENERA

To prove upper bounds for graphs on general surfaces it helps to have the following upper bound on the defective chromatic number of dense graphs.

Theorem 3.6. For all d and for all graphs with q edges it holds that $\chi_d < \sqrt{2q}/(d+1) + 2$.

Proof. Suppose that $\chi_d = k$. Label the vertices of G $v_1, v_2, \ldots, v_n$ such that v_i has the maximum degree Δ_i in the graph $G_i = G - \{v_1, v_2, \ldots, v_{i-1}\}$. (Note that $\Delta_1 \geq \Delta_2 \geq \cdots \geq \Delta_n$.) Then by Lovász (Theorem 1.1) $\Delta_1 \geq (k-1)(d+1)$. Similarly $\Delta_{d+2} \geq (k-2)(d+1)$, else one can color $v_1, v_2, \ldots, v_{d+1}$ with one color and G_{d+2} with $k-2$ colors. In general,

$\Delta_{j(d+1)+1} \geq (k-j-1)(d+1)$ for $0 \leq j \leq k-2$. It follows that

$$q = \sum_{i=1}^{n} \Delta_i \geq (k-1)(d+1) + \sum_{j=1}^{k-2}(k-j-1)(d+1)^2$$
$$= (d+1)(k-1)(dk+k-2d)/2. \tag{1}$$

In particular, $2q > (d+1)^2(k-2)^2$, whence the result. ∎

One can extract a sharper bound form Inequality (1) that is best possible for $d = 0$ and $d = 1$. If $d = 0$ it follows that $q \geq k(k-1)/2$, while the clique K_k has the fewest edges for a graph with $\chi_0 = k$. If $d = 1$ it follows that $q \geq 2(k-1)^2$. This is best possible because the graph constructed by taking K_{2k-1} and removing the edges of a maximum matching has $2(k-1)^2$ edges but is not $(k-1, 1)$-colorable. (It cannot have three vertices of the same color.)

Theorem 3.7. For genus γ, let $\chi_d(\gamma)$ be the maximum d-defect chromatic number of all graphs embeddable on the surface S_γ. Then:

$$\chi_d(\gamma) < \chi_0(\gamma)/(d+1) + 4.$$

Proof. Recall that $\chi_0 = \sqrt{12\gamma} + \frac{7}{2} + o(1)$, and the extremal graph is the complete graph (see [27]). We show that $\chi_d < \sqrt{12\gamma} + 7$.

Suppose first that $n \leq (d+1)\sqrt{12\gamma}$. Recall that in a graph of genus γ it holds that $q \leq 3n + 6\gamma$. So $2q \leq 6(d+1)\sqrt{12\gamma} + 12\gamma$. Thus by Theorem 3.6

$$\chi_d(G) < \sqrt{2q}/(d+1) + 2 < (\sqrt{12\gamma} + 3(d+1))/(d+1) + 2 = \sqrt{12\gamma}/(d+1) + 5.$$

For graphs with $n > (d+1)\sqrt{12\gamma}$, the proof is by induction on n. Such a graph has a vertex v of degree at most $(6n + 12\gamma)/n < 6 + \sqrt{12\gamma}/(d+1)$. So one can remove vertex v, color the graph $G - v$ by the induction hypothesis, and then re-insert vertex v and properly color it. ∎

The above theorem shows that the number of colors needed is only a few more than those needed for the maximum clique on that surface. For fixed number of colors, namely 3, Archdeacon [2] showed that a graph is approximately $(3, 3\gamma)$-colorable. The next theorem shows that this is improveable to $(3, c\sqrt{\gamma})$-colorable, which shows that the maximum defect needed for 3-colorability is within a constant factor of that needed for the maximum clique on that surface. We use virtually the same approach that Archdeacon used.

Lemma 3.2. Let $t > 12$, and suppose G is a graph with minimum degree at least 3, the vertices of G of degree less than t form an independent set, and G has a 2-cell embedding on the surface of genus γ. Then the number of vertices of degree at least t is at most $24(\gamma - 1)/(t - 12)$.

Proof. Let S denote the set of vertices of degree less than t and T the vertices of degree at least t. Now, in each region that is not a triangle add edges between vertices of T. One way to do this is, if $v_1, \ldots, v_4$ are four consecutive vertices on the boundary of the region with $v_1 \in T$, then if $v_3 \in T$ then join v_1 and v_3 by an edge inside the region, otherwise join v_2 and v_4, and repeat as necessary. The result is a triangulation of a multigraph H which has minimum degree at least 3, and in which the vertices S of degree less than t still form an independent set. (Multiple edges can arise between two vertices x and y with several common neighbors, when edges are added between them in each region.)

Let α denote the number of edges between S and T, and β denote the number of edges between vertices of T. Let v_i denote the number of vertices of degree i. Since S is an independent set, it follows that $\alpha = \sum_{i<t} iv_i$ and $\alpha + 2\beta = \sum_{i\geq t} iv_i$. Since the embedding is a triangulation, it also follows that $\alpha \leq 2\beta$. Hence

$$\begin{aligned}(t/2-6)|T| &= (t/2-6)\sum_{i\geq t} v_i \\ &\leq \sum_{3\leq i<t}(2i-6)v_i + \sum_{i\geq t}(i/2-6)v_i \\ &= \sum_{i<t}(i-6)v_i + \alpha + \sum_{i\geq t}(i/2-6)v_i \\ &\leq \sum_{i<t}(i-6)v_i + (\alpha/2+\beta) + \sum_{i\geq t}(i/2-6)v_i \\ &= \sum_{i}(i-6)v_i \\ &= 12\gamma - 12,\end{aligned}$$

where the last equality is Euler's formula for triangulations. ∎

Theorem 3.8. A graph of genus γ is $(3, \max(12, \sqrt{12\gamma}+6))$-colorable.

Proof. Let $t = \max(13, \sqrt{12\gamma}+7)$. The proof proceeds by induction on the number of edges. If G contains a vertex v of degree at most 2 then we easily obtain a suitable coloring of G from a coloring of $G - v$. Furthermore, if there is an edge e that joins two vertices of degree less than t then remove the edge, induct and conclude. Therefore we may assume that the minimum degree is at least 3 and that the vertices of degree less than t form an independent set S.

By the above lemma it follows that $T = V(G) - S$ has at most $24(\gamma-1)/(t-12)$ members. We form a 3-coloring by making all the members of S the first color, and then half the members of T receive the second color and half the third color. The defect is at most $|T|/2$ and so this gives the desired bound. ∎

4. HARDNESS RESULTS

We show in this section that determining whether or not a graph is $(2, d)$-colorable is NP-complete even for planar graphs. This extends a result of R. Cowen [11] who showed that (2, 1)-coloring is NP-complete in general graphs. We show that determining if a planar graph is (3, 1)-colorable is also NP-complete.

We also show that determining whether a graph of maximum degree 4 is (2, 1)-colorable is NP-complete, and in general so is determining whether a graph of maximum degree $2(d + 1)$ is $(2, d)$-colorable for $d \geq 1$. Thus there is no simple characterization of graphs for which equality holds in Theorem 1.1 and thus there is no equivalent of Brooks' theorem in general for defective colorings.

Not surprisingly, (k, d)-coloring is NP-complete in general graphs for all $k \geq 3$, and all $d \geq 0$. A simple reduction from proper coloring and the result of [23] shows that for any constant defect k, there exists an $\varepsilon > 0$ such that χ_d cannot be approximated within a factor of n^{ε} unless P = NP.

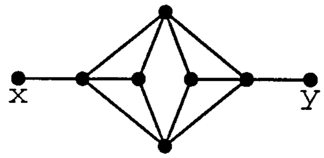

FIGURE 2. A regulator: x and y must have the same color in a (2, 1)-coloring.

4.1. Defective Coloring in the Plane

It is easy to (2, 0)-color any (planar) graph in linear time if such a coloring exists. Determining whether a planar graph is 3-colorable is NP-complete. We remarked earlier that the 4-color theorem gives a quadratic time algorithm for 4-coloring planar graphs (although the constants are terrible). Since, as was remarked in the introduction, the theorem in [10] provides a quadratic-time algorithm to (3, 2)-color any planar graph, together with the results of this section, this characterizes the complexity of defective coloring in the plane.

Theorem 4.9. To determine whether or not a graph is (2, 1)-colorable is NP-complete even for graphs of maximum degree 4 and for planar graphs.

Proof. We first show that (2, 1)-colorability is NP-hard for graphs of maximum degree 4 by reduction from 3-SAT, and then use an idea similar to that used in [29] to planarize the structure. We will show that for any 3-CNF ϕ, there exists a graph G_ϕ of maximum degree 4 constructible in polynomial time such that ϕ is satisfiable if and only if G_ϕ is (2, 1)-colorable.

We define a ''regulator'' as a gadget between two vertices x and y which forces them to have the same color but they have no defect within the gadget. One regulator consists of vertices $u_1, u_2, \ldots, u_6$ such that u_1 and u_2 are both adjacent to all four other vertices and u_3u_4 and u_5u_6 are edges. See Figure 2. When we connect x to u_3 and y to u_6, the only (2, 1)-coloring of this subgraph has $\{u_1, u_2, x, y\}$ as one color-class.

We need a vertex-gadget: a large subgraph that has a unique (2, 1)-coloring up to interchanging the names of the colors. One way to form a vertex-gadget is to string a series of vertices together with regulators, and use a $K_{2,3}$ as a ''oppositer'' as depicted in Figure 3. We use a double line to indicate a regulator.

Now for an OR-gate we use a 5-cycle. Say it's labelled $v_1v_2v_3v_4v_5v_1$. Two neighboring vertices v_1 and v_2 of the 5-cycle are joined by regulators to the vertices corresponding to the desired literals. Then the vertex v_4 at distance 2 from them must receive one of the colors that they do. We can join the output vertex by a regulator to another OR-gate and thus simulate an OR of three literals. Finally we joint the output vertex from all the second OR-gates by regulators. The subgraph associated with the clause $p \vee q \vee r$ is shown in Figure 4. The graph that results is G_ϕ and has degree at most 4. The number of vertices in G_ϕ is linear in the number m of literals in ϕ; so this reduction is polynomial.

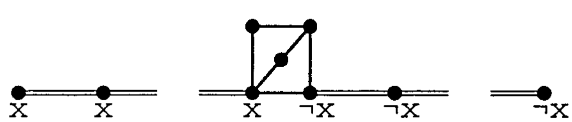

FIGURE 3. A vertex-gadget: x and $\neg x$ receive opposite colors.

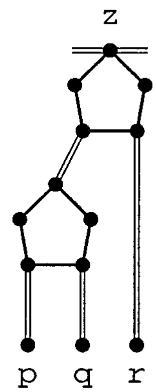

FIGURE 4. z can be colored 1 iff at least one of p, q or r is colored 1.

Suppose we have a (2, 1)-coloring of G_ϕ. Without loss of generality, assume that the output of each clause is colored 1. By construction, at least one of the inputs to each clause is colored 1 also. If we associate **1** with TRUE and **2** with FALSE, this coloring yields a satisfying assignment for ϕ. Conversely, if ϕ is satisfiable, then the truth assignment yields a (2, 1)-coloring for G_ϕ as follows: color the vertices associated with true variables with color 1 and the others with color 2. Then for each OR, if there is one 1-input, color the graph appropriately. It is easy to see that this is a (2, 1)-coloring, so we are done.

The graph constructed for the reduction above is unlikely to be planar. However, it can be made planar as follows. We can arrange the vertex-gadgets and the clauses so that the only edges that can cross are ones joining vertex-gadgets to OR-gates. Then, whenever two edges cross, we can uncross them as shown in Figure 5. It is easy to argue that x' must receive the same color as x, and y' must receive the same color as y. The number of times we might need to use the uncrosser is at most the number of pairs of edges in G_ϕ, so the resulting graph would have $O(m^4)$ vertices–-still polynomial. ∎

Notice that the planarizing structure in this construction increases the maximum degree of the graph to 5. We have been unable to find a reduction to planar graphs of maximum degree 4.

Theorem 4.10.

(a) For any positive integer d, deciding whether a planar graph is $(2, d)$-colorable is NP-complete.

(b) Deciding whether a planar graph is (3, 1)-colorable is NP-complete.

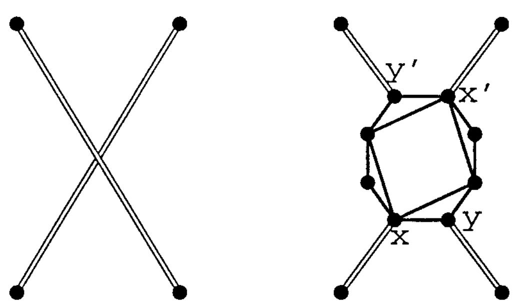

FIGURE 5. Uncrossing edges in G_ϕ.

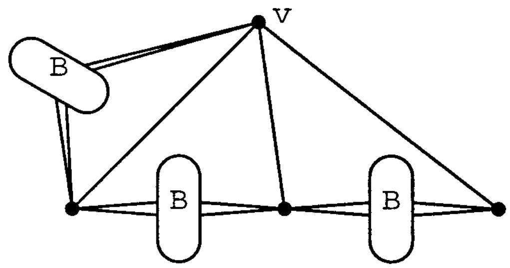

FIGURE 6. v has defect $d-1=3$ in D_v.

Proof. (a) The reduction is from (2, 1)-coloring in planar graphs. For each vertex x in G introduce the structure D_v defined as follows. The vertex set of D_v consists of the sets $B_1, B_2, \ldots, B_{d-1}$, each of cardinality $2d+1$, and the vertices $c_1, c_2, \ldots, c_{d-1}$. The only edges in D_v join c_i to all of B_i and B_{i+1} for $1 \leq i < d-1$, and c_{d-1} to all of B_{d-1}. Then v is joined to B_1 and all of the c_i. See Figure 6. In any $(2, d)$-coloring of D_v the vertices c_i must all have the same color. Furthermore, at least $d-1$ of each B_i must have the color opposite to the c_i. This means that v has defect at least $d-1$ in D_v. But by giving all the c_i the same color as v and all the B_i the opposite color one can ensure that v has defect exactly $d-1$ in D_v. Thus the resulting planar graph G' has a $(2, d)$-coloring if and only if the original graph G had a (2, 1)-coloring.

(b) The reduction is from planar 3-coloring (cf. Stockmeyer [29]). For any graph G in the plane, form the graph G' by joining to each vertex of G the 6-vertex Hajós subgraph H depicted in Figure 7. Since H is outerplanar, all its vertices can be joined to a single vertex of G and the resulting graph will still be planar. Furthermore, it is simple to check that H is not (2, 1)-colorable, so in any (3, 1)-coloring, all 3 colors must appear among the vertices of each copy of H, while H can be (3, 1)-colored so that a specified color appears only once thereby giving each vertex in G exactly one new defect. Thus G' will be (3, 1)-colorable if and only if G was (3, 0)-colorable. ∎

We remark that even though (2, 1)-coloring planar graphs is NP-complete, nonetheless these graphs form a class of planar graphs that are easy to 4-color in practice. For, it follows from Euler's formula that the average degree of any planar bipartite graph is less than 4. Since a (2, 1)-colorable graph is the edge-union of a bipartite graph and a matching, the average degree of any (2, 1)-colorable planar graph will be less than 5, and hence will always have a vertex v of degree 4 or less. So G can be colored by induction, using a Kempe-chain argument (from the 5-color theorem) to re-color in linear time in the case when v's four neighbors have different colors.

4.2. General Defective Coloring

These results are straightforward.

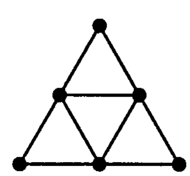

FIGURE 7. An outerplanar graph, H, which is not (2, 1)-colorable.

Theorem 4.11. (k, d)-colorability is NP-complete for any $k \geq 3$ and $d \geq 0$.

Proof. We reduce $(k, 0)$-colorability to (k, d)-colorability as follows. First consider the complete k-partite graph K, in which each part contains $kd + 1$ vertices. Because each part contains at least $kd + 1$ vertices, the only valid (k, d)-coloring of K assigns a different color to each part. Now form the graph G' from the graph G by connecting each vertex of G to d vertices in each part of K. The resulting graph G' is indeed (k, d)-colorable if and only if G is $(k, 0)$-colorable. ∎

We can also show that whether a graph of maximum degree $2(d + 1)$ is $(2, d)$-colorable is NP-hard. The reduction from (2, 1)-coloring in graphs of maximum degree 4 is by adding for each vertex v a copy of the complete bipartite graph $K_{2d+1,2d+1}$ and making v adjacent to $d-1$ vertices in each part of the new subgraph. This means there is not always a Brooks-type improvement on Lovász's bound. However we do not know what happens for 3 or more colors. For example, what is the complexity of (3, 1)-coloring in graphs of maximum degree 6? (We can only prove intractability for degree 7.) Thus we ask the following question:

Question 4.12. In general, what is the complexity of (k, d)-coloring in graphs of maximum degree $k(d + 1)$?

Theorem 4.13. For constant defect d, there exists an $\varepsilon > 0$ such that no polynomial-time algorithm can n^{ε}-approximate the d-defective chromatic number, unless P $=$ NP.

Proof. This is an immediate consequence of the results of Arora et al. [3] on the hardness of approximating the normal chromatic number. For, it is easy to transform a (k, d)-coloring of a graph G into an $(k(d+1), 0)$-coloring of G, by simply $(d+1)$-coloring each color class, because δ in a color class is at most d. This is an $O(k)$-coloring of G when d is constant. ∎

5. ALGORITHMS AND APPROXIMATE DEFECTIVE COLORING

Wigderson [30] gives the following algorithm to approximately color 3-colorable graphs. Pick a threshold δ. Take the node of highest degree and 2 color its neighborhood with two new colors. Remove its neighborhood. Continue until all nodes have degree at most δ. Then we can $\delta + 1$ color the remaining graph. Each round we eliminate at least δ nodes using 2 colors, so the total number of colors used is $2n/\delta + \delta + 1$, and we choose $\delta = O(\sqrt{n})$ to optimize. We now show how to modify this allowing for some defect, d.

The Wigderson algorithm is a 2-stage procedure, and it fits into the paradigm that has been used by nearly all subsequent algorithms for approximate 3-coloring (see [7, 5, 20, 6]).

(1) If the maximum degree of G is high, use the fact that the graph is 3-colorable to find a large independent set in the graph
(2) If the maximum degree of G is low, we can color with few colors.

We know of no way to improve on step (1), above, when the coloring is relaxed to allow defects, since in all cases, the original algorithms rely heavily on the fact that the (shared) neighborhood of a (set of) vertex (vertices) is 2-colorable, and finding the 2-coloring is easy. By contrast $(2, d)$ coloring is NP-complete for any constant $d > 0$, as we showed in the previous section. However, we can generalize both the first bound of Wigderson, and the more recent results by Karger, Motwani and Sudan [20], both of which improve the number of colors used in step (2), to a tradeoff for defects. The improvements in step (2) can then be inserted into the hybrid algorithms to achieve the best tradeoffs for defects.

The modification to Wigdgerson's algorithm is based on Lovász's coloring theorem as described in Section 1.1. First note, one can construct the Lovász coloring, using a simple greedy algorithm.

Theorem 5.14. Any graph of maximum degree Δ can be $(k, \lfloor \Delta/K \rfloor)$ colored in $O(\Delta E)$ time.

Proof. Begin with all vertices in the graph the same color. Then construct the Lovász coloring by repeatedly picking any vertex v with more than $\lfloor \Delta/k \rfloor$ self-colored neighbors, and flipping its color to a different color which at most $\lfloor \Delta/k \rfloor$ neighbors of v have (such a color must exist by the pigeonhole principle). The procedure terminates in at most E steps, since every time a vertex v is flipped in G, the number of monochromatic edges in G decreases by at least 1. ∎

For example, any 3-regular graph can be (2, 1)-colored in $O(E)$ time, and any 6-regular graph can be (3, 2)-colored in $O(E)$ time.

Theorem 5.15. There exists an $O(\Delta E)$-time algorithm to $(\lceil (\frac{8n}{d})^{.5} \rceil, d)$-color any 3-colorable graph.

Proof. We follow the algorithm of Wigderson until the maximum degree is δ. By Theorem 5.14, the remaining graph can be $(\frac{\delta}{d}, d)$ colored in $O(\delta E)$ time. The total number of colors is $2n/\delta + \delta/d$ which is optimized by choosing $\delta = \sqrt{2n/d}$. ∎

5.1. Generalizing the KMS Algorithm to Defects

We next show how to get a similar tradeoff for the better approximation algorithms of Karger, Motwani and Sudan [20]. We use the semidefinite program approach of [20] to obtain a vector 2-coloring. We then round to an integer defective coloring.

The approximation algorithms of Karger et al. [20] work as follows. First, the 3-coloring problem is relaxed to the *vector 3-coloring* problem, which is solved in polynomial time using semidefinite programming. The vector 3-coloring assigns unit vectors from $\mathbf{R}^n$ to the vertices so that two vertices that are adjacent in the graph have vectors whose dot product is at most $-\frac{1}{2}$.

Next, the vector 3-coloring is rounded to an ordinary coloring. One method used in [20] entails partitioning the space $\mathbf{R}^n$ by random hyperplanes and giving the vertices in each partition the same color. This results (with high probability) in what they call a *semicoloring,* which is an assignment of colors to the n vertices so that the set of vertices which are not properly colored is of size less than $n/4$. Recursion finished the process off.

A semicoloring bears considerable superficial resemblance to defective coloring, because it allows (many) defective (i.e., monochromatic) edges. However, a semicoloring is a *global* condition on defects that does not, in general, place any guarantee on the maximum local defect. We modify the definition to reflect defects.

Definition. A *d-defect semicoloring* is an assignment of colors to the n vertices, such that the number of vertices that have more than d adjacent neighbors of the same color, is less than $n/4$.

Theorem 5.16. There exists a polynomial-time algorithm to $(O((n/d)^{.387}), d)$-color a 3-colorable graph on n vertices.

Proof. We start with a vector 3-coloring. Then we select $r + O(1)$ independent random hyperplanes (where a random hyperplane is one with a random normal on the unit sphere in $\mathbf{R}^n$). Color each vertex according to which cell of the partition it lies in. The fact that the vectors assigned to each vertex form a vector 3-coloring implies that the probability that a random hyperplane cuts an edge (i.e., cuts the segment joining the ends of the two vectors) is at least $\frac{2}{3}$, by a lemma of [16].

Then, by Markov's inequality, with probability $\frac{1}{2}$, the number of uncut edges is at most $c(\frac{1}{3})^r n\Delta$, where Δ is the maximum degree of the graph. The number of vertices which are adjacent to at least d uncut edges is at most $2c(\frac{1}{3})^r n\Delta/d$, which is $o(n)$ when $r = \log_3 O(\Delta/d)$ (and less than $n/4$ by appropriate choice of constants).

If we repeat the entire process t times, then with probability at least $1 - \frac{1}{2}^t$ we will obtain a d-defect semicoloring with $O(2^r) = O((\Delta/d)^{\log_3 2})$ colors, which is approximately $O((\Delta/d)^{.631})$. The algorithm which produced the d-defect semicoloring is used recursively on the vertices with more than d adjacent neighbors of the same color. This colors the graph with defect d with less than a factor of 2 increase in the number of colors.

This approach is now combined with the Wigderson technique. Fix a threshold T. While the maximum degree of the graph is greater than T (or until half the vertices have been colored) pick a vertex of maximum degree, and 2-color its neighborhood (its neighborhood is 2-colorable, and 2-coloring is easy) with two new colors. Then one can finish with the above method when the degree falls below T. This yields a coloring with $O(n/T + (T/d)^{.631})$ colors, which is optimized by setting $T = n^{.631}d^{.387}$. ∎

We remark that a recent paper of Blum and Karger [8] achieves a $\tilde{O}(n^{2/9})$-coloring for 3-colorable graphs, by combining a modification of some complicated improvements for step 1, due to Blum [5], with a more complicated version of [20] that uses a ''random center'' rather than random hyperplane method for rounding. With a little work, one can obtain the corresponding $\tilde{O}((n/d)^{2/9}, d)$-coloring.

Also, Theorem 5.16 can be generalized to χ-colorable graphs for $\chi > 3$. The guarantee on the size θ of the angle between the unit vectors corresponding to adjacent vectors is now only $\theta \geq \arccos(-1/(\chi - 1))$. A lemma of [16] says that the probability that an edge is cut by a random hyperplane is θ/π. Thus the probability that an edge is not cut by r independent random hyperplanes is given by $(1-\theta/\pi)^r = (1/\alpha)^r$, say. With appropriate choice of constants, just as in the proof of Theorem 5.16, (since Δ is always less than n) we immediately obtain an d-defective semicoloring using $\tilde{O}((n/d)^{\log_\alpha 2})$ colors. This yields an $(\tilde{O}((n/d)^{\log_\alpha 2}), d)$-coloring.

6. APPLICATIONS AND OPEN PROBLEMS

The two most immediate open problems are Questions 2.5 and 4.12 listed in the text. The first asks whether every toroidal graph is (4, 1)-colorable. This would complete the characterization of defective colorings on the torus. The other asks for the complexity of (k, d)-coloring in graphs of maximum degree $k(d + 1)$. This is known to be easy for $d = 0$ (by Brooks' theorem) and is now known to be hard for $k = 2$ and $d > 0$.

The most obvious application of defective coloring is a generalization of the application of coloring to scheduling. For the scheduling problem where vertices represent jobs (say users on a computer system), and edges represent conflicts (needing to access one or more of the same files), allowing a defect means tolerating some threshold of conflict: each user may find the maximum slowdown incurred for retrieval of data with 2 conflicting other users on the system acceptable, and with more than 2 unacceptable. One might generalize this still further: to model different tolerances at different vertices. Some jobs may be more tolerant of interference than others, or all conflicts could not be equally expensive. This could partially be modeled by allowing multiple edges, or equivalently weights on the conflict edges. Notice that the Lovász coloring result (Theorem 5.14). would still apply in this case. In addition, if different colors correspond to different time periods in the schedule, it is possible that some jobs may not be able to schedule in all time-slots; rather each job may have a different subset of slots in which it is allowed to be scheduled. This is the defective version of the ''list-coloring'' problem, and would allow the modeling of more complicated constraints.

Another approach involves looking at alternative definitions of defective coloring. One possibility would be to allow some *total* number of monochromatic edges, rather than the stronger requirement of a maximum threshold of monochromatic edges at each vertex. One specific generalization is to allow different defects for different colors. For example we might use the notation [0, 1]-coloring to denote a coloring of the vertices with two colors such that the first color is an independent set and the second color has defect at most 1. One can show that even this simple extension of bipartiteness is NP-hard for planar graphs. The generalization of Theorem 1.1 to defects which are bounded as a function of the vertex and color has been explored in [4, 8, 21, 32].

ACKNOWLEDGMENTS

The first and second authors pretentiously thank Cafe Sha Sha in Greenwich Village, on whose napkins the proof of Theorem 2.3 was first scrawled. The first and third authors wish to thank DIMACS: both these authors worked on these problems while visiting DIMACS in the Summer of 1994. Thanks to Martin Farach, Steve Mahaney, Mike Saks, and Carsten Thomassen for helpful discussions. Thanks to Alan Goldman and Peter Shor for helpful comments.

LJC was supported in part by an NSF postdoctoral fellowship, and by ONR Young Investigator Grant N00014-96-1-0829. WDG was supported in part by Foundation for Research Development.

References

[1] J. Andrews and M. Jacobson, On a generalization of chromatic number, *Congressus Numer.* **47** (1985), 33–48.

[2] D. Archdeacon, A note on defective coloring of graphs in surfaces, *J. Graph Theory* **11** (1987), 517–519.

[3] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy, Proof verification and hardness of approximation problems, in *Proceedings of the 33rd Annual Symposium on Foundations of Computer Science,* pp. 14–23, 1992.

[4] C. Bernardi, On a theorem about vertex colorings of graphs, *Discrete Math.* **64** (1987), 95–96.

[5] A. Blum, New approximation algorithms for graph coloring, *J. ACM* **41** (1994), 470–516.

[6] A. Blum and D. Karger, unpublished manuscript (1996).

[7] B. Berger and J. Rompel, A better performance guarantee for approximate graph coloring, *Algorithmica* **5**(4) (1990), 459–466.

[8] O. Borodin and A. Kostochka, On an upper bound of a graph's chromatic number depending on the graph's degree and density, *J. Combinatorial Theory, Ser. B* **23** (1977), 247–250.

[9] N. Chiba, T. Nishizeki, and N. Saito, A linear-time algorithm for 5-coloring a planar graph, *J. Algorithms* **2** (1981), 317–327.

[10] L. Cowen, R. Cowen, and D. Woodall, Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valence, *J. Graph Theory* **10** (1986), 187–195.

[11] R. Cowen, Some connections between set theory and computer science, in G. Gottlob, A. Leitsch, and D. Mundici (eds.), *Proceedings of the third Kurt Godel colloquium on Computational Logic and proof theory,* Lecture Notes in Computer Science, pages 14–22. Springer-Verlag (1993).

[12] G. Fredrickson, On linear-time algorithms for five-coloring planar graphs, *Information Processing Lett.* **19** (1984), 219–224.

[13] M. Frick, A survey of (m, k)-colorings, in J. Gimbel, J. W. Kennedy, and L. V. Quintas (eds.), *Quo Vadis, Graph Theory?,* volume 55 of Annals of Discrete Mathematics, pages 45–58. Elsevier Science Publishers, New York (1993).

[14] M. Frick and M. Henning, Extremal results on defective colorings of graphs, *Discrete Math.* **126** (1994), 151–158.

[15] W. Goddard, Acyclic colorings of planar graphs, *Discrete Math* **91** (1991), 91–94.

[16] M. Goemans and D. Williamson, .878-approximation algorithms for MAX CUT and MAX 2SAT, in *Proceedings of the 26th Annual ACM Symposium on Theory of Computing* (1994).

[17] M. Halldorsson, A still better performance guarantee for approximate graph coloring, *Information Processing Lett.* **45** (1993), 19–23.

[18] F. Harary and K. Jones, Conditional colorability II: Bipartite variations, *Congressus Numer.* **50** (1985), 205–218.

[19] T. Jensen and B. Toft, *Graph Coloring Problems,* John Wiley and Sons, New York (1995).

[20] D. Karger, R. Motwani, and M. Sudan, Approximate graph coloring by semidefinite programming, in *Proceedings of the 35th Annual Symposium on Foundations of Computer Science* (1994).

[21] J. Lawrence, Covering the vertex set of a graph with subgraphs of smaller degree, *Discrete Math.* **21** (1978), 271–273.

[22] L. Lovász, On decomposition of graphs, *Studia Sci. Math. Hungar.* **1** (1966), 237–238.

[23] C. Lund and M. Yannakakis, On the hardness of approximating minimization problems, in *Proceedings of the 25th Annual ACM Symposium on Theory of Computing,* pages 286–293 (1993).

[24] D. Matula, Y. Shiloah, and R. Tarjan, Two linear-time algorithms for five-coloring a planar graph, technical report STAN-CS-80-830, Dept. of CS, Stanford U.

[25] B. Mohar, Embedding graphs in an arbitrary surface in linear time, in *Proceedings of the 28th Annual ACM Symposium on Theory of Computing* (1996).

[26] K. Poh, On the linear vertex-arboricity of a planar graphs, *J. Graph Theory* **14** (1990), 73–75.

[27] G. Ringel, *Map Color Theorem,* Springer-Verlag, Berlin (1974).

[28] N. Robertson, D. Saunders, P. Seymour, and R. Thomas, Efficiently four-coloring planar graphs, in *Proceedings of the 28th Annual ACM Symposium on Theory of Computing* (1996).

[29] L. Stockmeyer, Planar 3-colorability is NP-complete, *SIGACT news* **5** (1973), 19–25.

[30] A. Wigderson, Improving the performance for approximate graph coloring, *J. ACM* **30** (1983), 729–735.

[31] M. Williams, A linear algorithm for coloring planar graphs with five colors, *Comput. J.,* **28** (1985), 78–81.

[32] D. Woodall, Improper colourings of graphs, in R. Nelson and R. J. Wilson (eds.), *Graph Colourings,* Longman Scientific and Technical (1990).

Received March 29, 1995