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Best proximity point theorems with Suzuki distances

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Abstract

In this paper, we define the weak *P*-property and the α-ψ-proximal contraction by p in which p is a τ-distance on a metric space. Then, we prove some best proximity point theorems in a complete metric space X with generalized distance. Also we define two kinds of α-p-proximal contractions and prove some best proximity point theorems.

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1 Introduction

Let us assume that A and B are two nonempty subsets of a metric space (X,d) and $T : A \longrightarrow B$. Clearly $T(A) \cap A \neq \emptyset$ is a necessary condition for the existence of a fixed point of T. The idea of the best proximity point theory is to determine an approximate solution x such that the error of equation $d(x, Tx) = 0$ is minimum. A solution x for the equation $d(x, Tx) = d(A, B)$ is called a best proximity point of T. The existence and convergence of best proximity points have been generalized by several authors [1–8] in many directions. Also, Akbar and Gabeleh [9, 10], Sadiq Basha [11] and Pragadeeswarar and Marudai [12] extended the best proximity points theorems in partially ordered metric spaces (see also [13–18]). On the other hand, Suzuki [19] introduced the concept of τ-distance on a metric space and proved some fixed point theorems for various contractive mappings by τ-distance. In this paper, by using the concept of τ-distance, we prove some best proximity point theorems.

2 Preliminaries

Let A, B be two nonempty subsets of a metric space (X,d). The following notations will be used throughout this paper:

$$d(y,A) := \inf\{d(x,y) : x \in A\},$$

$$d(A,B) := \inf\{d(x,y) : x \in A \text{ and } y \in B\},$$

$$A_0 := \{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\},$$

$$B_0 := \{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\}.$$

We recall that $x \in A$ is a best proximity point of the mapping $T : A \longrightarrow B$ if $d(x, Tx) = d(A,B)$. It can be observed that a best proximity point reduces to a fixed point if the underlying mapping is a self-mapping.

Definition 2.1 ([20]) Let (A,B) be a pair of nonempty subsets of a metric space X with $A \neq \emptyset$. Then the pair (A,B) is said to have the P-property if and only if

$$\left.\begin{aligned} d(x_1,y_1) = d(A,B), \\ d(x_2,y_2) = d(A,B) \end{aligned}\right\} \quad\Longrightarrow\quad d(x_1,x_2) = d(y_1,y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is clear that, for any nonempty subset A of X, the pair (A,A) has the P-property.

Definition 2.2 ([5]) A is said to be approximately compact with respect to B if every sequence $\{x_n\}$ of A, satisfying the condition that $d(y,x_n) \longrightarrow d(y,A)$ for some y in B, has a convergent subsequence.

Remark 2.3 ([5]) Every set is approximately compact with respect to itself.

Samet *et al.* [21] introduced a class of contractive mappings called α-ψ-contractive mappings. Let Ψ be the family of nondecreasing functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where $\psi^n(t)$ is the nth iterate of ψ.

Lemma 2.4 ([21]) *For every function $\psi : [0,\infty) \longrightarrow [0,\infty)$, the following holds*:
if ψ is nondecreasing, then, for each $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.

Definition 2.5 ([1]) Let $T : A \longrightarrow B$ and $\alpha : A \times A \longrightarrow [0,\infty)$. We say that T is α-proximal admissible if

$$\left.\begin{aligned} \alpha(x_1,x_2) \geq 1, \\ d(u_1,Tx_1) = d(A,B), \\ d(u_2,Tx_2) = d(A,B) \end{aligned}\right\} \quad\Longrightarrow\quad \alpha(u_1,u_2) \geq 1$$

for all $x_1, x_2, u_1, u_2 \in A$.

Remark 2.6 Let '$\preceq$' be a partially ordered relation on A and $\alpha : A \times A \longrightarrow [0,\infty)$ be defined by

$$\alpha(x,y) = \begin{cases} 1, & x \preceq y, \\ 0, & \text{otherwise.} \end{cases}$$

If T is α-proximal admissible, then T is said to be proximally increasing. In other words, T is proximally increasing if it satisfies the condition that

$$\left.\begin{aligned} x_1 \preceq x_2, \\ d(u_1,Tx_1) = d(A,B), \\ d(u_2,Tx_2) = d(A,B) \end{aligned}\right\} \quad\Longrightarrow\quad u_1 \preceq u_2$$

for all $x_1, x_2, u_1, u_2 \in A$.

Definition 2.7 ([19]) Let X be a metric space with metric d. A function $p : X \times X \longrightarrow [0,\infty)$ is called τ-distance on X if there exists a function $\eta : X \times [0,\infty) \longrightarrow [0,\infty)$ such that the following are satisfied:

(τ_1) $p(x,z) \leq p(x,y) + p(y,z)$ for all $x, y, z \in X$;
(τ_2) $\eta(x,0) = 0$ and $\eta(x,t) \geq t$ for all $x \in X$ and $t \in [0,\infty)$, and η is concave and continuous in its second variable;
(τ_3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n,(z_n,x_m)) : m \geq n\} = 0$ imply $p(w,x) \leq \liminf_n p(w,x_n)$ for all $w \in X$;
(τ_4) $\lim_n \sup\{p(x_n,y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n,t_n) = 0$ imply $\lim_n \eta(y_n,t_n) = 0$
(τ_5) $\lim_n \eta(z_n,p(z_n,x_n)) = 0$ and $\lim_n \eta(z_n,p(z_n,y_n)) = 0$ imply $\lim_n d(x_n,y_n) = 0$.

Remark 2.8 (τ_2) can be replaced by the following $(\tau_2)'$.

$(\tau_2)'$ $\inf\{\eta(x,t) : t > 0\} = 0$ for all $x \in X$, and η is nondecreasing in its second variable.

Remark 2.9 If (X,d) is a metric space, then the metric d is a τ-distance on X.

In the following examples, we define $\eta : X \times [0,\infty) \longrightarrow [0,\infty)$ by $\eta(x,t) = t$ for all $x \in X$, $t \in [0,\infty)$. It is easy to see that p is a τ-distance on a metric space X.

Example 2.10 Let (X,d) be a metric space and c be a positive real number. Then $p : X \times X \longrightarrow [0,\infty)$ by $p(x,y) = c$ for $x,y \in X$ is a τ-distance on X.

Example 2.11 Let $(X,\|\cdot\|)$ be a normed space. $p : X \times X \longrightarrow [0,\infty)$ by $p(x,y) = \|x\| + \|y\|$ for $x,y \in X$ is a τ-distance on X.

Example 2.12 Let $(X,\|\cdot\|)$ be a normed space. $p : X \times X \longrightarrow [0,\infty)$ by $p(x,y) = \|y\|$ for $x,y \in X$ is a τ-distance on X.

Definition 2.13 Let (X,d) be a metric space and p be a τ-distance on X. A sequence $\{x_n\}$ in X is called p-Cauchy if there exists a function $\eta : X \times [0,\infty) \longrightarrow [0,\infty)$ satisfying (τ_2)-(τ_5) and a sequence z_n in X such that $\lim_n \sup\{\eta(z_n,(z_n,x_m)) : m \geq n\} = 0$.

The following lemmas are essential for the next sections.

Lemma 2.14 ([19]) *Let (X,d) be a metric space and p be a τ-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then it is a Cauchy sequence. Moreover, if $\{y_n\}$ is a sequence satisfying* $\lim_n \sup\{p(x_n,y_m) : m \geq n = 0\}$, *then $\{y_n\}$ is also a p-Cauchy sequence and* $\lim_n d(x_n,y_n) = 0$.

Lemma 2.15 ([19]) *Let (X,d) be a metric space and p be a τ-distance on X. If $\{x_n\}$ in X satisfies* $\lim_n p(z,x_n) = 0$ *for some $z \in X$, then $\{x_n\}$ is a p-Cauchy sequence. Moreover, if $\{y_n\}$ in X also satisfies* $\lim_n p(z,y_n) = 0$, *then* $\lim_n d(x_n,y_n) = 0$. *In particular, for $x,y,z \in X$, $p(z,x) = 0$ and $p(z,y) = 0$ imply $x = y$.*

Lemma 2.16 ([19]) *Let (X,d) be a metric space and p be a τ-distance on X. If a sequence $\{x_n\}$ in X satisfies* $\lim_n \sup\{p(x_n,x_m) : m \geq n\} = 0$, *then $\{x_n\}$ is a p-Cauchy sequence. Moreover, if $\{y_n\}$ in X satisfies* $\lim_n p(x_n,y_n) = 0$, *then $\{y_n\}$ is also a p-Cauchy sequence and* $\lim_n d(x_n,y_n) = 0$.

The next result is an immediate consequence of Lemma 2.14 and Lemma 2.16.

Corollary 2.17 *Let* (X,d) *be a metric space and* p *be a* τ*-distance on* X. *If a sequence* $\{x_n\}$ *in* X *satisfies* $\lim_n \sup\{p(x_n,x_m) : m \geq n\} = 0$, *then* $\{x_n\}$ *is a Cauchy sequence.*

3 Some best proximity point theorems

Now, we define the weak P-property with respect to a τ-distance as follows.

Definition 3.1 Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Also let p be a τ-distance on X. Then the pair (A,B) is said to have the weak P-property with respect to p if and only if

$$\left.\begin{aligned} d(x_1,y_1) &= d(A,B), \\ d(x_2,y_2) &= d(A,B) \end{aligned}\right\} \quad \Longrightarrow \quad p(x_1,x_2) \leq p(y_1,y_2),$$

where $x_1,x_2 \in A_0$ and $y_1,y_2 \in B_0$.

It is clear that, for any nonempty subset A of X, the pair (A,A) has the weak P-property with respect to p.

Remark 3.2 ([22]) If $p = d$, then (A,B) is said to have the weak P-property where $A_0 \neq \emptyset$.

It is easy to see that if (A,B) has the P-property, then (A,B) has the weak P-property.

Example 3.3 Let $X = \mathbf{R}^2$ with the usual metric and p_1, p_2 be two τ-distances defined in Example 2.11 and Example 2.12, respectively. Consider the following:

$$\begin{aligned} A &= \bigl\{(a,b) \in \mathbf{R}^2 \mid a = 0, 2 \leq b \leq 3\bigr\}, \\ B &= \bigl\{(a,b) \in \mathbf{R}^2 \mid a = 1, b \leq 1\bigr\} \cup \bigl\{(a,b) \in \mathbf{R}^2 \mid a = 1, b \geq 4\bigr\}. \end{aligned}$$

Then (A,B) has the weak P-property with respect to p_1 but has not the weak P-property with respect to p_2.

By the definition of A and B, we obtain

$$d\bigl((0,2),(1,1)\bigr) = d\bigl((0,3),(1,4)\bigr) = d(A,B) = \sqrt{2},$$

where $(0,2),(0,3) \in A$ and $(1,1),(1,4) \in B$. We have

$$\begin{aligned} p_1\bigl((0,2),(0,3)\bigr) = 5 \quad &\text{and} \quad p_1\bigl((1,1),(1,4)\bigr) = \sqrt{2} + \sqrt{17}, \\ p_1\bigl((0,3),(0,2)\bigr) = 5 \quad &\text{and} \quad p_1\bigl((1,4),(1,1)\bigr) = \sqrt{17} + \sqrt{2}. \end{aligned}$$

Therefore (A,B) has the weak P-property with respect to p_1. On the other hand, we have

$$p_2\bigl((0,3),(0,2)\bigr) = 2 \quad \text{and} \quad p_2\bigl((1,4),(1,1)\bigr) = \sqrt{2}.$$

This implies that (A,B) has not the weak P-property with respect to p_2.

Definition 3.4 Let (X,d) be a metric space and let p be a τ-distance on X. A mapping $T: A \longrightarrow B$ is said to be an α-ψ-proximal contraction with respect to p if

$$\alpha(x,y)p(Tx,Ty) \leq \psi\bigl(p(x,y)\bigr) \quad \text{for all } x,y \in A,$$

where $\alpha: A \times A \longrightarrow [0,\infty)$ and $\psi \in \Psi$.

Remark 3.5 ([1]) If $p = d$, then T is said to be an α-ψ-proximal contraction.

Example 3.6 Let (X,d) be a metric space and A, B be two subsets of X. Let p be the τ-distance defined in Example 2.10. Consider the following:

$$\psi(t) = \frac{t}{2} \quad \text{for all } t \geq 0,$$

$$\alpha_1(x,y) = k_1, \quad \text{where } k_1 \in \mathbf{R}, 0 \leq k_1 \leq \frac{1}{2},$$

$$\alpha_2(x,y) = k_2, \quad \text{where } k_2 \in \mathbf{R}, k_2 > \frac{1}{2}.$$

Then $T: A \longrightarrow B$ is an α_1-ψ-proximal contraction with respect to p, but it is not an α_2-ψ-proximal contraction with respect to p.

Definition 3.7 $g: A \longrightarrow A$ is said to be a τ-distance preserving with respect to p if

$$p(gx_1, gx_2) = p(x_1,x_2)$$

for all x_1 and x_2 in A.

We first prove the following lemma. Then we state our results.

Lemma 3.8 *Let A and B be nonempty, closed subsets of a metric space (X,d) such that A_0 is nonempty. Let p be a τ-distance on X and $\alpha: A \times A \longrightarrow [0,\infty)$. Suppose that $T: A \longrightarrow B$ and $g: A \longrightarrow A$ satisfy the following conditions*:

(a) *T is α-proximal admissible.*
(b) *g is a τ-distance preserving with respect to p.*
(c) *$\alpha(gu,gv) \geq 1$ implies that $\alpha(u,v) \geq 1$ for all $u,v \in A$.*
(d) *$T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.*
(e) *There exist $x_0, x_1 \in A$ such that*

$$d(gx_1, Tx_0) = d(A,B) \quad \textit{and} \quad \alpha(x_0,x_1) \geq 1.$$

Then there exists a sequence $\{x_n\}$ in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A,B) \quad \textit{and} \quad \alpha(x_n,x_{n+1}) \geq 1 \quad \textit{for all } n \in \mathbf{N} \cup \{0\}.$$

Proof By condition (e) there exist $x_0, x_1 \in A$ such that

$$d(gx_1, Tx_0) = d(A,B) \quad \text{and} \quad \alpha(x_0,x_1) \geq 1. \tag{1}$$

Since $Tx_1 \in T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B). \tag{2}$$

T is α-proximal admissible, therefore by (1) and (2) we have

$$\alpha(gx_1, gx_2) \geq 1.$$

By condition (c) we obtain

$$\alpha(x_1, x_2) \geq 1.$$

Continuing this process, we can find a sequence $\{x_n\}$ in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A, B) \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbf{N} \cup \{0\}. \tag{3}$$

This completes the proof of the lemma. □

The following result is a special case of Lemma 3.8 obtained by setting α defined in Remark 2.6.

Corollary 3.9 *Let A and B be nonempty, closed subsets of a metric space (X, d) such that A_0 is nonempty. Let '$\preceq$' be a partially ordered relation on A and p be a τ-distance on X. Suppose that $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions*:

(a) *T is proximally increasing.*
(b) *g is a τ-distance preserving with respect to p.*
(c) *$gu \preceq gv$ implies that $u \preceq v$ for all $u, v \in A$.*
(d) *$T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.*
(e) *There exist $x_0, x_1 \in A$ such that*

$$d(gx_1, Tx_0) = d(A, B) \quad \textit{and} \quad x_0 \preceq x_1.$$

Then there exists a sequence $\{x_n\}$ in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A, B) \quad \textit{and} \quad x_n \preceq x_{n+1} \quad \textit{for all } n \in \mathbf{N} \cup \{0\}.$$

The following result is a spacial case of Lemma 3.8 if g is the identity map.

Corollary 3.10 *Let A and B be nonempty, closed subsets of a metric space (X, d) such that A_0 is nonempty and $\alpha : A \times A \longrightarrow [0, \infty)$. Suppose that $T : A \longrightarrow B$ satisfies the following conditions*:

(a) *T is α-proximal admissible.*
(b) *$T(A_0) \subseteq B_0$.*
(c) *There exist $x_0, x_1 \in A$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1.$$

Then there exists a sequence $\{x_n\}$ *in* A_0 *such that*

$$d(x_{n+1}, Tx_n) = d(A,B) \quad \textit{and} \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \textit{for all } n \in \mathbf{N} \cup \{0\}.$$

Theorem 3.11 *Let* A *and* B *be nonempty, closed subsets of a complete metric space* (X,d) *such that* A_0 *is nonempty. Let* $\alpha : A \times A \longrightarrow [0,\infty)$ *and* $\psi \in \Psi$. *Also suppose that* p *is a* τ*-distance on* X *and* $T : A \longrightarrow B$ *satisfies the following conditions*:

(a) $T(A_0) \subseteq B_0$ *and* (A,B) *has the weak P-property with respect to* p.

(b) T *is* α*-proximal admissible.*

(c) *There exist* $x_0, x_1 \in A$ *such that*

$$d(x_1, Tx_0) = d(A,B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1.$$

(d) T *is a continuous* α-ψ*-proximal contraction with respect to* p.

Then T *has a best proximity point in* A.

Proof By Corollary 3.10 there exists a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A,B) \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbf{N} \cup \{0\}. \tag{4}$$

(A,B) satisfies the weak P-property with respect to p, therefore by (4) we obtain that

$$p(x_n, x_{n+1}) \leq p(Tx_{n-1}, Tx_n) \quad \text{for all } n \in \mathbf{N}. \tag{5}$$

Also, by the definition of T, we have

$$\alpha(x_{n-1}, x_n) p(Tx_{n-1}, Tx_n) \leq \psi\bigl(p(x_{n-1}, x_n)\bigr) \quad \text{for all } n \in \mathbf{N}.$$

On the other hand, we have $\alpha(x_{n-1}, x_n) \geq 1$ for all $n \in \mathbf{N}$, which implies that

$$p(Tx_{n-1}, Tx_n) \leq \psi\bigl(p(x_{n-1}, x_n)\bigr) \quad \text{for all } n \in \mathbf{N}. \tag{6}$$

From (5) and (6), we get that

$$p(x_n, x_{n+1}) \leq \psi\bigl(p(x_{n-1}, x_n)\bigr) \quad \text{for all } n \in \mathbf{N}. \tag{7}$$

If there exists $n_0 \in \mathbf{N}$ such that $p(x_{n_0}, x_{n_0-1}) = 0$, then, by the definition of ψ, we obtain that $\psi(p(x_{n_0-1}, x_{n_0})) = 0$. Therefore by (7) we have $p(x_n, x_{n+1}) = 0$ for all $n > n_0$. Thus by Lemma 3.8 the sequence $\{x_n\}$ is Cauchy.

Now, let $p(x_{n-1}, x_n) \neq 0$ for all $n \in \mathbf{N}$. By the monotony of ψ and using induction in (7), we obtain

$$p(x_n, x_{n+1}) \leq \psi^n\bigl(p(x_0, x_1)\bigr) \quad \text{for all } n \in \mathbf{N}. \tag{8}$$

By the definition of ψ, we have $\sum_{k=1}^{\infty} \psi^k(p(x_0, x_1)) < \infty$. So, for all $\varepsilon > 0$, there exists some positive integer $h = h(\varepsilon)$ such that

$$\sum_{k \geq h}^{\infty} \psi^k\bigl(p(x_0, x_1)\bigr) < \varepsilon.$$

Now let $m > n > h$. By the triangle inequality and (8), we have

$$p(x_n, x_m) \leq \sum_{k=n}^{m-1} p(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \psi^k\bigl(p(x_0, x_1)\bigr) \leq \sum_{k\geq h} \psi^k\bigl(p(x_0, x_1)\bigr) < \varepsilon.$$

This implies that

$$\limsup_n \bigl\{p(x_n, x_m) : m \geq n\bigr\} = 0.$$

By Corollary 2.17 $\{x_n\}$ is a Cauchy sequence in A. Since X is a complete metric space and A is a closed subset of X, there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$.

T is continuous, therefore, by letting $n \longrightarrow \infty$ in (4), we obtain

$$d(x, Tx) = d(A, B).$$

This completes the proof of the theorem. □

The following result is the special case of Theorem 3.11 obtained by setting $p = d$.

Corollary 3.12 ([1]) *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\alpha : A \times A \longrightarrow [0, \infty)$ and $\psi \in \Psi$. Suppose that $T : A \longrightarrow B$ is a nonself mapping satisfying the following conditions*:

(a) *$T(A_0) \subseteq B_0$ and (A, B) has the P-property.*

(b) *T is α-proximal admissible.*

(c) *There exist $x_0, x_1 \in A$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1.$$

(d) *T is a continuous α-ψ-proximal contraction.*

Then there exists an element $x^* \in A_0$ such that

$$d\bigl(x^*, Tx^*\bigr) = d(A, B).$$

Theorem 3.13 *Let A and B be nonempty, closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Also suppose that p is a τ-distance on X and $T : A \longrightarrow B$ satisfies the following conditions*:

(a) *$T(A_0) \subseteq B_0$ and (A, B) has the weak P-property with respect to p.*

(b) *There exists $r \in [0, 1)$ such that*

$$p(Tx, Ty) \leq rp(x, y), \quad \forall x, y \in A.$$

(c) *T is continuous.*

Then T has a best proximity point in A. Moreover, if $d(x, Tx) = d(x^*, Tx^*) = d(A, B)$ for some $x, x^* \in A$, then $p(x, x^*) = 0$.

Proof Define $\alpha : A \times A \longrightarrow [0,\infty)$ and $\psi : [0,\infty) \longrightarrow [0,\infty)$ by $\alpha(x,y) = 1$ for all $x, y \in A$ and $\psi(t) = t$ for all $t \geq 0$. Therefore by Theorem 3.11, T has a best proximity point in A. Now let x, x^* be best proximity points in A. Therefore we have

$$d(x, Tx) = d\bigl(x^*, Tx^*\bigr) = d(A,B).$$

The pair (A,B) has the weak P-property with respect to p, hence by the definition of T we obtain that

$$p\bigl(x,x^*\bigr) \leq p\bigl(Tx, Tx^*\bigr) \leq rp\bigl(x,x^*\bigr).$$

Hence $p(x,x^*) = 0$ and this completes the proof of the theorem. □

The next result is an immediate consequence of Theorem 3.13 by taking $A = B$ and $p = d$.

Corollary 3.14 (Banach's contraction principle) *Let (X,d) be a complete metric space and A be a nonempty closed subset of X. Let $T : A \longrightarrow A$ be a contractive self-map. Then T has a unique fixed point x^* in A.*

4 α-p-Proximal contractions

Definition 4.1 Let A, B be subsets of a metric space (X,d) and p be a τ-distance on X. A mapping $T : A \longrightarrow B$ is said to be an α-p-proximal contraction of the first kind if there exists $r \in [0,1)$ such that

$$\left.\begin{array}{r} \alpha(x_1,x_2) \geq 1, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{array}\right\} \quad \Longrightarrow \quad p(u_1,u_2) \leq rp(x_1,x_2),$$

where $\alpha : A \times A \longrightarrow [0,\infty)$ and $u_1, u_2, x_1, x_2 \in A$.

Also if T is an α-p-proximal contraction of the first kind, then

(i) T is said to be an ordered p-proximal contraction of the first kind if '$\preceq$' is a partially ordered relation on A and α is defined in Remark 2.6.
(ii) T is said to be p-proximal contraction of the first kind if $\alpha(x,y) = 1$ for all $x, y \in A$.

Remark 4.2 ([11]) If T is an ordered p-proximal contraction of the first kind and $p = d$, then T is said to be an ordered proximal contraction of the first kind.

Remark 4.3 If T is a p-proximal contraction of the first kind and $p = d$, then T is said to be a proximal contraction of the first kind (see [5]).

Definition 4.4 Let A, B be subsets of a metric space (X,d) and p be a τ-distance on X. A mapping $T : A \longrightarrow B$ is said to be an α-p-proximal contraction of the second kind if there exists $r \in [0,1)$ such that

$$\left.\begin{array}{r} \alpha(x_1,x_2) \geq 1, \\ d(u_1, Tx_1) = d(A,B), \\ d(u_2, Tx_2) = d(A,B) \end{array}\right\} \quad \Longrightarrow \quad p(Tu_1,Tu_2) \leq rp(Tx_1,Tx_2),$$

where $\alpha : A \times A \longrightarrow [0,\infty)$ and $u_1, u_2, x_1, x_2 \in A$.

Also if T is an α-p-proximal contraction of the second kind, then

(i) T is said to be an ordered p-proximal contraction of the second kind if '$\preceq$' is a partially ordered relation on A and α is defined in Remark 2.6.

(ii) T is said to be a p-proximal contraction of the second kind if $\alpha(x,y) = 1$ for all $x, y \in A$.

Remark 4.5 If T is an ordered p-proximal contraction of the second kind and $p = d$, then T is said to be an ordered proximal contraction of the second kind.

Remark 4.6 If T is a p-proximal contraction of the second kind and $p = d$, then T is said to be a proximal contraction of the second kind.

Example 4.7 Let $X = \mathbf{R}$ with the usual metric and p be the τ-distance defined in Example 2.11. Given $A = [-3,-2] \cup [2,3]$, $B = [-1,1]$ and $T : A \longrightarrow B$ by

$$T(x) = \begin{cases} x+2, & -3 \leq x \leq -2, \\ x-2, & 2 \leq x \leq 3, \end{cases}$$

then T is a p-proximal contraction of the first and second kind.

It is easy to see that

$$d\bigl(-2, T(-3)\bigr) = d\bigl(2, T(3)\bigr) = d(A,B) = 1.$$

If $r \in [\frac{2}{3}, 1)$, then we have

$$p(-2,2) \leq rp(-3,3),$$
$$p(2,-2) \leq rp(3,-3).$$

Hence T is a p-proximal contraction of the first kind. Also,

$$p\bigl(T(-2), T(2)\bigr) \leq rp\bigl(T(-3), T(3)\bigr),$$
$$p\bigl(T(2), T(-2)\bigr) \leq rp\bigl(T(3), T(-3)\bigr)$$

for all $r \in [0,1)$. This implies that T is a p-proximal contraction of the second kind.

Example 4.8 Let $X = \mathbf{R}$ with the usual metric and p be the τ-distance defined in Example 2.12. Let '$\preceq$' be the usual partially ordered relation in $\mathbf{R}$. Given $A = \{-2\} \cup [2,3]$, $B = [-1,1]$ and $T : A \longrightarrow B$ by

$$T(x) = \begin{cases} -1, & x = -2, \\ x-2, & 2 \leq x \leq 3, \end{cases}$$

then T is an ordered p-proximal contraction of the first and second kind, but it is not a p-proximal contraction of the first and second kind.

It is easy to see that

$$d\big(-2, T(-2)\big) = d\big(2, T(3)\big) = d(A, B) = 1 \quad \text{and} \quad -2 \preceq 3.$$

If $r \in [\frac{2}{3}, 1)$, then we have

$$p(-2, 2) \leq rp(-2, 3).$$

$p(2, -2) \nleq rp(3, -2)$, but it is not necessary because $3 \npreceq -2$. Hence T is an ordered p-proximal contraction of the first kind. But T is not a p-proximal contraction of the first kind because $p(2, -2) \nleq rp(3, -2)$ for all $r \in [0, 1)$. Also,

$$p\big(T(-2), T(2)\big) \leq rp\big(T(-2), T(3)\big)$$

for all $r \in [0, 1)$. Notice that $p(T(2), T(-2)) \nleq rp(T(3), T(-2))$, but it is not necessary because $3 \npreceq -2$. This implies that T is an ordered p-proximal contraction of the second kind. But T is not a p-proximal contraction of the second kind because $p(T(2), T(-2)) \nleq rp(T(3), T(-2))$ for all $r \in [0, 1)$.

Theorem 4.9 *Let A and B be nonempty, closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let p be a w-distance on X and $\alpha : A \times A \longrightarrow [0, \infty)$. Suppose that $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions*:

(a) *T is an α-proximal admissible and continuous α-p-proximal contraction of the first kind.*
(b) *g is a continuous τ-distance preserving with respect to p.*
(c) *$\alpha(gu, gv) \geq 1$ implies that $\alpha(u, v) \geq 1$ for all $u, v \in A$.*
(d) *$T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.*
(e) *There exist $x_0, x_1 \in A$ such that*

$$d(gx_1, Tx_0) = d(A, B) \quad \textit{and} \quad \alpha(x_0, x_1) \geq 1.$$

Then there exists an element $x \in A$ such that

$$d(gx, Tx) = d(A, B).$$

Proof By Lemma 3.8 there exists a sequence $\{x_n\}$ in A_0 such that

$$d(gx_{n+1}, Tx_n) = d(A, B) \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbf{N} \cup \{0\}. \tag{9}$$

We will prove the convergence of a sequence $\{x_n\}$ in A. T is an α-p-proximal contraction of the first kind and (3) holds, hence, for any positive integer n, we have

$$p(gx_n, gx_{n+1}) \leq rp(x_n, x_{n-1}).$$

Also g is a τ-distance preserving with respect to p, so we get that

$$p(x_n, x_{n+1}) \leq rp(x_n, x_{n-1}) \leq \cdots \leq r^n p(x_0, x_1)$$

for every $n \in \mathbf{N}$. Hence, if $m > n$,

$$\begin{aligned} p(x_n, x_m) &\le p(x_n, x_{n+1}) + \cdots + p(x_{m-1}, x_m) \\ &\le r^n p(x_0, x_1) + \cdots + r^{m-1} p(x_0, x_1) \\ &\le \frac{r^n}{1-r} p(x_0, x_1). \end{aligned}$$

This implies that

$$\lim_n \sup\{p(x_n, x_m) : m \ge n\} = 0.$$

By Corollary 2.17, $\{x_n\}$ is a Cauchy sequence in A. Since X is a complete metric space and A is a closed subset of X, there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$.

T and g are continuous, therefore by letting $n \longrightarrow \infty$ in (3), we obtain

$$d(gx, Tx) = d(A, B).$$

This completes the proof of the theorem. □

The next result is an immediate consequence of Theorem 4.9 by setting α defined in Remark 2.6.

Corollary 4.10 *Let A and B be nonempty, closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let '$\preceq$' be a partially ordered relation on A and p be a τ-distance on X. Suppose that $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions:*

(a) *T is a proximally increasing and continuous ordered p-proximal contraction of the first kind.*
(b) *g is a continuous τ-distance preserving with respect to p.*
(c) *$gu \preceq gv$ implies that $u \preceq v$ for all $u, v \in A$.*
(d) *$T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.*
(e) *There exist $x_0, x_1 \in A$ such that*

$$d(gx_1, Tx_0) = d(A, B) \quad \text{and} \quad x_0 \preceq x_1.$$

Then there exists an element $x \in A$ such that

$$d(gx, Tx) = d(A, B).$$

Theorem 4.11 *Let A and B be nonempty, closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let p be a τ-distance on X. Suppose that $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions:*

(a) *T is a continuous p-proximal contraction of the first kind.*
(b) *g is a continuous τ-distance preserving with respect to p.*
(c) *$T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.*

Then there exists an element $x \in A$ such that

$$d(gx, Tx) = d(A, B).$$

Moreover, if $d(gx, Tx) = d(gx^*, Tx^*) = d(A, B)$ for some $x, x^* \in A$, then $p(x, x^*) = 0$.

Proof By Theorem 4.9 there exists an element $x \in A$ such that

$$d(gx, Tx) = d(A, B).$$

Now let x^* be in A such that

$$d\big(gx^*, Tx^*\big) = d(A, B).$$

T is a p-proximal contraction of the first kind and g is a τ-distance preserving with respect to p, therefore

$$p\big(x, x^*\big) \leq rp\big(x, x^*\big).$$

Hence $p(x, x^*) = 0$ and this completes the proof of the theorem. □

The next result is obtained by taking $p = d$ in Theorem 4.11.

Corollary 4.12 ([5]) *Let X be a complete metric space. Let A and B be nonempty, closed subsets of X. Further, suppose that A_0 and B_0 are nonempty. Let $T : A \longrightarrow B$ and $g : A \longrightarrow A$ satisfy the following conditions*:

(a) *T is a continuous proximal contraction of the first kind.*
(b) *g is an isometry.*
(c) *$T(A_0) \subseteq B_0$.*
(d) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique element $x \in A$ such that

$$d(gx, Tx) = d(A, B).$$

The following result is a best proximity point theorem for nonself α-p-proximal contraction of the second kind.

Theorem 4.13 *Let A and B be nonempty, closed subsets of a complete metric space (X, d) such that A is approximately compact with respect to B and A_0 is nonempty. Let p be a τ-distance on X and $\alpha : A \times A \longrightarrow [0, \infty)$. Suppose that $T : A \longrightarrow B$ satisfies the following conditions*:

(a) *T is an α-proximal admissible and continuous α-p-proximal contraction of the second kind.*
(b) *$T(A_0) \subseteq B_0$.*
(c) *There exist $x_0, x_1 \in A$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \text{and} \quad \alpha(x_0, x_1) \geq 1.$$

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Proof By Corollary 3.10 there exists a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = d(A,B) \quad \text{and} \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all } n \in \mathbf{N} \cup \{0\}. \tag{10}$$

We will prove the convergence of a sequence $\{x_n\}$ in A. T is an α-p-proximal contraction of the second kind and (10) holds, hence, for any positive integer n, we have

$$p(Tx_n, Tx_{n+1}) \leq rp(Tx_{n-1}, Tx_n) \leq \cdots \leq r^n p(Tx_0, Tx_1)$$

for every $n \in \mathbf{N}$. Hence, if $m > n$,

$$\begin{aligned} p(Tx_n, Tx_m) &\leq p(Tx_n, Tx_{n+1}) + \cdots + p(Tx_{m-1}, Tx_m) \\ &\leq r^n p(Tx_0, Tx_1) + \cdots + r^{m-1} p(Tx_0, Tx_1) \\ &\leq \frac{r^n}{1-r} p(Tx_0, Tx_1). \end{aligned}$$

This implies that

$$\lim_n \sup\bigl\{p(Tx_n, Tx_m) : m \geq n\bigr\} = 0.$$

By Corollary 2.17, $\{Tx_n\}$ is a Cauchy sequence in B. Since X is a complete metric space and B is a closed subset of X, there exists $y \in B$ such that $\lim_{n\to\infty} Tx_n = y$. By the triangle inequality, we have

$$\begin{aligned} d(y,A) &\leq d(y,x_n) \\ &\leq d(y, Tx_{n-1}) + d(Tx_{n-1}, x_n) \\ &= d(y, Tx_{n-1}) + d(A,B) \\ &\leq (y, Tx_{n-1}) + d(y,A). \end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} d(y,x_n) = d(y,A).$$

Since A is approximately compact with respect to B, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some $x \in A$. Therefore

$$d(x,y) = \lim_{k\to\infty} d(x_{n_k}, Tx_{n_k-1}) = d(A,B).$$

This implies that $x \in A_0$. T is continuous and $\{Tx_n\}$ is convergent to y, therefore

$$\lim_{n_k\to\infty} Tx_{n_k} = Tx = y.$$

Thus, it follows that

$$d(x, Tx) = \lim_{n_k\to\infty} d(x_{n_k}, Tx_{n_k-1}) = d(A,B).$$

This completes the proof of the theorem. □

The next result is an immediate consequence of Theorem 4.13 by setting α defined in Remark 2.6.

Corollary 4.14 *Let A and B be nonempty, closed subsets of a complete metric space (X, d) such that A is approximately compact with respect to B and A_0 is nonempty. Let '$\preceq$' be a partially ordered relation on A and p be a τ-distance on X. Suppose that $T : A \longrightarrow B$ satisfies the following conditions*:

(a) *T is a proximally increasing and continuous ordered p-proximal contraction of the second kind.*
(b) $T(A_0) \subseteq B_0$.
(c) *There exist $x_0, x_1 \in A$ such that*

$$d(x_1, Tx_0) = d(A, B) \quad \textit{and} \quad x_0 \preceq x_1.$$

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Theorem 4.15 *Let A and B be nonempty, closed subsets of a complete metric space (X, d) such that A is approximately compact with respect to B, and let p be a τ-distance on X. Further, suppose that A_0 is nonempty. Let $T : A \longrightarrow B$ satisfy the following conditions*:

(a) *T is a continuous p-proximal contraction of the second kind.*
(b) $T(A_0) \subseteq B_0$.

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, if $d(x, Tx) = d(x^*, Tx^*) = d(A, B)$ for some $x, x^* \in A$, then $p(Tx, Tx^*) = 0$.

Proof By Theorem 4.13 there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Now let x^* be an element in A such that

$$d\left(x^*, Tx^*\right) = d(A, B).$$

We will show that $p(Tx, Tx^*) = 0$. T is a p-proximal contraction of the second kind, therefore

$$p\left(Tx, Tx^*\right) \leq rp\left(Tx, Tx^*\right).$$

Hence $p(Tx, Tx^*) = 0$ and this completes the proof of the theorem. □

The following result is obtained by taking $p = d$ in Theorem 4.15.

Corollary 4.16 ([5]) *Let A and B be nonempty, closed subsets of a complete metric space such that A is approximately compact with respect to B. Further, suppose that A_0 and B_0 are nonempty. Let $T : A \longrightarrow B$ satisfy the following conditions*:

(a) *T is a continuous proximal contraction of the second kind.*

(b) *$T(A_0)$ is contained in B_0.*

Then there exists an element $x \in A$ such that

$$d(x, Tx) = d(A, B).$$

Moreover, if x^* is another best proximity point of T, then Tx and Tx^* are identical.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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