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Modelling of continuous elastic systems by using the Finite Element Method

Modelización de sistemas elásticos continuos mediante el Método de Elementos Finitos

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Abstract

Propagation of mechanical waves in unidimensional systems is a fundamental part of physics, necessary for learning subjects such as acoustics and vibrations. The vibration of transverse waves in strings is the easiest case of elastic system. Usually, this is the first continuous elastic system in which students apply fundamental mathematical concepts as vibration mode, equation of motion and boundary condition. In this work the use of simulation methods is proposed to reinforce the understanding of vibratory and acoustic simple phenomena. This will be applied to the case of a string, a beam and a membrane of finite length with different physical characteristics and boundary conditions.

La propagación de ondas mecánicas en sistemas unidimensionales es una parte fundamental de la física, necesaria para el aprendizaje de asignaturas como acústica y vibraciones. La vibración de ondas transversales en cuerdas es el caso más sencillo de sistema elástico. Habitualmente, este es el primer sistema elástico continuo en el cual los alumnos aplican conceptos matemáticos fundamentales como modo de vibración, ecuación de movimiento y condición de contorno. En este trabajo se propone el uso de los métodos de simulación para reforzar la comprensión de fenómenos simples en acústica y vibraciones. Lo aplicaremos a los casos de cuerda vibrante, barras y membranas de longitud finita con diferentes características físicas y condiciones de contorno.

Keywords: Acoustics; vivration; continuum elastic system; finite Element Method: vibrating string, beam
Palabras clave: Acústica; vibración; sistema elástico continuo; método de elementos finitos, cuerda vibrante; membrana; barra

1. Introduction

Models are essential to the production, dissemination and acceptance of scientific knowledge (Giere, 1988; Gilbert, 1991; Tomas, 1988). The value of models in the field of science education has been increasingly recognized among the science education reform movements (Gobert, 2000). Both the design of and implementation of experimental practices in modern science are often based on the use of computational modeling (Gilbert, 2004). Simulation softwares have evolved significantly over the last years, which has gifted these new technologies plenty of increasingly important advantages, such as an easy visualization of some phenomena and the possibility of having an intermediate step between the theoretical and the experimental phase of the work. In this sense, the method can be applied in many fields like biomechanics (Kinzl, 2013), the analysis of surface acoustic wave devices (Mohibul et al., 2017) or the development of fatigue tests (Yu et al., 2017), among many others, as well as in the validation of experimental results obtained.

Nevertheless, the computational cost can be huge due to the number of parameters to be studied or the wide range of some of them and it will always depend on the power of the device to be used. Thus, it is necessary to reach a compromise that allows getting a result that satisfies the requirements of the study with a reasonable time consumption, for which it is indispensable to optimize the available resources. The more complete is the model, the more computational power will be required to calculate its solution.

In this work authors show the application of modelling in the field of vibrations, applying simulation techniques to solve numerically the equations of different simple continuous elastic systems. These systems, understood as those systems whose characteristics are continuous along their whole domain (as opposite to discrete systems), are strings, beams and membranes. The main goal is to show the utility of this learning tool for science and engineering students at universities as a way to secure fundamental mathematical concepts due to its simplicity and easy use and, by extension, as a teaching tool for a more easy explanation and visualization of some phenomena.

Models presented in this work are implemented using COMSOL Multiphysics, which is a software based on the Finite Element Method (FEM). It is applicable to all fields of engineering and science (Oladejo, Abu & Adewale, 2012). This method is based in the discretization of one or more domains and calculating the approximated solution of the equations that govern the phenomena to be studied for each of the resulting elements.

The application of this method is particularly interesting when hand calculations cannot provide accurate results (Oladejo, Abu & Adewale, 2012), the model to be developed has a complex geometry or the range of parameters to be studied is very wide. In this last case, FEM allows analysing different parameters simultaneously in order to choose the optimum one.

2. The vibrating string

Usually in textbooks of mechanical systems, the first model of a continuous system presented to the reader is the vibrating string. It is an idealized representation of a one dimensional system that allows the propagation of transverse waves, that is, waves with displacement perpendicular to the propagation direction. An application of this model is for the acoustic behaviour of chordophones, as the vibration of the string is the main physical mechanism for the description of the sound radiation of these musical instruments.

The model consists of an homogeneous string of length L with tension T and linear mass density per unit length μ. It is assumed to be infinitesimally thin and completely flexible. If all these suppositions are satisfied, transverse waves are supported by the system and the wave equation describing the transverse displacement of the string $y(x,t)$ is given by:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = c^2 \frac{\partial^2 y(x,t)}{\partial x^2} \tag{1}$$

where the speed of the travelling wave is related to the intrinsic properties of the string, $c = \sqrt{T/\mu}$. Remark that in this system, all the frequencies travel with the same speed as $\omega/k = c$, and thus, transverse waves propagating in a string are non-dispersive.

Several discrete solutions (called 'modes') can be found for the transverse displacement at spatial position x and time t when the wave equation 1 is solved assuming that both ends of the string are fixed, $y(x=0,t) = y(x=L,t) = 0$:

$$y_n(x,t) = \tilde{A}_n e^{j\omega_n t} sin \frac{n\pi}{2L} x, \tag{2}$$

where n is an integer number corresponding to each mode, $\tilde{A}_n$ is the amplitude and ω_n is the angular frequency of the nth mode.

The discreteness of solutions shows that only some specific frequencies (called 'natural frequencies' or 'eigenfrequencies') are allowed to propagate in the system. The lowest fundamental frequency of a vibrating string with fixed ends is termed the fundamental frequency, and the higher natural frequencies are termed overtones (Kinsler et al., 2000). Natural frequencies depend on the specific properties of the string and can be calculated as follows:

$$f_n = \frac{nc}{2L}. \tag{3}$$

The analytical procedure describe below for the vibrating string is consistent with modelling the the system and solving it with a numerical method. Indeed checking that analytical solutions match with numerical calculations is a significant pedagogical support for the student. Here a server with two Intel Xeon E5-2680v2 10C/20T 2.8GHz 25MB processors and 256 Gb DDR3-1866 R ECC memory has been used for implementing all the simulations. Due to the simplicity of this model, the number of elements in the mesh is very small (201 elements). The computation time is particularly short, taking only a few seconds in our work station. For any standard computer it will be also very short.

The mode shapes for the unidimensional string with fixed ends are obtained numerically and represented for n=1, 2 and 3 in Figure 4. The transverse displacement $y(x,t)$ of a string with length $L = 1$ m is represented for each position x at a specific value of time t. Remark that the boundary conditions are satisfied for all the modes as always a minimum of transverse displacement (called 'node') appears in both ends.

3. Vibrations in beams

As in the case of the vibrating string the propagation of waves in beams is applicable to musical instruments: idiophones (as bars of xylophones or marimbas). Other applications are piezoelectric tubes, system calibration.

Since it is true that one of the dimensions of beams is usually greater than the others, in this case can not be considered as infinitely greater and more than physically significant dimension

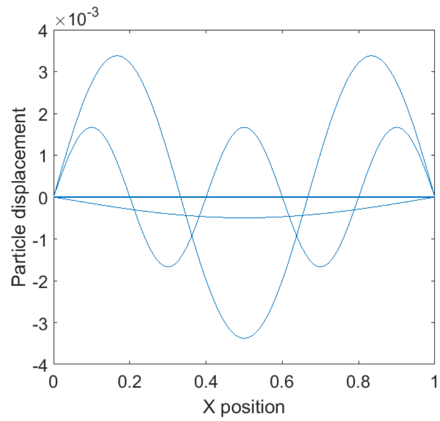

Figure 1: First three mode shapes of a one-dimensional vibrating string of length $L = 1m$ with fixed boundaries at $x = 0$ and $x = L$.

can be considered. Thus, the model has to be developed in the three dimensions of the space for more confident results.

Three main types of waves propagate in beams: longitudinal waves, transverse waves and torsional waves. Polarization of transverse waves is identical to those present at the vibrating string. Longitudinal waves are those in which particles move parallel to the propagation axis of the wave and torsional waves are those in which the vibration causes the rotation of the beam on its own axis. The wave equation of every type of waves propagating in beams are as follows:

Longitudinal waves,

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2} \tag{4}$$

Transverse waves,

$$\frac{\partial^2 y}{\partial t^2} = -b^2 \frac{\partial^4 y}{\partial x^4} \tag{5}$$

Torsional waves,

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2} \tag{6}$$

where ξ is the particle displacement in the longitudinal plane, y is the particle displacement in the transverse plane and ϕ is the rotation angle.

Two boundary conditions are considered in this work discussed in the following section: a beam with both ends fixed and a beam with a mass-loaded end.

3.1. Case 1: Clamped - Clamped

The clamped-clamped boundary condition implies that both ends of the beam are fixed to a rigid wall with theoretically infinite mass. In this way, both the displacement and the velocity in these points are equal to 0,

$$\begin{gathered} u(0) = u'(0) = 0 \\ u(L) = u'(L) = 0 \end{gathered}$$

The fundamental frequency of the beam is also determined by this boundary condition and it is given by Eq. 7.

$$\xi_n\left(x,t\right)=\left[A_n\cos\left(\omega_n t\right)+B_n\sin\left(\omega_n t\right)\right]\sin\left(k_n x\right) \tag{7}$$

where A_n and B_n are amplitude coefficients, n is the mode number and ω_n is the angular frequency for the nth mode.

If the step to numerical methods is given, it is possible to implement a 3-D eigenfrequency model considering, as in the case of the string, an homogeneous beam with length L and linear density and supposing that the magnitudes to be studied are linear. With these requirements, a model consisting of 7089 elements is created.

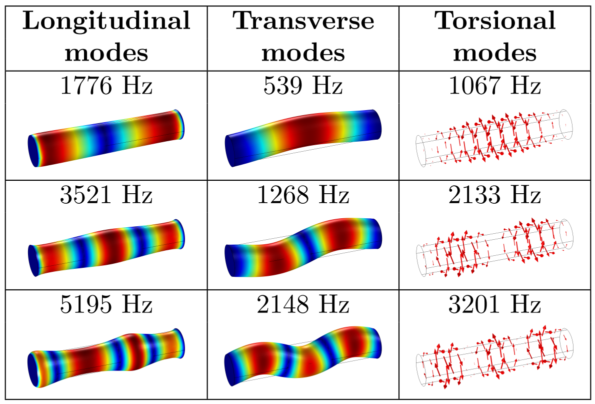

Figure 2: Vibrational modes of beams with clamped boundary condition.

Results of the eigenfrequencies of a $L = 1$ m beam made of steel with circular cross-section from the numerical simulations can be observed in Figure **2**. With this method, it is possible to identify the fundamental frequency (n=1) of the system as well as its overtones. One can also ascertain that, successive modes are result of multiplying the fundamental frequency by n, getting a result which is very close to the analytical solution with a small error due to the numerical approximations of the FEM. The error is greater in the case of transverse waves due to the fact that in this wave equation the fourth derivative of the displacement is considered, resulting in a greater dispersion and, thus, the numerical approximation is less precise.

Blue regions of every figure in Table **2** represent points with no particle displacement. As the clamped-clamped boundary condition has been applied to the model, no particle displacement is produced at both ends of the beam.

3.2. Case 2: Free - Mass-loaded

Not only fundamental equations, but also boundary conditions have an importante influence on the wave propagation behaviour in the system, in particular in its eigenfrequencies. With the purpose of studying the influence of the boundary conditions in the natural frequencies of vibrations of a system, Figure **3** shows the difference between the eigenfrequencies of a given beam with both ends free and in the case that of one of the ends is mass-loaded with a point mass.

As can be seen in Figure **3** the frequency of the successive modes using the free-free boundary condition is approximately the double, which corresponds to the fundamental frequency and its harmonics. Note that, as was mentioned above, FEM is able to calculate approximate solutions.

n	f [Hz] Free - Free	f [Hz] Free - Mass-loaded
1	283	204
2	739	568
3	1356	1073

Figure 3: Influence of the boundary conditions in the eigenfrequencies of a beam.

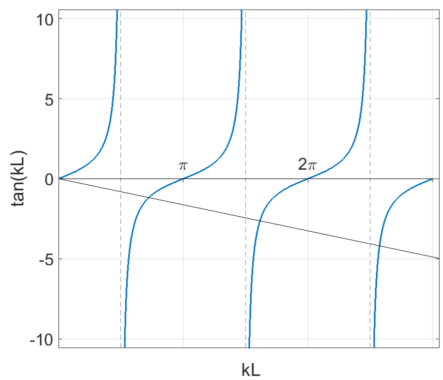

Figure 4: Graphic representation of functions $tan(kL)$ and kL. The cut points correspond to the eigenfrequencies.

On the other hand, if one of the ends is mass-loaded this trend is broken. When a point mass is added, frequencies of the modes decrease and the ratio between the first fundamental frequency and the harmonics changes. As an example, the ratio between the first and the second mode is 2.11 and between the first and the second mode is 3.26.

This fact can be explained by the resolution of Eq. 8. A graphic representation of the equation is shown in Figure 4. Both, the function $tan(kL)$ and the function kL are plotted. The curved lines correspond to the left hand side of equation 8. The straight line drawn from the point 0 with a slope m/m_b, corresponds to the right hand side. The cut points correspond to the wave numbers that satisfy the equation, and thus, the engenfrequencies of the system (the fundamental frequency and overtones). Different to the vibrating string, these solutions are not harmonic.

$$tan(kL) = -\left(\frac{m}{m_b}\right) kL \tag{8}$$

where m and m_b are the added mass and the mass of the beam, respectively, k is the wavenumber and L is the length of the beam.

Mode shape displacement for the fundamental modes with eigenfrequencies 204 Hz and 283 Hz are represented for each boundary condition can be seen in Figure 5.

4. Vibrations in membranes

Membranes are widely used in many applications. Some of the most well-known are their use for loudspeaker manufacturing, as the cone is a circular membrane, and in the building of percussion instruments. Likewise, membranes are also used in research (e.g., in the field of acoustics membranes have been studied for sound absorption).

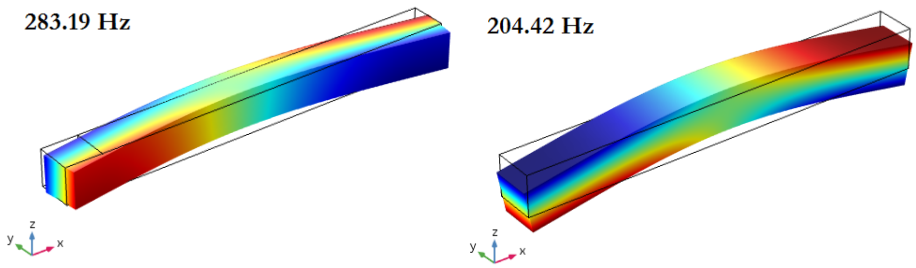

Figure 5: Mode shape for free - mass-loaded (up) and free - free (down) boundary conditions.

4.1. Rectangular membranes

In this example, the case of a rectangular membrane with all its boundaries clamped is studied. If the membrane is assumed to be very thin, only flexural modes will be found (displacement is perpendicular to the plane of the membrane). These modes can be calculated analytically by using the following equation:

$$f_{mn} = \frac{\omega_{mn}}{2\pi} = \frac{c}{2}\sqrt{\frac{n^2}{L_x^2} + \frac{m^2}{L_z^2}} \tag{9}$$

where ω_{mn} is the angular frequency of the mn mode, c is the wave speed and L_x and L_z are the length of the membrane in the x and z axis, respectively.

The particle displacement $\tilde{y}$ in flexural modes can be analytically calculated as follows:

$$\tilde{y} = \tilde{A} sin(k_x x) sin(k_z z) e^{j\omega t} \tag{10}$$

where $\tilde{A}$ is the amplitude, $k = \omega/c$ is the wavenumber and x and z are the position of the membraney the x and z axis, respectively.

Figure 6 shows the numerical results of the simulations implemented for the case described. As in the case of the beam with clamped ends, no displacement is produced in the boundaries of the membrane, which is a proof that the simulation is accomplishing the boundary conditions imposed.

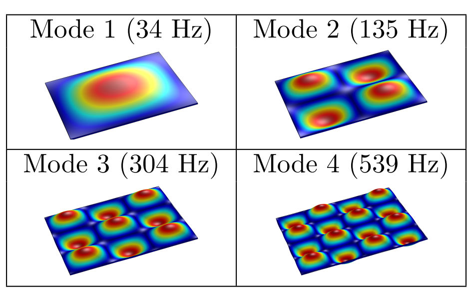

Figure 6: Vibrational modes of rectangular membranes.

4.2. Circular membranes

In this section, numerical simulations have been performed considering a circular membrane of radius r and fixed boundary conditions in a similar way to rectangular membranes. In the case of circular membranes, the analytical equation that provides the particle displacement in

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flexural waves is Eq. 9, which is composed by a radial term $J_n(kr)$, an angular term $cos(m\Theta)$ and an harmonic term $cos(\omega t)$.

$$y = A_{mn} J_n(kr) cos(m\Theta) cos(\omega t) \tag{11}$$

where $\tilde{A}$ is the amplitude, $k = \omega/c$ is the wavenumber, ω is the angular frequency, t is the time and r and Θ are the radial and angular components, respectively.

In membranes, the modes are labelled in terms of the number of nodes appearing in the angular and radial components of the analytical equation (i.e., the mode (2,1) has two angular nodes and 1 radial mode. Keeping this in mind, performing a numerical simulation of a circular membrane gives the results shown in Table 7.

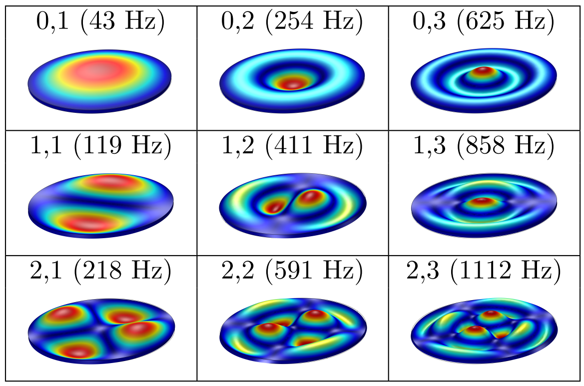

Figure 7: Vibrational modes of circular membranes.

As in the case of the beam, one can observe that there is no displacement in the boundaries of the membrane, which is a proof that boundary conditions have been respected by the model.

5. Conclusions

The application of the FEM has a great interest for the resolution of fundamental equations. In this work simple continuous mechanical systems are analysed like the vibrating string, the beam and membranes with different boundary conditions. The method allows an easy visualization of the phenomena that take place in the study case, which is specially useful for a better understanding and prediction.

Results given in this work are a good example of this application. Unidimensional systems, as the vibrating string, are quickly computed while more complex systems require more time and means. Examples shown also demonstrate the importance of the boundary conditions comparing, for the case of beams, the variation of the modes considering free, clamped or mass-loaded ends.

In some cases, an experimental set-up of a work is very expensive or its study is very complex, and FEM can also be a previous step between the theoretical/analytical and the experimental for interpreting results and study the case in depth.

The main disadvantage of this method is the great computational cost derived from the big number of parameters to be studied in the model or the wide range of some of them.

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