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The boundary manifold of a complex line arrangement

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Abstract We study the topology of the boundary manifold of a line arrangement in $\mathbb{CP}^2$, with emphasis on the fundamental group G and associated invariants. We determine the Alexander polynomial $\Delta(G)$, and more generally, the twisted Alexander polynomial associated to the abelianization of G and an arbitrary complex representation. We give an explicit description of the unit ball in the Alexander norm, and use it to analyze certain Bieri-Neumann-Strebel invariants of G. From the Alexander polynomial, we also obtain a complete description of the first characteristic variety of G. Comparing this with the corresponding resonance variety of the cohomology ring of G enables us to characterize those arrangements for which the boundary manifold is formal.

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Keywords line arrangement, graph manifold, fundamental group, twisted Alexander polynomial, BNS invariant, cohomology ring, holonomy Lie algebra, characteristic variety, resonance variety, tangent cone, formality

For Fred Cohen on the occasion of his sixtieth birthday.

1 Introduction

1.1 The boundary manifold

Let $\mathcal{A}$ be an arrangement of hyperplanes in the complex projective space $\mathbb{CP}^m$, $m > 1$. Denote by $V = \bigcup_{H\in\mathcal{A}} H$ the corresponding hypersurface, and by $X = \mathbb{CP}^m \setminus V$ its complement. Among the origins of the topological study of arrangements are seminal results of Arnol'd [2] and Cohen [8], who independently computed the cohomology of the configuration space of n ordered points in $\mathbb{C}$, the complement of the braid arrangement. The cohomology ring of the complement of an arbitrary arrangement $\mathcal{A}$ is by now well known. It is isomorphic to the Orlik-Solomon algebra of $\mathcal{A}$, see Orlik and Terao [34] as a general reference.

In this paper, we study a related topological space, namely the *boundary manifold* of $\mathcal{A}$. By definition, this is the boundary $M = \partial N$ of a regular neighborhood of the variety V in $\mathbb{CP}^m$. Unlike the complement X, an open manifold with the homotopy type of a CW-complex of dimension at most m, the boundary manifold M is a compact (orientable) manifold of dimension $2m-1$.

In previous work [7], we have shown that the cohomology ring of M is functorially determined by that of X and the ambient dimension. In particular, $H_*(M;\mathbb{Z})$ is torsion-free, and the respective Betti numbers are related by $b_k(M) = b_k(X) + b_{2m-k-1}(X)$. So we turn our attention here to another topological invariant, the fundamental group. The inclusion map $M \to X$ is an $(m-1)$-equivalence, see Dimca [9]. Consequently, for an arrangement $\mathcal{A}$ in $\mathbb{CP}^m$ with $m \geq 3$, the fundamental group of the boundary is isomorphic to that of the complement. In light of this, we focus on arrangements of lines in $\mathbb{CP}^2$.

1.2 Fundamental group

Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be a line arrangement in $\mathbb{CP}^2$. The boundary manifold M is a graph manifold in the sense of Waldhausen [41], modeled on a certain weighted graph $\Gamma_{\mathcal{A}}$. This structure, which we review in §2, has been used by a number of authors to study the manifold M. For instance, Jiang and Yau [22, 23] investigate the relationship between the topology of M and the combinatorics of $\mathcal{A}$, and Hironaka [20] analyzes the relationship between the fundamental groups of M and X.

If $\mathcal{A}$ is a pencil of lines, then M is a connected sum of n copies of $S^1 \times S^2$. Otherwise, M is aspherical, and so the homotopy type of M is encoded in its fundamental group. Using the graph manifold structure, and a method due to Hirzebruch [21], Westlund finds a presentation for the group $G = \pi_1(M)$ in [42]. In §3, we build on this work to find a *minimal* presentation for the fundamental group, of the form

$$G = \langle x_j, \gamma_{i,k} \mid R_j, R_{i,k} \rangle, \tag{1}$$

where x_j corresponds to a meridian loop around line ℓ_j, for $1 \leq j \leq n = b_1(X)$, and $\gamma_{i,k}$ corresponds to a loop in the graph $\Gamma_{\mathcal{A}}$, indexed by a pair $(i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})$, where $|\mathbf{nbc}_2(\mathsf{d}\mathcal{A})| = b_2(X)$. The relators R_j, $R_{i,k}$ (indexed in the same way) are certain products of commutators in the generators. In other words, G is a commutator-relators group, with both generators and relators equal in number to $b_1(M)$.

1.3 Twisted Alexander polynomial and related invariants

Since M is a graph manifold, the group $G = \pi_1(M)$ may be realized as the fundamental group of a graph of groups. In §4 and §5, this structure is used to calculate the twisted Alexander polynomial $\Delta^\phi(G)$ associated to G and an arbitrary complex representation $\phi\colon G \to \mathrm{GL}_k(\mathbb{C})$. In particular, we show that the classical multivariable Alexander polynomial, arising from the trivial representation of G, is given by

$$\Delta(G) = \prod_{v \in \mathcal{V}(\Gamma_{\mathcal{A}})} (t_v - 1)^{m_v - 2}, \tag{2}$$

where $\mathcal{V}(\Gamma_{\mathcal{A}})$ is the vertex set of $\Gamma_{\mathcal{A}}$, m_v denotes the multiplicity or degree of the vertex v, and $t_v = \prod_{i \in v} t_i$.

Twisted Alexander polynomials inform on invariants such as the Alexander and Thurston norms, and Bieri-Neumann-Strebel (BNS) invariants. As such, they are a subject of current interest in 3-manifold theory. In the case where G is a link group, a number of authors, including Dunfield [12] and Friedl and Kim [18], have used twisted Alexander polynomials to distinguish between the Thurston and Alexander norms. This is not possible for (complex representations of) the fundamental group of the boundary manifold of a line arrangement. In §6, we show that the unit balls in the norms on $H^1(G;\mathbb{R})$ corresponding to any two twisted Alexander polynomials are equivalent polytopes. Analysis of the structure of these polytopes also enables us to calculate the number of components of the BNS invariant of G and the Alexander invariant of G.

1.4 Cohomology ring and graded Lie algebras

In §7, we revisit the cohomology ring of the boundary manifold M, in our 3-dimensional context. From [7], we know that $H^*(M;\mathbb{Z})$ is isomorphic to $\widehat{A}$, the "graded double" of $A = H^*(X;\mathbb{Z})$. In particular, $\widehat{A}^1 = A^1 \oplus \bar{A}^2$, where $\bar{A}^k = \mathrm{Hom}(A^k, \mathbb{Z})$. This information allows us to identify the 3-form η_M which encodes all the cup-product structure in the Poincaré duality algebra $H^*(M;\mathbb{Z})$. If $\{e_j\}$ and $\{f_{i,k}\}$ denote the standard bases for A^1 and A^2, then

$$\eta_M = \sum_{(i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})} e_{I(i,k)} \wedge e_k \wedge \bar{f}_{i,k}, \tag{3}$$

where $I(i,k) = \{j \mid \ell_j \supset \ell_i \cap \ell_k,\ 1 \le j \le n\}$ and $e_J = \sum_{j \in J} e_j$.

The explicit computations described in (1) and (3) facilitate analysis of two Lie algebras attached to our space M: the graded Lie algebra $\mathrm{gr}(G)$ associated to the lower central series of G, and the holonomy Lie algebra $\mathfrak{h}(\widehat{A})$ arising from the multiplication map $\widehat{A}^1 \otimes \widehat{A}^1 \to \widehat{A}^2$. For the complement X, the corresponding Lie algebras are isomorphic over the rationals, as shown by Kohno [26]. For the boundary manifold, though, such an isomorphism no longer holds, as we illustrate by a concrete example in §9. This indicates that the manifold M, unlike the complement X, need not be formal, in the sense of Sullivan [38].

1.5 Jumping loci and formality

The non-formality phenomenon identified above is fully investigated in §8 and §9 by means of two types of varieties attached to M: the characteristic varieties $V^1_d(M)$ and the resonance varieties $\mathcal{R}^1_d(M)$. Our calculation of $\Delta(G)$ recorded in (2) enables us to give a complete description of the first characteristic variety of M, the set of all characters $\phi \in \mathrm{Hom}(G, \mathbb{C}^*)$ for which the corresponding local system cohomology group $H^1(M; \mathbb{C}_\phi)$ is non-trivial:

$$V^1_1(M) = \bigcup_{v \in \mathcal{V}(\Gamma_\mathcal{A}), m_v \geq 3} \{t_v - 1 = 0\}. \tag{4}$$

The resonance varieties of M are the analogous jumping loci for the cohomology ring $H^*(M; \mathbb{C})$. Unlike the resonance varieties of the complement X, the varieties $\mathcal{R}^1_d(M)$, for d sufficiently large, may have non-linear components. Nevertheless, the first resonance variety $\mathcal{R}^1_1(M)$ is very simple to describe: with a few exceptions, it is equal to the ambient space, $H^1(M; \mathbb{C})$. Comparing the tangent cone to $V^1_1(M)$ at the identity to $\mathcal{R}^1_1(M)$, and making use of a recent result of Dimca, Papadima, and Suciu [10], we conclude that the boundary manifold of a line arrangement $\mathcal{A}$ is formal precisely when $\mathcal{A}$ is a pencil or a near-pencil.

2 Boundary manifolds of line arrangements

Let $\mathcal{A} = \{\ell_0, \ldots, \ell_n\}$ be an arrangement of lines in $\mathbb{CP}^2$. The boundary manifold of $\mathcal{A}$ may be realized as the boundary of a regular neighborhood of the curve $C = \bigcup_{i=0}^n \ell_i$ in $\mathbb{CP}^2$. In this section, we record a number of known results regarding this manifold.

2.1 The boundary manifold

Choose homogeneous coordinates $\mathbf{x} = (x_0 \colon x_1 \colon x_2)$ on $\mathbb{CP}^2$. For each i, $0 \le i \le n$, let $f_i = f_i(x_0, x_1, x_2)$ be a linear form which vanishes on the line ℓ_i of $\mathcal{A}$. Then $Q = Q(\mathcal{A}) = \prod_{i=0}^{n} f_i$ is a homogeneous polynomial of degree $n+1$, with zero locus C. The *complement* of $\mathcal{A}$ is the open manifold $X = X(\mathcal{A}) = \mathbb{CP}^2 \setminus C$.

A closed, regular neighborhood N of C may be constructed as follows. Define $\phi \colon \mathbb{CP}^2 \to \mathbb{R}$ by $\phi(\mathbf{x}) = |Q(\mathbf{x})|^2 / \|\mathbf{x}\|^{2(n+1)}$, and let $N = \phi^{-1}([0,\delta])$ for $\delta > 0$ sufficiently small. Alternatively, triangulate $\mathbb{CP}^2$ with C as a subcomplex, and take N to be the closed star of C in the second barycentric subdivision. As shown by Durfee [13] in greater generality, these approaches yield isotopic neighborhoods, independent of the choices made in the respective constructions. The *boundary manifold* of $\mathcal{A}$ is the boundary of such a regular neighborhood:

$$M = M(\mathcal{A}) = \partial N. \tag{5}$$

This compact, connected, orientable 3-manifold will be our main object of study. We start with a couple of simple examples.

Example 2.2 Let $\mathcal{A}$ be a pencil of $n+1$ lines in $\mathbb{CP}^2$, defined by the polynomial $Q = x_1^{n+1} - x_2^{n+1}$. The complement X of $\mathcal{A}$ is diffeomorphic to $(\mathbb{C} \setminus \{n \text{ points}\}) \times \mathbb{C}$, so has the homotopy type of a bouquet of n circles. On the other hand, $\mathbb{CP}^2 \setminus N = (D^2 \setminus \{n \text{ disks}\}) \times D^2$; hence M is diffeomorphic to the n-fold connected sum $\sharp^n S^1 \times S^2$.

Example 2.3 Let $\mathcal{A}$ be a near-pencil of $n+1$ lines in $\mathbb{CP}^2$, defined by the polynomial $Q = x_0(x_1^n - x_2^n)$. In this case, $M = S^1 \times \Sigma_{n-1}$, where $\Sigma_g = \sharp^g S^1 \times S^1$ denotes the orientable surface of genus g, see [7] and Example 3.10.

2.4 Blowing up dense edges

A third construction, which sheds light on the structure of M as a 3-manifold, may also be used to obtain the topological type of the boundary manifold. This involves blowing up (certain) singular points of C. Before describing it, we establish some notation.

An edge of $\mathcal{A}$ is a non-empty intersection of lines of $\mathcal{A}$. An edge F is said to be *dense* if the subarrangement $\mathcal{A}_F = \{\ell_j \in \mathcal{A} \mid F \subseteq \ell_j\}$ of lines containing F is not a product arrangement. Hence, the dense edges are the lines of $\mathcal{A}$, and the intersection points $\ell_{j_1} \cap \cdots \cap \ell_{j_k}$ of multiplicity $k \ge 3$. Denote the set of dense

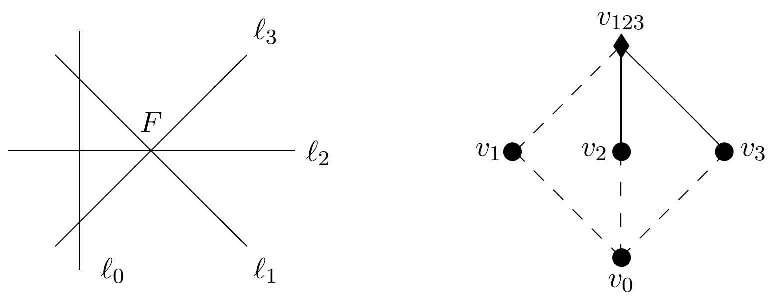

Figure 1: A near-pencil of 4 lines and its associated graph Γ (with maximal tree $\mathcal{T}$ in dashed lines)

edges of $\mathcal{A}$ by $\mathsf{D}(\mathcal{A})$, and let $F_1, \dots, F_r$ be the 0-dimensional dense edges. We will occasionally denote the dense edge $\bigcap_{j \in J} \ell_j$ by F_J.

Blowing up $\mathbb{CP}^2$ at each 0-dimensional dense edge of $\mathcal{A}$, we obtain an arrangement $\tilde{\mathcal{A}} = \{L_i\}_{i=0}^{n+r}$ in $\widetilde{\mathbb{CP}}^2$ consisting of the proper transforms $L_i = \tilde{\ell}_i$, $0 \le i \le n$, of the lines of $\mathcal{A}$, and exceptional lines $L_{n+j} = \tilde{F}_j$, $1 \le j \le r$, arising from the blow-ups.

By construction, the curve $\tilde{C} = \bigcup_{i=0}^{n+r} L_i$ in $\widetilde{\mathbb{CP}}^2$ is a divisor with normal crossings. Let U_i be a tubular neighborhood of L_i in $\widetilde{\mathbb{CP}}^2$. For sufficiently small neighborhoods, we have $U_i \cap U_j = \emptyset$ if $L_i \cap L_j = \emptyset$. Then, rounding corners, $N(\tilde{C}) = \bigcup_{i=0}^{n+r} U_i$ is a regular neighborhood of $\tilde{C}$ in $\widetilde{\mathbb{CP}}^2$. Contracting the exceptional lines of $\tilde{\mathcal{A}}$ gives rise to a homeomorphism $M \cong \partial N(\tilde{C})$.

2.5 Graph manifold structure

This last construction realizes the boundary manifold M of $\mathcal{A}$ as a *graph manifold*, in the sense of Waldhausen [41]. The underlying graph $\Gamma_{\mathcal{A}}$ may be described as follows. The vertex set $\mathcal{V}(\Gamma_{\mathcal{A}})$ is in one-to-one correspondence with the dense edges of $\mathcal{A}$ (i.e., the lines of $\tilde{\mathcal{A}}$). Label the vertices of $\Gamma_{\mathcal{A}}$ by the relevant subsets of $\{0, 1, \dots, n\}$: the vertex corresponding to ℓ_i is labeled v_i, and, if F_J is a 0-dimensional dense edge (i.e., an exceptional line in $\widetilde{\mathcal{A}}$), label the corresponding vertex v_J. If ℓ_i and ℓ_j meet in a double point of $\mathcal{A}$, we say that ℓ_i and ℓ_j are transverse, and (sometimes) write $\ell_i \pitchfork \ell_j$. The graph $\Gamma_{\mathcal{A}}$ has an edge $e_{i,j}$ from v_i to v_j, $i < j$, if the corresponding lines ℓ_i and ℓ_j are transverse, and an edge $e_{J,i}$ from v_J to v_i if $\ell_i \supset F_J$. See Figure 1 for an illustration.

Let m_v denote the multiplicity (i.e., degree) of the vertex v of $\Gamma_{\mathcal{A}}$. Note that, if v corresponds to the line L_i of $\tilde{\mathcal{A}}$, then m_v is given by the number of lines $L_j \in \tilde{\mathcal{A}} \setminus \{L_i\}$ which intersect L_i. The graph manifold structure of the boundary manifold $M = \partial N(\tilde{C})$ may be described as follows. If $v \in \mathcal{V}(\Gamma_{\mathcal{A}})$ corresponds to $L_i \in \tilde{\mathcal{A}}$, then the vertex manifold, M_v, is given by

$$M_v = \partial U_i \setminus \{\mathrm{Int}(U_j \cap \partial U_i) \mid L_j \cap L_i \neq \emptyset\} \cong S^1 \times \big(\mathbb{CP}^1 \setminus \bigcup_{j=1}^{m_v} B_j\big), \tag{6}$$

where $\mathrm{Int}(X)$ denotes the interior of X, and the B_j are open, disjoint disks. Note that the boundary of M_v is a disjoint union of m_v copies of the torus $S^1 \times S^1$. The boundary manifold M is obtained by gluing together these vertex manifolds along their common boundaries by standard longitude-to-meridian orientation-preserving attaching maps.

Graph manifolds are often aspherical. As noted in Example 2.2, if $\mathcal{A}$ is a pencil, then the boundary manifold of $\mathcal{A}$ is a connected sum of $S^1 \times S^2$'s, hence fails to be a $K(\pi, 1)$-space. Pencils are the only line arrangements for which this failure occurs.

Proposition 2.6 ([7]) *Let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2$. The boundary manifold $M = M(\mathcal{A})$ is aspherical if and only if $\mathcal{A}$ is essential, that is, not a pencil.*

3 Fundamental group of the boundary

Using the graph manifold structure described in the previous section, and a method due to Hirzebruch [21], Westlund [42] obtained a presentation for the fundamental group of the boundary manifold of a projective line arrangement. In this section, we recall this presentation, and use it to obtain a minimal presentation.

3.1 The group of a weighted graph

Let Γ be a loopless graph with $N+1$ vertices. Identify the vertex set of Γ with $\{0, 1, \dots, N\}$, and assume that there is a weight $w_i \in \mathbb{Z}$ given for each vertex. Identify the edge set $\mathcal{E}$ of Γ with a subset of $\{(i,j) \mid 0 \leq i < j \leq N\}$ in the obvious manner. Direct Γ arbitrarily.

We associate a group $G(\Gamma)$ to the weighted graph Γ, as follows. Let $\mathcal{T}$ be a maximal tree in Γ, let $\mathcal{C} = \mathcal{E} \setminus \mathcal{T}$, and order the edges in $\mathcal{C}$. Note that $g = |\mathcal{C}| = b_1(\Gamma)$ is the number of (linearly independent) cycles in Γ. The group $G(\Gamma)$ has presentation

$$G(\Gamma) = \left\langle \begin{array}{l} x_0, x_1, \dots, x_N \\ \gamma_1, \dots, \gamma_g \end{array} \,\middle|\, \begin{array}{ll} [x_i, x_j^{u_{i,j}}], & (i,j) \in \mathcal{E} \\ \prod_{j=1}^N x_j^{u_{i,j}}, & 0 \le i \le N \end{array} \right\rangle, \tag{7}$$

where

$$u_{i,j} = \begin{cases} w_i & \text{if } i = j, \\ \gamma_k & \text{if } (i,j) \text{ is the } k\text{-th element of } \mathcal{C}, \\ \gamma_k^{-1} & \text{if } (j,i) \text{ is the } k\text{-th element of } \mathcal{C}, \\ 1 & \text{if } (i,j) \text{ or } (j,i) \text{ belongs to } \mathcal{T}, \\ 0 & \text{otherwise.} \end{cases}$$

Here $[a,b] = aba^{-1}b^{-1}$, $a^0 = 1$ is the identity element of G, and $a^b = b^{-1}ab$ for $b \ne 0$. Note that if $i \ne j$ and $u_{i,j} \ne 0$, then $u_{j,i} = u_{i,j}^{-1}$.

Now let $\mathcal{A}$ be an arrangement of $n+1$ lines in $\mathbb{CP}^2$, with associated graph $\Gamma_{\mathcal{A}}$, and consider the group $G(\Gamma_{\mathcal{A}})$. Recall that the vertices of $\Gamma_{\mathcal{A}}$ are in one-to-one correspondence with the lines $\{L_i \mid 0 \le i \le n+r\}$ of the arrangement $\tilde{\mathcal{A}}$ in $\widetilde{\mathbb{CP}}^2$. If L_i is the proper transform of the line $\ell_i \in \mathcal{A}$, let p_i denote the number of 0-dimensional dense edges of $\mathcal{A}$ contained in ℓ_i, and assign the weight $w_i = 1 - p_i$ to the corresponding vertex v_i of $\Gamma_{\mathcal{A}}$. If L_i is an exceptional line, arising from blowing up the dense edge F_J of $\mathcal{A}$, assign the weight $w_J = -1$ to the corresponding vertex v_J of $\Gamma_{\mathcal{A}}$. Note that the weights of the vertices of $\Gamma_{\mathcal{A}}$ are the self-intersection numbers of the corresponding lines L_i in $\widetilde{\mathbb{CP}}^2$.

Theorem 3.2 ([42]) *Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{CP}^2$ with boundary manifold M. Then the fundamental group of M is isomorphic to the group $G(\Gamma_{\mathcal{A}})$ associated to the weighted graph $\Gamma_{\mathcal{A}}$.*

The presentation provided by this result may be simplified, so as to obtain a presentation with $b_1(M) = b_2(M)$ generators and relators, realizing $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$ as a commutator-relators group. The presentation from Theorem 3.2 depends on a number of choices: the orderings of the lines of $\mathcal{A}$ and the vertices of $\Gamma_{\mathcal{A}}$, the orientation of the edges of $\Gamma_{\mathcal{A}}$, and the choice of maximal tree $\mathcal{T}$. As noted in [42], different choices yield isomorphic groups. To simplify the presentation, we will fix orderings and orientations, and work with a specific maximal tree. Our choice of tree will make transparent the relationship between the Betti numbers of the boundary manifold M and the complement X of $\mathcal{A}$.

3.3 Simplifying the presentation

Recall that the lines $\{\ell_i\}_{i=0}^n$ of $\mathcal{A}$ are ordered. Designate $\ell_0 \in \mathcal{A}$ as the line at infinity in $\mathbb{CP}^2$. Let $\hat{\mathcal{A}}$ be the central arrangement in $\mathbb{C}^3$ corresponding to $\mathcal{A} \subset \mathbb{CP}^2$, and let $\mathsf{d}\mathcal{A}$ be the decone of $\hat{\mathcal{A}}$ with respect to ℓ_0. Incidence with ℓ_0 gives a partition

$$\Pi_0 = (I_1 \mid I_2 \mid \cdots \mid I_f) \tag{8}$$

of the remaining lines of $\mathcal{A}$, where I_k is maximal so that $\ell_0 \cap \bigcap_{i \in I_k} \ell_i$ is an edge of $\mathcal{A}$. Reorder these remaining lines if necessary to insure that $I_1 = \{1, \dots, i_1\}$, $I_2 = \{i_1+1, \dots, i_2\}$, etc., and that lines ℓ_i transverse to ℓ_0 come last. In terms of the decone $\mathsf{d}\mathcal{A}$ of $\mathcal{A}$ with respect to ℓ_0, this insures that members of parallel families of lines in $\mathsf{d}\mathcal{A}$ are indexed consecutively.

Order the vertices of $\Gamma_{\mathcal{A}}$ by $v_{J_1}, \dots, v_{J_r}, v_1, \dots, v_n, v_0$, where the v_{J_k} are ordered lexicographically. In particular, the vertices corresponding to dense edges $F \subset \ell_0$ come first. Recall that the edge $e_{i,j}$ is oriented from v_i to v_j if $\ell_i \pitchfork \ell_j$ are transverse and $i < j$, and that $e_{J,i}$ is oriented from v_J to v_i if the 0-dimensional dense edge F_J is contained in ℓ_i.

Let $\mathcal{T}$ be the tree in $\Gamma_{\mathcal{A}}$ consisting of the following edges:

$$\mathcal{T} = \{e_{0,i} \mid \ell_0 \pitchfork \ell_i\} \cup \{e_{J,i} \mid F_J \subset \ell_0 \cap \ell_i\} \cup \{e_{J,i} \mid F_J \subset \ell_i,\ i = \min J\}.$$

It is readily checked that $\mathcal{T}$ is maximal. The edges of $\Gamma_{\mathcal{A}}$ not in the tree $\mathcal{T}$ are

$$\mathcal{C} = \{e_{i,j} \mid \ell_i \pitchfork \ell_j,\ 1 \le i < j \le n\} \cup \{e_{J,i} \mid F_J \subset \ell_i,\ i \ne \min J,\ 0 \notin J\}.$$

The edges in $\mathcal{C}$ are in one-to-one correspondence with the set $\mathbf{nbc}_2(\mathsf{d}\mathcal{A})$ of pairs of elements of the decone $\mathsf{d}\mathcal{A}$ which have nonempty intersection and contain *no broken circuits*, see [34]. It is well known that the cardinality of the set $\mathbf{nbc}_2(\mathsf{d}\mathcal{A})$ is equal to $b_2(X)$, the second Betti number of the complement of $\mathcal{A}$.

Now consider the group $G(\mathcal{A}) = G(\Gamma_{\mathcal{A}})$ associated to the graph $\Gamma_{\mathcal{A}}$. Denote the generators corresponding to the vertices of $\Gamma_{\mathcal{A}}$ by x_i, $0 \le i \le n$, and x_{J_k}, $1 \le k \le r$, where $\{F_{J_1}, \dots, F_{J_r}\}$ are the 0-dimensional dense edges of $\mathcal{A}$. Since the edges of $\mathcal{C}$ correspond to elements $(i,j) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})$, we denote the associated generators of $G(\mathcal{A})$ by $\gamma_{i,j}$. We modify the notation of the presentation (7) accordingly, writing R_J, $u_{J,i}$, w_J etc.

Lemma 3.4 *All commutator relators in the presentation* (7) *of* $G(\mathcal{A}) = G(\Gamma_{\mathcal{A}})$ *involving the generator* x_0 *are redundant.*

Proof If $\ell_0 \cap \ell_i$ is a double point of $\mathcal{A}$ for some i, $1 \le i \le n$, then for this i, we have the commutator relators $[x_p, x_i^{u_{p,i}}]$ for $1 \le p < i$ and $\ell_p \pitchfork \ell_i$, $[x_i, x_q^{u_{i,q}}]$ for $i < q \le n$ and $\ell_i \pitchfork \ell_q$, and $[x_J, x_i^{u_{J,i}}]$ for $F_J \subset \ell_i$. Here, $u_{p,i} = \gamma_{p,i}$, $u_{i,q} = \gamma_{i,q}$, $u_{J,i} = 1$ if $i = \min J$ (by our choice of tree), and $u_{J,i} = \gamma_{k,i}$ if $k = \min J < i$. We also have the relator

$$R_i = x_{J_1}^{u_{i,J_1}} \cdots x_{J_r}^{u_{i,J_r}} \cdot x_1^{u_{i,1}} \cdots x_{i-1}^{u_{i,i-1}} \cdot x_i^{w_i} \cdot x_{i+1}^{u_{i,i+1}} \cdots x_n^{u_{i,n}} \cdot x_0^{u_{i,0}}.$$

By our choice of tree, we have $u_{i,0} = 1$. If $\ell_i \cap \ell_j$ is not a double point of $\mathcal{A}$, there is no edge joining v_i and v_j, and $u_{i,j} = 0$. Similarly, if $F_J \not\subset \ell_i$, then $u_{i,J} = 0$.

Since $u_{p,i} = u_{i,p}^{-1}$ and $u_{J,i} = u_{i,J}^{-1}$, the commutator relators $[x_p, x_i^{u_{p,i}}]$ and $[x_J, x_i^{u_{J,i}}]$ are equivalent to $[x_p^{u_{i,p}}, x_i]$ and $[x_J^{u_{i,J}}, x_i]$. It follows that $R_i = a \cdot x_0$, where x_i commutes with a. Hence $x_i = x_i^{u_{i,0}}$ commutes with x_0.

If $F_J \subset \ell_0$, then $J = \{i_1, \dots, i_q\}$ and $i_1 = 0$. In this instance, we have relators $R_J = x_J^{-1} \cdot x_1^{u_{J,1}} \cdots x_n^{u_{J,n}} \cdot x_0^{u_{J,0}}$ and $[x_J, x_{i_p}^{u_{J,i_p}}]$ for $2 \le p \le q$. If $F_J \not\subset \ell_i$, then $u_{J,i} = 0$. By our choice of tree, $u_{J,i_p} = 1$ for $1 \le p \le q$. It follows that $R_J = x_J^{-1} \cdot x_{i_2} \cdots x_{i_q} \cdot x_0$, and x_J commutes with x_{i_p} for $2 \le p \le q$. Hence $x_J = x_J^{u_{J,0}}$ commutes with x_0. □

Now observe that the relators of type $R_J = x_J^{-1} \cdot \prod_{k=1}^n x_k^{u_{J,k}} \cdot x_0^{u_{J,0}}$ may be used to express the generators x_J in terms of x_i, $0 \le i \le n$. If $F_J = \ell_{j_1} \cap \dots \cap \ell_{j_q}$ and $j_1 = 0$, then as noted above, $R_J = x_J^{-1} \cdot x_{j_2} \cdots x_{j_q} \cdot x_0$. If $j_1 \ge 1$, then $u_{J,k} = 0$ for $k \ne j_p$, $u_{J,j_1} = 1$, and $u_{J,j_p} = \gamma_{j_1,j_p}$ for $2 \le p \le q$. So we have

$$x_J = \begin{cases} x_{j_2} \cdots x_{j_q} \cdot x_0 & \text{if } F_J \subset \ell_0, \\ x_{j_1} \cdot x_{j_2}^{\gamma_{j_1,j_2}} \cdots x_{j_q}^{\gamma_{j_1,j_q}} & \text{if } F_J \not\subset \ell_0. \end{cases} \tag{9}$$

For each p, $1 \le p \le q$, we have $F_J \subset \ell_{j_p}$ and the corresponding commutator relator $[x_J, x_{j_p}^{\gamma_{j_1,j_p}}]$. In light of (9), this may be expressed as

$$[z_J,\, x_{j_p}^{\gamma_{j_1,j_p}}], \tag{10}$$

where $z_J = x_{j_1} \cdot x_{j_2}^{\gamma_{j_1,j_2}} \cdots x_{j_q}^{\gamma_{j_1,j_q}}$ if $j_1 \ge 1$, and $z_J = x_{j_2} \cdots x_{j_q} \cdot x_0 = x_0 \cdot x_{j_2} \cdots x_{j_q}$ if $j_1 = 0$.

Note that the relator (10) in case $p = 1$ (with $\gamma_{j_1,j_1} = 1$) is a consequence of those for $2 \le p \le q$. Thus, we obtain a presentation for $G(\mathcal{A})$ with generators x_i, $0 \le i \le n$, and $\gamma_{i,j}$, $(i,j) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})$, the relators recorded in (10), together with the relators $[x_i, x_j^{\gamma_{i,j}}]$, $1 \le i < j \le n$, corresponding to double points $\ell_i \cap \ell_j$ of $\mathsf{d}\mathcal{A}$, and $R_i = \prod_{F_J \subset \ell_i} x_J^{u_{i,J}} \cdot \prod_{k=1}^n x_k^{u_{i,k}} \cdot x_0$, where x_J is given by (9), the order is irrelevant in the first product, and $0 \le i \le n$.

Lemma 3.5 *If $F_J = \ell_{j_1} \cap \cdots \cap \ell_{j_q}$ and $F_J \subset \ell_0$, then all the commutator relators recorded in* (10) *are redundant.*

Proof We have $j_1 = 0$ and, by Lemma 3.4, the assertion holds in the case $j_p = 0$. So for $j_p \neq 0$, we must show that the relator $[x_J,\, x_{j_p}]$ is a consequence of other relators, where $x_J = x_0 \cdot x_{j_2} \cdots x_{j_q}$.

For fixed $j_p \neq 0$, we have relators $[x_i, x_{j_p}^{\gamma_{i,j_p}}]$ and $[x_{j_p}, x_k^{\gamma_{j_p,k}}]$ for $i < j_p < k$, and $\ell_i \pitchfork \ell_{j_p}$, $\ell_{j_p} \pitchfork \ell_k$. The first of these is equivalent to $[x_i^{\gamma_{i,j_p}^{-1}}, x_{j_p}]$. From (10), we also have relators $[x_{j_p}, x_{J_l}^{u_{j_p,J_l}}]$ if $F_{J_l} \subset \ell_{j_p}$, where x_{J_l} is given by (9), $u_{j_p,J_l} = 1$ if $j_p = \min J_l$, and $u_{j_p,J_l} = \gamma_{j,j_p}^{-1}$ if $j_p > j = \min J_l$. Note that if $J_l \neq J$, then the word x_{J_l} does not involve the generator x_0. Additionally, we have the relator R_{j_p}, which may be expressed as

$$R_{j_p} = x_J \cdot \prod_{J_l \neq J} x_{J_l}^{u_{j_p,J_l}} \cdot \prod_{i<j_p} x_i^{\gamma_{i,j_p}^{-1}} \cdot x_{j_p}^{w_{j_p}} \cdot \prod_{j_p<k} x_k^{\gamma_{j_p,k}},$$

where the first product is over all J_l with $F_{J_l} \subset \ell_{j_p}$ with $F_{J_l} \not\subset \ell_0$, and the last two products are over all i, $1 \le i < j_p$, and k, $j_p < k \le n$, for which $\ell_i \pitchfork \ell_{j_p}$ and $\ell_{j_p} \pitchfork \ell_k$.

The above commutator relators imply that $R_{j_p} = x_J \cdot a$, where x_{j_p} commutes with a. Hence x_{j_p} commutes with x_J. The result follows. $\square$

3.6 A commutator-relators presentation

There are now $|\mathbf{nbc}_2(\mathsf{d}\mathcal{A})| = b_2(X)$ remaining commutator relators: those given by (10) corresponding to dense edges $F_J = \bigcap_{j\in J} \ell_j$ with $F_J \not\subset \ell_0$, and the relators $[x_i, x_j^{\gamma_{i,j}}]$, $1 \le i < j \le n$, corresponding to double points $\ell_i \pitchfork \ell_j$ of $\mathsf{d}\mathcal{A}$. Note that all of these commutator relators may be expressed as $[z_J,\, x_j^{\gamma_{i,j}}]$, where $\bigcap_{j\in J} \ell_j$ is an edge of $\mathsf{d}\mathcal{A}$, $i = \min(J)$, and $j \in J \setminus \min(J)$.

There also remain the relators

$$R_i = \prod_{F_J \subset \ell_i} x_J^{u_{i,J}} \cdot \prod_{k=1}^{n} x_k^{u_{i,k}} \cdot x_0^{u_{i,0}},$$

for $0 \le i \le n$. We obtain a minimal presentation for $G(\mathcal{A})$ by eliminating the generator x_0 using the relator R_0. By our choice of tree, this relator is given by

$$R_0 = \prod_{F_J \subset \ell_0} x_J \cdot x_0^{w_0} \cdot x_1^{u_{0,1}} \cdots \cdot x_n^{u_{0,n}},$$

where $u_{0,i} = 1$ if $\ell_0 \pitchfork \ell_i$, $u_{0,i} = 0$ otherwise, and $x_J = x_0 \cdot x_{j_2} \cdots x_{j_q}$ if $F_J = \ell_0 \cap \ell_{j_2} \cap \cdots \cap \ell_{j_q}$. The chosen ordering of the lines of $\mathcal{A}$ implies that $\{j_2, \dots, j_q\} = I_k$, where $(I_1 \mid \cdots \mid I_t)$ is the partition of $\{1, \dots, n\}$ induced by incidence with ℓ_0. Simplifying using the commutation relations reveals that

$$R_0 = x_0 \cdot x_1 \cdots x_n. \tag{11}$$

Consequently, we write $x_0 = (x_1 \cdots x_n)^{-1}$ and delete the relation R_0.

Now, if $\ell_0 \cap \ell_i$ is a double point of $\mathcal{A}$, then $R_i = Y_i \cdot x_0$, where Y_i is a word in the x_j, $j \neq 0$, and the $\gamma_{i,j}$. If $\ell_0 \cap \ell_i = F_J \in \mathsf{D}(\mathcal{A})$, then by our ordering of the vertices of $\Gamma_{\mathcal{A}}$, $R_i = x_J \cdot Z_i = x_0 \cdot x_{j_2} \cdots x_{j_q} \cdot Z_i$, where $J = \{0, j_2, \dots, j_q\}$. Conjugating by x_0, we can write $R_i = Y_i \cdot x_0$, where Y_i is a word as above, in this instance as well.

The next result summarizes the above simplifications. If $(i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})$, let $F_{I(i,k)}$ be the corresponding edge of $\mathsf{d}\mathcal{A}$. For an edge F_I of $\mathsf{d}\mathcal{A}$, with $i = \min I$, and $j \in I \setminus \min I$, let $\gamma_{I,j} = \gamma_{i,j}$. If $\ell_0 \cap \ell_p \cap \cdots \cap \ell_q$ is an edge of $\mathcal{A}$, set $\zeta_{0,j} = x_p \cdots x_q$ for each j, $p \leq j \leq q$. Note that if ℓ_0 and ℓ_j are transverse, then $\zeta_{0,j} = x_j$.

Proposition 3.7 *The fundamental group of the boundary manifold M of $\mathcal{A}$ has presentation*

$$G(\mathcal{A}) = \left\langle \begin{array}{ll} x_j, & 1 \leq j \leq n \\ \gamma_{i,k}, & (i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A}) \end{array} \,\middle|\, \begin{array}{ll} R_j, & 1 \leq j \leq n \\ R_{i,k}, & (i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A}) \end{array} \right\rangle,$$

where

$$R_j = \zeta_{0,j} \cdot \prod_{\substack{F_I \in \mathsf{D}(\mathsf{d}\mathcal{A}) \\ j \in I \setminus \min I}} (\gamma_{I,j} z_I \gamma_{I,j}^{-1} x_j^{-1}) \cdot \prod_{\substack{F_I \in \mathsf{D}(\mathsf{d}\mathcal{A}) \\ j = \min I}} (x_j^{-1} z_I) \cdot \prod_{\substack{\ell_i \pitchfork \ell_j \\ 1 \leq i < j}} x_i^{\gamma_{i,j}^{-1}} \cdot \prod_{\substack{\ell_j \pitchfork \ell_k \\ j < k \leq n}} x_k^{\gamma_{j,k}} \cdot (x_1 \cdots x_n)^{-1}$$

and

$$R_{i,k} = [z_{I(i,k)}, x_k^{\gamma_{i,k}}].$$

Proof It follows from the preceding discussion that the group $G(\mathcal{A})$ has such a presentation with the relators $R_{i,k}$ as asserted. So it is enough to show that the relators R_j admit the above description.

Fix j, $1 \leq j \leq n$, and consider the line ℓ_j of $\mathcal{A}$. Assume that

$$\begin{array}{rl} \text{(i)} & j \in J, \text{ where } J = [p,q] \text{ and } \ell_0 \cap \ell_p \cap \cdots \cap \ell_q \text{ is an edge of } \mathcal{A}; \\ \text{(ii)} & j \in J_t \setminus \min J_t \text{ for } 1 \leq t \leq a \text{ and } F_{J_t} \text{ is a dense edge of } \mathsf{d}\mathcal{A}; \\ \text{(iii)} & j = \min K_t \text{ for } 1 \leq t \leq b \text{ and } F_{K_t} \text{ is a dense edge of } \mathsf{d}\mathcal{A}; \\ \text{(iv)} & \ell_j \pitchfork \ell_{i_t} \text{ for } 1 \leq t \leq c \text{ and } 1 \leq i_t < j; \text{ and} \\ \text{(v)} & \ell_j \pitchfork \ell_{k_t} \text{ for } 1 \leq t \leq d \text{ and } j < k_t \leq n. \end{array} \tag{12}$$

Note that ℓ_j contains either $a+b$ or $a+b+1$ dense edges of $\mathcal{A}$, depending on whether ℓ_j is transverse to ℓ_0 or not. Consequently, the weight of the vertex $v_j \in \Gamma_{\mathcal{A}}$ is

$$w_j = \begin{cases} 1-a-b & \text{if } \ell_j \pitchfork \ell_0, \\ -a-b & \text{otherwise.} \end{cases}$$

With these data, the preceding discussion and our conventions regarding the graph $\Gamma_{\mathcal{A}}$ and the group $G(\mathcal{A})$ imply that the relator R_j is given by

$$R_j = \zeta_{0,j} \cdot \prod_{t=1}^{a} z_{J_t}^{\gamma_{j,J_t}} \cdot \prod_{t=1}^{b} z_{K_t}^{\gamma_{j,K_t}} \cdot \prod_{t=1}^{c} x_{i_t}^{\gamma_{j,i_t}} \cdot x_j^{-a-b} \cdot \prod_{t=1}^{d} x_{k_t}^{\gamma_{j,k_t}} \cdot x_0.$$

The commutator relators $R_{i,k}$ imply that x_j commutes with each of $z_{J_t}^{\gamma_{j,J_t}}$, $z_{K_t}^{\gamma_{j,K_t}}$, $x_{i_t}^{\gamma_{j,i_t}}$, $x_{k_t}^{\gamma_{j,k_t}}$ for all relevant t. Furthermore, $\gamma_{j,J} = \gamma_{J,j}^{-1}$ if $j \in J \setminus \min J$, $\gamma_{j,K} = 1$ if $j = \min K$, and $\gamma_{j,i} = \gamma_{i,j}^{-1}$ if $i < j$. Using these facts, the relator R_j may be expressed as

$$R_j = \zeta_{0,j} \cdot \prod_{t=1}^{a} (z_{J_t}^{\gamma_{J_t,j}^{-1}} \cdot x_j^{-1}) \cdot \prod_{t=1}^{b} (x_j^{-1} \cdot z_{K_t}) \cdot \prod_{t=1}^{c} x_{i_t}^{\gamma_{i_t,j}^{-1}} \cdot \prod_{t=1}^{d} x_{k_t}^{\gamma_{j,k_t}} \cdot x_0.$$

Recalling that $x_0 = (x_1 \cdots x_n)^{-1}$, this is easily seen to be equivalent to the expression given in the statement of the Proposition. □

Remark If $(i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})$ and $I = I(i,k) = \{i_1, \dots, i_q\}$, the relators $[z_I, x_{i_p}^{\gamma_{i_1,i_p}}]$, $2 \le p \le q$, are equivalent to the family $[x_{i_1}, x_{i_2}^{\gamma_{i_1,i_2}}, \dots, x_{i_q}^{\gamma_{i_1,i_q}}]$ of "Randell relations" familiar from presentations of the fundamental group of the complement of an arrangement.

Corollary 3.8 *The group $G(\mathcal{A})$ is a commutator-relators group.*

Proof By Proposition 3.7, the group $G(\mathcal{A}) = \pi_1(M)$ admits a presentation with $b_1 = b_1(M)$ generators. The conclusion follows from this, together with the fact that $H_1(M)$ is free abelian of rank b_1, see [30, Proposition 2.7]. □

Remark This result may also be established directly, by showing that each relator R_j is a product of commutators. Using the Randell relations noted above, one can show that $R_j = x_{\rho_{i,1}}^{v_{i,1}} \cdots x_{\rho_{i,n}}^{v_{i,n}} \cdot x_0$, where $\{\rho_{i,1}, \dots, \rho_{i,n}\}$ is a permutation of $[n]$ and $v_{p,q}$ is a word in the generators $\gamma_{i,j}$. This may be expressed as a product of commutators using the fact that $x_0 = (x_1 \cdots x_n)^{-1}$.

3.9 Some computations

We conclude this section with a few examples illustrating how the presentation from Proposition 3.7 works in practice.

Example 3.10 Let $\mathcal{A}$ be a near-pencil of $n+1$ lines, with defining polynomial $Q = x_0(x_1^n - x_2^n)$ and boundary manifold M. The graph $\Gamma_{\mathcal{A}}$ has vertices $v_0, v_1, \dots, v_n$ corresponding to the lines, and one more vertex $v_{n+1} = v_F$ corresponding to the multiple point $F = \ell_1 \cap \cdots \cap \ell_n$. The weights of the vertices are $w_0 = 1$, $w_1 = \cdots = w_n = 0$, and $w_{n+1} = -1$. The edge set is $\mathcal{E}$ consists of edges $e_{0,i}$ and $e_{i,n+1}$ for $1 \le i \le n$. Fix the maximal tree $\mathcal{T} = \{e_{0,1}, \dots, e_{0,n}, e_{1,n+1}\}$, indicated by dashed edges in Figure 1.

By Proposition 3.7, the fundamental group of M has presentation

$$G(\mathcal{A}) = \langle x_1,\ x_j,\ \gamma_{1,j} \mid z\zeta^{-1},\ x_j\gamma_{1,j}z\gamma_{1,j}^{-1}x_j^{-1}\zeta^{-1},\ [z, x_j^{\gamma_{1,j}}]\rangle,$$

where $z = x_1 \cdot x_2^{\gamma_{1,2}} \cdots x_n^{\gamma_{1,n}}$, $\zeta = x_1 \cdot x_2 \cdots x_n$, and $2 \le j \le n$.

The elements $\zeta, x_2, \dots, x_n, \gamma_{1,2}, \dots, \gamma_{1,n}$ generate the group $G(\mathcal{A})$, and it is readily checked that ζ is central. Also, conjugating the relator R_1 by x_1 yields

$$[\gamma_{1,2}^{-1}, x_2] \cdot x_2[\gamma_{1,3}^{-1}, x_3]x_2^{-1} \cdot \cdots \cdot (x_2 \cdots x_{n-1})[\gamma_{1,n}^{-1}, x_n](x_2 \cdots x_{n-1})^{-1}.$$

It follows that $G(\mathcal{A})$ is isomorphic to the direct product of a cyclic group $\mathbb{Z} = \langle c\rangle$ with a genus $n-1$ surface group

$$\pi_1(\Sigma_{n-1}) = \langle g_1, \dots, g_{2n-2} \mid [g_1, g_2] \cdots [g_{2n-3}, g_{2n-2}]\rangle.$$

An explicit isomorphism $\mathbb{Z} \times \pi_1(\Sigma_{n-1}) \xrightarrow{\simeq} G(\mathcal{A})$ is given by

$$c \mapsto \zeta, \quad g_i \mapsto \begin{cases} x_2 \cdots x_k \cdot \gamma_{1,k+1}^{-1} \cdot (x_2 \cdots x_k)^{-1}, & \text{if } i = 2k-1, \\ x_2 \cdots x_k \cdot x_{k+1} \cdot (x_2 \cdots x_k)^{-1}, & \text{if } i = 2k. \end{cases}$$

Example 3.11 Let $\mathcal{A}$ be an arrangement of $n+1$ lines in general position. The graph $\Gamma_{\mathcal{A}}$ is the complete graph on $n+1$ vertices. Here, there are no 0-dimensional dense edges and all vertices have weight 1.

Using the maximal tree $\mathcal{T} = \{e_{0,i} \mid 1 \le i \le n\}$ (indicated by dashed edges in Figure 2), Proposition 3.7 yields a presentation for $G(\mathcal{A})$ with generators x_i $(1 \le i \le n)$ and $\gamma_{i,j}$ $(1 \le i < j \le n)$, and relators

$$\begin{aligned} R_j &= x_j \cdot x_1^{\gamma_{1,j}^{-1}} \cdots x_{j-1}^{\gamma_{j-1,j}^{-1}} \cdot x_{j+1}^{\gamma_{j,j+1}} \cdots x_n^{\gamma_{j,n}} \cdot x_n^{-1} \cdots x_1^{-1} && (1 \le j \le n), \\ R_{i,j} &= [x_i, x_j^{\gamma_{i,j}}] && (1 \le i < j \le n). \end{aligned}$$

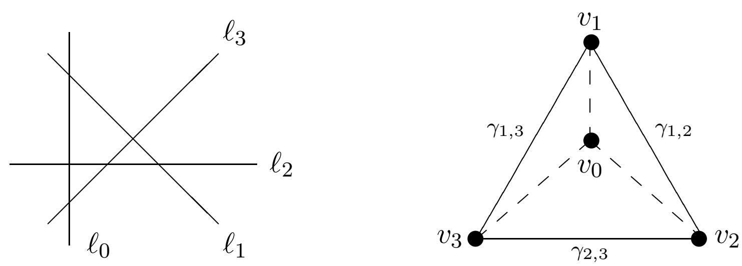

Figure 2: A general position arrangement and its associated graph

4 Twisted Alexander polynomials

A finitely generated module K over a Noetherian ring R admits a finite presentation, $R^r \xrightarrow{\psi} R^s \to K \to 0$. Let $E_i(K)$ denote the i-th elementary ideal of K, the ideal of R generated by the codimension i minors of the matrix ψ. It is well known that the elementary ideals do not depend on the choice of presentation, so are invariants of the module K.

Let $\Lambda = \mathbb{F}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ be the ring of Laurent polynomials in n variables over a field $\mathbb{F}$. Since Λ is a unique factorization domain, there is a unique minimal principal ideal that contains the elementary ideal $E_0(K)$. Define the order, $\operatorname{ord}(K)$, of the module K to be a generator of this principal ideal. Note that $\operatorname{ord}(K)$ is defined up to multiplication by a unit in Λ, which necessarily is of the form $ct_1^{l_1} \cdots t_n^{l_n}$, for some $l_i \in \mathbb{Z}$ and $c \in \mathbb{F}^*$.

Now let G be a group of type FL, and $\alpha\colon G \to H$ a homomorphism to a finitely generated, free abelian group. Note that if $\operatorname{rank}(H) = n$, then $\mathbb{F}[H] \cong \Lambda$. Let $\phi\colon G \to \mathrm{GL}_k(\mathbb{F})$ be a representation. With these data, the vector space $\Lambda^k_{\phi,\alpha} = \mathbb{F}^k \otimes_{\mathbb{F}} \Lambda$ admits the structure of a (left) G-module: if $\gamma \in G$ and $v \otimes q \in \Lambda^k_{\phi,\alpha}$, then

$$\gamma \cdot (v \otimes q) = (\phi(\gamma)v) \otimes (\alpha(\gamma)q).$$

Following [24], define the twisted Alexander modules of G (with respect to α and ϕ) to be the homology groups of G with coefficients in $\Lambda^k_{\phi,\alpha}$: if $C_*(G)$ is a finite, free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, then

$$H_i(G; \Lambda^k_{\phi,\alpha}) = H_i(C_*(G) \otimes_{\mathbb{Z}G} \Lambda^k_{\phi,\alpha}). \tag{13}$$

Note that $H_i(G; \Lambda^k_{\phi,\alpha})$ carries the structure of a (finitely generated) right Λ-module. Define the twisted Alexander polynomial $\Delta_i^{\phi,\alpha}(G)$ to be the order of this module:

$$\Delta_i^{\phi,\alpha}(G) = \operatorname{ord}\big(H_i(G; \Lambda^k_{\phi,\alpha})\big). \tag{14}$$

If $\theta\colon G \twoheadrightarrow G'$ is an epimorphism, $\alpha = \alpha' \circ \theta$, and $\phi = \phi' \circ \theta$, then $\Delta_1^{\phi',\alpha'}(G')$ divides $\Delta_1^{\phi,\alpha}(G)$, see [25].

In the case where $\alpha\colon G \twoheadrightarrow H_1(G)/\operatorname{Tors}(H_1(G))$ is the projection onto the maximal torsion-free abelian quotient, we suppress α and write simply Λ_ϕ^k and $\Delta_i^\phi(G)$. Note that if $\phi\colon G \to \mathrm{GL}_1(\mathbb{F})$ is the trivial representation, then $\Delta_1^\phi(G)$ is the classical Alexander polynomial $\Delta(G)$. Up to a monomial change of variables, $t_i \mapsto t_1^{a_{i,1}} \cdots t_n^{a_{i,n}}$, where $(a_{i,j}) \in \mathrm{GL}_n(\mathbb{Z})$, this Laurent polynomial is an invariant of the isomorphism type of the group G. In what follows, we will focus our attention on the case $\mathbb{F} = \mathbb{C}$.

Lemma 4.1 *Let G be a finitely generated free abelian group, and $\phi\colon G \to \mathrm{GL}_k(\mathbb{C})$ a representation. Then the twisted Alexander module $H_i(G;\Lambda_\phi^k)$ vanishes for $i \geq 1$, and $\operatorname{ord}\big(H_0(G;\Lambda_\phi^k)\big) = 1$.*

Proof Let $n = \operatorname{rank}(G)$. Denote the generators of G by $t_1, \dots, t_n$, and identify $\mathbb{C}[G] \cong \Lambda$.

The proof is by induction on k. If $k = 1$, the chain complex $C_*(G) \otimes_{\mathbb{Z}G} \Lambda_\phi^1$ may be realized as the standard Koszul complex in the variables $z_i = \phi(t_i) \cdot t_i$. Consequently, $H_i(G;\Lambda_\phi^1) = H_i(C_*(G) \otimes_{\mathbb{Z}G} \Lambda_\phi^1) = 0$ for $i \geq 1$, and $H_0(G;\Lambda_\phi^1) = \mathbb{C}$ has order 1.

Suppose $k > 1$. Since G is abelian, the automorphisms $\phi(t_i) \in \mathrm{GL}_k(\mathbb{C})$, $1 \leq i \leq n$, all commute. Consequently, they have a common eigenvector, say v. Let λ_i be the eigenvalue of $\phi(t_i)$ with eigenvector v, and let $\{w_1, \dots, w_{k-1}\}$ be a basis for $\langle v \rangle^\perp$. With respect to the basis $\{v, w_1, \dots, w_{k-1}\}$ for $\mathbb{C}^k$, the matrix A_i of $\phi(t_i)$ is of the form

$$A_i = \begin{pmatrix} \lambda_i & * \\ 0 & \bar{A}_i, \end{pmatrix}$$

where $\bar{A}_i$ is an invertible $(k-1) \times (k-1)$ matrix. Define representations $\phi'\colon G \to \mathbb{C}^*$ and $\phi''\colon G \to \mathrm{GL}_{k-1}(\mathbb{C})$ by $\phi'(t_i) = \lambda_i$ and $\phi''(t_i) = \bar{A}_i$. Then we have a short exact sequence of G-modules

$$0 \longrightarrow \Lambda_{\phi'}^1 \longrightarrow \Lambda_\phi^k \longrightarrow \Lambda_{\phi''}^{k-1} \longrightarrow 0\ ,$$

and a corresponding long exact sequence in homology

$$\cdots \longrightarrow H_i(G;\Lambda_{\phi'}^1) \longrightarrow H_i(G;\Lambda_\phi^k) \longrightarrow H_i(G;\Lambda_{\phi''}^{k-1}) \longrightarrow \cdots .$$

Using this sequence, the case $k = 1$, and the inductive hypothesis, we conclude that $H_i(G;\Lambda_\phi^k) = 0$ for $i \geq 1$, and that $\operatorname{ord} H_0(G;\Lambda_\phi^k) = 1$. □

Let Γ be a connected, directed graph, and let $\mathcal{V}=\mathcal{V}(\Gamma)$ and $\mathcal{E}=\mathcal{E}(\Gamma)$ denote the vertex and edge sets of Γ. A graph of groups is such a graph, together with vertex groups $\{G_v \mid v\in\mathcal{V}\}$, edge groups $\{G_e \mid e\in\mathcal{E}\}$, and monomorphisms $\theta_0\colon G_e\to G_v$ and $\theta_1\colon G_e\to G_w$ for each directed edge $e=(v,w)$. Choose a maximal tree T for Γ. The fundamental group $G=G(\Gamma)$ (relative to T) is the group generated by the vertex groups G_v and the edges e of Γ not in T, with the additional relations $e\cdot\theta_1(x)=\theta_0(x)\cdot e$, for $x\in G_e$ if $e\in\Gamma\setminus T$, and $\theta_1(y)=\theta_0(y)$, for $y\in G_e$ if $e\in T$.

Theorem 4.2 *Let $(\Gamma,\{G_e\}_{e\in\mathcal{E}(\Gamma)},\{G_v\}_{v\in\mathcal{V}(\Gamma)})$ be a graph of groups, with fundamental group G, vertex groups of type FL, and free abelian edge groups. Assume that the inclusions $G_e\hookrightarrow G$ induce monomorphisms in homology. If $\phi\colon G\to \mathrm{GL}_k(\mathbb{C})$ is a representation, then*

(i) $H_i(G;\Lambda^k_\phi)=\bigoplus_{v\in\mathcal{V}} H_i(G_v;\Lambda^k_\phi)$ *for* $i\ge 2$, *and*

(ii) $\operatorname{ord}\big(H_1(G;\Lambda^k_\phi)\big)=\operatorname{ord}\big(\bigoplus_{v\in\mathcal{V}} H_1(G_v;\Lambda^k_\phi)\big)$.

Proof For simplicity, we will suppress the coefficient module Λ^k_ϕ for the duration of the proof. Given a graph of groups, there is a Mayer-Vietoris sequence

$$\cdots\longrightarrow \bigoplus_{e\in\mathcal{E}} H_i(G_e)\longrightarrow \bigoplus_{v\in\mathcal{V}} H_i(G_v)\longrightarrow H_i(G)\xrightarrow{\partial}\bigoplus_{e\in\mathcal{E}} H_{i-1}(G_e)\longrightarrow\cdots$$

see [5, Ch. VII, §9]. Since the edges groups are free abelian and the inclusions $G_e\hookrightarrow G$ induce monomorphisms in homology, we may apply Lemma 4.1 to conclude that $H_i(G_e)=0$ for all $i\ge 1$. Assertion (i) follows.

Lemma 4.1 also implies that $\operatorname{ord}\big(H_0(G_e)\big)=1$, for each $e\in\mathcal{E}$. Consequently, $\operatorname{ord}\big(\bigoplus_{e\in\mathcal{E}} H_0(G_e)\big)=1$. The above Mayer-Vietoris sequence reduces to

$$0\longrightarrow \bigoplus_{v\in\mathcal{V}} H_1(G_v)\longrightarrow H_1(G)\xrightarrow{\partial}\bigoplus_{e\in\mathcal{E}} H_0(G_e)\longrightarrow\cdots.$$

From this, we obtain a short exact sequence

$$0\longrightarrow \bigoplus_{v\in\mathcal{V}} H_1(G_v)\longrightarrow H_1(G)\xrightarrow{\partial}\operatorname{Im}(\partial)\longrightarrow 0\,.$$

Since $\operatorname{Im}(\partial)$ is a submodule of $\bigoplus_{e\in\mathcal{E}} H_0(G_e)$, and the latter has order 1, we have $\operatorname{ord}\big(\operatorname{Im}(\partial)\big)=1$ as well. Assertion (ii) follows. □

5 Alexander polynomials of line arrangements

Let $\mathcal{A}=\{\ell_0,\dots,\ell_n\}$ be an arrangement of $n+1$ lines in $\mathbb{CP}^2$, with boundary manifold M. Since M is a graph manifold, the fundamental group $G=\pi_1(M)$

is the fundamental group of a graph of groups. Recall from Section 2 that, in the graph manifold structure, the vertex manifolds are of the form $M_v \cong S^1 \times \left(\mathbb{CP}^1 \setminus \bigcup_{j=1}^m B_j\right)$, where the B_j are disjoint disks and m is the multiplicity (degree) of the vertex v of $\Gamma_{\mathcal{A}}$, and these vertex manifolds are glued together along tori. Consequently, the vertex groups are of the form $\mathbb{Z} \times F_{m-1}$, and the edge groups are free abelian of rank 2.

The edge groups are generated by meridian loops about the lines L_i of $\tilde{\mathcal{A}}$ in $\widetilde{\mathbb{CP}}^2$. In terms of the generators x_i of G, these generators are of the form x_i^y or $x_{i_1}^{y_1} \cdots x_{i_k}^{y_k}$ if L_i is the proper transform of $\ell_i \in \mathcal{A}$ or L_i is the exceptional line arising from blowing up the dense edge F_I of $\mathcal{A}$, where $I = \{i_1, \dots, i_k\}$. By (11), $x_0 x_1 \cdots x_n = 1$ in G. This fact may be used to check that the inclusions of the edge groups in G induce monomorphisms in homology. Therefore, Theorem 4.2 may be applied to calculate twisted Alexander polynomials of G. We first record a number of preliminary facts.

Lemma 5.1 *Let $G = \mathbb{Z} \times F_{m-1}$, and let $\phi\colon G \to \mathrm{GL}_k(\mathbb{C})$ be a representation. Then the twisted Alexander polynomial $\Delta_1^\phi(G)$ is given by*

$$\Delta_1^\phi(G) = \left[p(A,t)\right]^{m-2},$$

where t is the image of a generator z of the center $\mathbb{Z}$ of G under the abelianization map, and $p(A,t)$ is the characteristic polynomial of the automorphism $A = \phi(z)$ in the variable t. In particular, the classical Alexander polynomial is $\Delta(G) = (t-1)^{m-2}$.

Proof Write $G = \mathbb{Z} \times F_{m-1} = \langle z, y_1, \dots, y_{m-1} \mid [z, y_1], \dots, [z, y_{m-1}] \rangle$. Applying the Fox calculus to this presentation yields a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$,

$$(\mathbb{Z}G)^{m-1} \xrightarrow{\partial_2} (\mathbb{Z}G)^m \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0\ ,$$

where $\epsilon\colon \mathbb{Z}G \to \mathbb{Z}$ is the augmentation map, and the matrices of ∂_1 and ∂_2 are given by $[\partial_1] = \begin{pmatrix} z-1 & y_1 - 1 & \cdots & y_{m-1}-1 \end{pmatrix}^\top$ and

$$[\partial_2] = \begin{pmatrix} 1-y_1 & z-1 & 0 & \cdots & 0 \\ 1-y_2 & 0 & z-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-y_{m-1} & 0 & 0 & \cdots & z-1 \end{pmatrix}$$

A calculation with this resolution yields the result. □

Let $\Gamma_{\mathcal{A}}$ denote the graph underlying the graph manifold structure of the boundary manifold M of the line arrangement $\mathcal{A} = \{\ell_i\}_{i=0}^n$ in $\mathbb{CP}^2$ and the graph of groups structure of the fundamental group $G = \pi_1(M)$. For a vertex v of $\Gamma_{\mathcal{A}}$ with multiplicity m_v, in the identification $G_v = \mathbb{Z} \times F_{m_v-1}$ of the vertex groups of G, the center $\mathbb{Z}$ of G_v is generated by z_v, an meridian loop about the corresponding line L_i of $\widetilde{\mathcal{A}}$. Denoting the images of the generators x_i of G under the abelianization $\alpha\colon G \to G/G'$ by t_i, there is a choice of generator z_v so that

$$\alpha(z_v) = t_v = \begin{cases} t_i & \text{if } v = v_i,\ 0 \le i \le n, \\ t_{i_1} \cdots t_{i_k} & \text{if } v = v_I, \text{ where } I = \{i_1, \dots, i_k\} \text{ and } F_I \in \mathsf{D}(\mathcal{A}). \end{cases}$$

If $I = \{i_1, \dots, i_k\}$, we subsequently write $t_I = t_{i_1} \cdots t_{i_k}$.

Theorem 4.2 and Lemma 5.1 yield the following result.

Theorem 5.2 *Let $\mathcal{A}$ be an essential line arrangement in $\mathbb{CP}^2$, let $\Gamma_{\mathcal{A}}$ be the associated graph, and let G be the fundamental group of the boundary manifold M of $\mathcal{A}$. If $\phi\colon G \to \mathrm{GL}_k(\mathbb{C})$ is a representation, then the twisted Alexander polynomial $\Delta_1^{\phi}(G)$ is given by*

$$\Delta_1^{\phi}(G) = \prod_{v \in \mathcal{V}(\Gamma_{\mathcal{A}})} \big[p(A_v, t_v)\big]^{m_v-2},$$

where t_v is the image of a generator of the center $\mathbb{Z}$ of G_v under the abelianization map, and $p(A_v, t_v)$ is the characteristic polynomial of the automorphism $A_v = \phi(z_v)$ in the variable t_v. In particular, the classical Alexander polynomial of G is

$$\Delta(G) = \prod_{v \in \mathcal{V}(\Gamma_{\mathcal{A}})} (t_v - 1)^{m_v-2}.$$

Remark By gluing formulas of Meng-Taubes [32] and Turaev [39], with appropriate identifications, Milnor torsion is multiplicative when gluing along tori. Since Milnor torsion coincides with the Alexander polynomial for a 3-manifold M with $b_1(M) > 1$, the calculation of $\Delta(G)$ in Theorem 5.2 above may alternatively be obtained using these gluing formulas, see Vidussi [40, Lemma 7.4].

The above formula for $\Delta(G)$ is also reminiscent of the Eisenbud-Neumann formula for the Alexander polynomial $\Delta_L(t)$ of a graph (multi)-link L, see [14, Theorem 12.1]. For example, if L is the n-component Hopf link (i.e., the singularity link of a pencil of $n \ge 2$ lines), then $\Delta_L(t) = (t_1 \cdots t_n - 1)^{n-2}$.

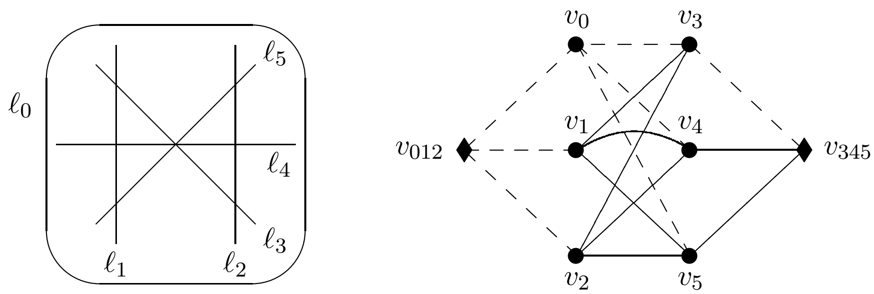

Figure 3: The Falk arrangement $\mathcal{F}_1$ and its associated graph

Recall from (11) that the meridian generators x_i of G corresponding to the lines ℓ_i of $\mathcal{A}$, $0 \le i \le n$, satisfy the relation $x_0x_1\cdots x_n = 1$. Consequently, $t_0t_1\cdots t_n = 1$ and the twisted Alexander polynomial $\Delta_1^{\phi}(G)$ may be viewed as an element of $\Lambda = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. In particular, in the classical Alexander polynomial, if $I = \{0\} \cup J$, then $t_I - 1 \doteq t_{[n]\setminus J} - 1$, since Alexander polynomials are defined up to multiplication by units. In what follows, we make substitutions such as these without comment.

In light of Theorem 5.2, we focus on the classical Alexander polynomial for the remainder of this section.

Example 5.3 In [15], Falk considered a pair of arrangements whose complements are homotopy equivalent, even though the two intersection lattices are not isomorphic. In this example, we analyze the respective boundary manifolds.

The Falk arrangements $\mathcal{F}_1$ and $\mathcal{F}_2$ have defining polynomials

$$Q(\mathcal{F}_1) = x_0(x_1 + x_0)(x_1 - x_0)(x_1 + x_2)x_2(x_1 - x_2)$$

and

$$Q(\mathcal{F}_2) = x_0(x_1 + x_0)(x_1 - x_0)(x_2 + x_0)(x_2 - x_0)(x_2 + x_1 - x_0).$$

These arrangements, and the associated graphs, are depicted in Figures 3 and 4.

By Theorem 5.2, the fundamental groups, $G_i = \pi_1(M(\mathcal{F}_i))$, of the boundary manifolds of these arrangements have Alexander polynomials

$$\Delta_1 = [(t_1 - 1)(t_2 - 1)(t_3 - 1)(t_4 - 1)(t_5 - 1)(t_{[5]} - 1)(t_{345} - 1)]^2 \tag{15}$$

and

$$\Delta_2 = [(t_1 - 1)(t_2 - 1)(t_3 - 1)(t_4 - 1)]^2(t_5 - 1)^3(t_{[5]} - 1)(t_{345} - 1)(t_{125} - 1),$$

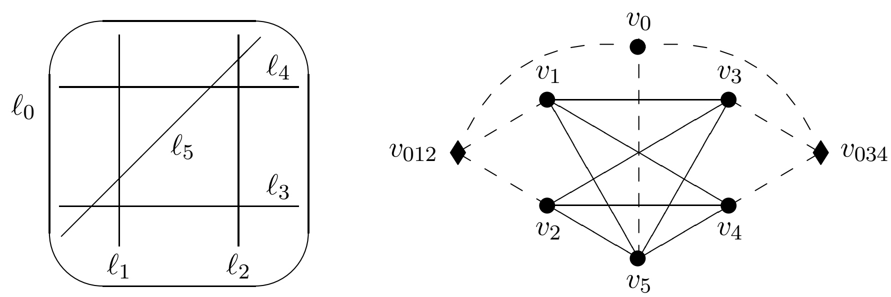

Figure 4: The Falk arrangement $\mathcal{F}_2$ and its associated graph

where $\Delta_i = \Delta(G_i)$. Since these polynomials have different numbers of distinct factors, there is no monomial isomorphism of $\Lambda = \mathbb{C}[t_1^{\pm 1}, \dots, t_5^{\pm 1}]$ taking Δ_1 to Δ_2. Hence, the groups G_1 and G_2 are not isomorphic, and the boundary manifolds $M(\mathcal{F}_1)$ and $M(\mathcal{F}_2)$ are not homotopy equivalent. It follows that the complements of the two Falk arrangements are not homeomorphic—a result obtained previously by Jiang and Yau [23] by invoking the classification of Waldhausen graph manifolds.

Note that the number of distinct factors in the Alexander polynomial $\Delta(G_2)$ above is equal to the number of vertices in the graph $\Gamma_{\mathcal{F}_2}$, while $\Delta(G_1)$ has fewer factors than $|\mathcal{V}(\Gamma_{\mathcal{F}_1})|$. In general, the cardinality of $\mathcal{V}(\Gamma_{\mathcal{A}})$ is equal to $|\mathsf{D}(\mathcal{A})|$, the number of dense edges of $\mathcal{A}$. We record several families of arrangements for which the Alexander polynomial $\Delta(G)$ is "degenerate", i.e., the number of distinct factors is less than the number of dense edges.

Example 5.4 Let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2$, with boundary manifold M, and let $G = \pi_1(M)$. If $I = \{i_1, \dots, i_k\}$, recall that $t_I = t_{i_1} \cdots t_{i_k}$. In particular, write $t_{[k]} = t_1 \cdots t_k$ and $t_{[i,j]} = t_i t_{i+1} \cdots t_{j-1} t_j$. If Q is a defining polynomial for $\mathcal{A}$, order the lines of $\mathcal{A}$ (starting with 0) as indicated in Q.

(1) If $Q = x_1^{n+1} - x_2^{n+1}$, then $\mathcal{A}$ is a pencil with $|\mathsf{D}(\mathcal{A})| = n+1$ dense edges, and $G = F_n$ is a free group of rank n. Thus, $\Delta(G) = 0$ if $n \neq 1$, and $\Delta(G) = 1$ if $n = 1$.

(2) If $Q = x_0(x_1^n - x_2^n)$, then $\mathcal{A}$ is a near-pencil with $|\mathsf{D}(\mathcal{A})| = n+2$, while $\Delta(G) = (t_{[n]} - 1)^{n-2}$ has a single (distinct) factor.

(3) If $Q = x_0(x_0^m - x_1^m)(x_0^n - x_2^n)$, then $|\mathsf{D}(\mathcal{A})| = m+n+3$. Writing $J = [m+1, m+n]$, $\Delta(G)$ is given by

$$[(t_1 - 1) \cdots (t_m - 1)(t_{[m]} - 1)]^{n-1} [(t_{m+1} - 1) \cdots (t_{m+n} - 1)(t_J - 1)]^{m-1}.$$

(4) If $Q = x_0(x_0^m - x_2^m)(x_1^n - x_2^n)$, then $|\mathsf{D}(\mathcal{A})| = m+n+3$. Writing $J = [m+1, m+n]$ and $k = m+n-3$, $\Delta(G)$ is given by

$$[(t_1-1)\cdots(t_m-1)(t_{[m+n]}-1)]^{n-1}[(t_{m+1}-1)\cdots(t_{m+n}-1)]^m(t_J-1)^k.$$

Note that, after a change of coordinates, the Falk arrangement $\mathcal{F}_1$ is of this form.

The arrangements recorded in Example 5.4 (3) and (4) have the property that there are two 0-dimensional dense edges which exhaust the lines of the arrangement. That is, there are edges $F = \bigcap_{i\in I}\ell_i$ and $F' = \bigcap_{i\in I'}\ell_i$ so that $\mathcal{A} = \{\ell_i \mid i \in I \cup I'\}$. We say F and F' cover $\mathcal{A}$. This condition insures that the Alexander polynomial is degenerate.

Proposition 5.5 *Let $\mathcal{A}$ be an arrangement of $n+1$ lines in $\mathbb{CP}^2$ that is not a pencil or a near-pencil. If $\mathcal{A}$ has two 0-dimensional dense edges which cover $\mathcal{A}$, then the number of distinct factors in the Alexander polynomial of the boundary manifold of $\mathcal{A}$ is $|\mathsf{D}(\mathcal{A})|-1$. Otherwise, the number of distinct factors is $|\mathsf{D}(\mathcal{A})|$.*

Proof If $\mathcal{A}$ satisfies the hypotheses of the proposition, it is readily checked that, up to a coordinate change, $\mathcal{A}$ is one of the arrangements recorded in Example 5.4 (3) and (4). So assume that these hypotheses do not hold.

If $\mathcal{A}$ has no 0-dimensional dense edges, then $\mathcal{A}$ is a general position arrangement. Since $\mathcal{A}$ is, by assumption, not a near-pencil, the cardinality of $\mathcal{A}$ is at least 4, i.e., $n \geq 3$. In this instance, the Alexander polynomial of the boundary manifold,

$$\Delta(G) = [(t_1-1)\cdots(t_n-1)(t_{[n]}-1)]^{n-2},$$

has $n+1 = |\mathsf{D}(\mathcal{A})|$ factors.

Suppose $\mathcal{A}$ has one 0-dimensional dense edge. Since $\mathcal{A}$ is not a pencil or near pencil, there are at least two lines of $\mathcal{A}$ which do not contain the dense edge. Write $\mathcal{A} = \{\ell_0, \ell_1, \ldots, \ell_n\}$, where $\bigcap_{i=1}^k \ell_i$, $k \geq 3$, is the unique 0-dimensional dense edge. Since $\mathcal{A}$ has a single 0-dimensional dense edge, the subarrangement $\{\ell_0, \ell_{k+1}, \ldots, \ell_n\}$ is in general position. By Theorem 5.2, the Alexander polynomial of the boundary of $\mathcal{A}$ is

$$\Delta(G) = \prod_{i=1}^n (t_i-1)^{m_i-2}\cdot(t_{[n]}-1)^{m_0-2}\cdot(t_{[k]}-1)^{k-2},$$

and one can check that $m_i \geq 3$ for each i, $0 \leq i \leq n$.

Now consider the case where $\mathcal{A}$ has two 0-dimensional dense edges, but they do not cover $\mathcal{A}$. Either there is a line of $\mathcal{A}$ containing both dense edges, or not. Assume first there is no such line. Write $\mathcal{A} = \{\ell_i\}_{i=0}^n$, and assume without loss that the two dense edges are $\bigcap_{i=0}^k \ell_i$ and $\bigcap_{i=k+1}^m \ell_i$, where $k \geq 2$, $m-k \geq 3$, and $m < n$. By Theorem 5.2,

$$\Delta(G) = \prod_{i=1}^n (t_i - 1)^{m_i-2} \cdot (t_{[n]} - 1)^{m_0-2} \cdot (t_{[k+1,n]} - 1)^{k-1} \cdot (t_{[k+1,m]} - 1)^{m-k-2},$$

and one can check that $m_i \geq 3$ for each i, $0 \leq i \leq n$.

If there is a line of $\mathcal{A}$ containing both 0-dimensional dense edges, we can assume that $\mathcal{A} = \{\ell_i\}_{i=0}^n$, and the two dense edges are $\bigcap_{i=0}^k \ell_i$ and $\ell_0 \cap \bigcap_{i=k+1}^m \ell_i$, where $k \geq 2$, $m-k \geq 2$, and $m < n$. By Theorem 5.2,

$$\Delta(G) = \prod_{i=1}^n (t_i - 1)^{m_i-2} \cdot (t_{[n]} - 1)^{m_0-2} \cdot (t_{[k+1,n]} - 1)^{k-1} \cdot (t_{[k]} t_{[m+1,n]} - 1)^{m-k-1},$$

and one can check that $m_i \geq 3$ for each i, $0 \leq i \leq n$.

Finally, suppose that $\mathcal{A} = \{\ell_i\}_{i=0}^n$ has at least three 0-dimensional dense edges. If $\bigcap_{i=0}^k \ell_i$ is a dense edge, this assumption implies that $\bigcap_{i=k+1}^n \ell_i$ cannot be a dense edge. Consequently, the factors of the Alexander polynomial corresponding to 0-dimensional dense edges are relatively prime, and are prime to the factor $(t_{[n]} - 1)^{m_0-2}$ corresponding to the line ℓ_0 of $\mathcal{A}$.

To complete the argument, it suffices to show that $m_i \geq 2$ for each i, $0 \leq i \leq n$. For a line ℓ_i of $\mathcal{A}$, this may be established by choosing 0-dimensional dense edges F_1, F_2, F_3 of $\mathcal{A}$, and considering whether F_j is contained in ℓ_i or not. $\square$

6 Alexander balls

Let M be a 3-manifold with positive first Betti number, and let $G = \pi_1(M)$. Let $H = H_1(M)/\operatorname{Tors}(H_1(M))$, and denote by $\alpha\colon G \twoheadrightarrow H$ the projection onto the maximal torsion-free abelian quotient. Write $\operatorname{rank}(H) = n$, and identify $\mathbb{F}[H] \cong \Lambda = \mathbb{F}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Let $\phi\colon G \to \mathrm{GL}_k(\mathbb{F})$ be a linear representation, and $\Delta^\phi = \Delta_1^\phi(G)$ the corresponding twisted Alexander polynomial. Assume that $\Delta^\phi \neq 0$, and write $\Delta^\phi = \sum c_i g_i$, where $0 \neq c_i \in \mathbb{F}$ and $g_i \in H$.

Following McMullen [31], we use the twisted Alexander polynomial Δ^ϕ to define a norm on $H^1(M;\mathbb{R}) = \operatorname{Hom}(H_1(M), \mathbb{R})$. For $\xi \in H^1(M;\mathbb{R})$, define

$$\|\xi\|_A^\phi := \sup_{i,j} \xi(g_i - g_j), \tag{16}$$

the supremum over all $\{g_i, g_j\}$ for which $c_i c_j \neq 0$. This defines a seminorm on $H^1(G;\mathbb{R})$, the *twisted Alexander norm* of M and ϕ. The unit ball $\mathbb{B}_A^\phi$ in the twisted Alexander norm is the polytope dual to $\mathcal{N}(\Delta^\phi)$, the Newton polytope of the twisted Alexander polynomial Δ^ϕ.

One also has the *Thurston norm* on $H^1(M;\mathbb{R})$. If Σ is a compact, connected surface, let $\chi\_(\Sigma) = -\chi(\Sigma)$ if $\chi(\Sigma) \leq 0$, and set $\chi\_(\Sigma) = 0$ otherwise. If Σ is a surface with connected components Σ_i, set $\chi\_(\Sigma) = \sum \chi\_(\Sigma_i)$. For $\xi \in H^1(M)$, define

$$\|\xi\|_T := \inf \{\chi\_(\Sigma) \mid \Sigma \text{ dual to } \xi\}, \tag{17}$$

the infimum over all properly embedded oriented surfaces Σ. The Thurston norm extends continuously to $H^1(M;\mathbb{R})$. Let $\mathbb{B}_T$ denote the unit ball in the Thurston norm, a polytope in $H^1(M;\mathbb{R})$.

As shown by Friedl and Kim [18], extending a result of McMullen [31], the twisted Alexander norm provides a lower bound for the Thurston norm,

$$\frac{1}{k}\| \bullet \|_A^\phi \leq \| \bullet \|_T.$$

Consequently, the unit ball in the Thurston norm is contained in the unit ball in the twisted Alexander norm, $\mathbb{B}_T \subset \mathbb{B}_A^\phi$. For certain link complements, one can exhibit representations ϕ for which the twisted Alexander ball $\mathbb{B}_A^\phi$ differs from $\mathbb{B}_A$, the unit ball in the (classical) Alexander norm [18], thereby distinguishing the Alexander and Thurston norms. Such a distinction is not possible in the case where M is the boundary manifold of a line arrangement.

Theorem 6.1 *Let $\mathcal{A}$ be an essential line arrangement in $\mathbb{CP}^2$, with boundary manifold M. If ϕ_1 and ϕ_2 are complex representations of the group $G = \pi_1(M)$, then the twisted Alexander balls $\mathbb{B}_A^{\phi_1}$ and $\mathbb{B}_A^{\phi_2}$ are equivalent.*

Proof Let $\phi\colon G \to \mathrm{GL}_k(\mathbb{C})$ be a representation. We will show that the twisted Alexander ball $\mathbb{B}_A^\phi$ and the (classical) Alexander ball $\mathbb{B}_A$ are equivalent. Let $\Delta = \Delta(G)$ be the Alexander polynomial and $\Delta^\phi = \Delta_1^\phi(G)$ be the twisted Alexander polynomial associated to the representation ϕ. Since the Alexander balls $\mathbb{B}_A$ and $\mathbb{B}_A^\phi$ are the polytopes dual to the respective Newton polytopes of the Alexander polynomials, it suffices to show that $\mathcal{N}(\Delta)$ and $\mathcal{N}(\Delta^\phi)$ are equivalent.

By Theorem 5.2, the Alexander polynomials Δ and Δ^ϕ are given by

$$\Delta = \prod_{v \in \mathcal{V}(\Gamma_\mathcal{A})} (t_v - 1)^{m_v - 2} \quad \text{and} \quad \Delta^\phi = \prod_{v \in \mathcal{V}(\Gamma_\mathcal{A})} \big[p(A_v, t_v)\big]^{m_v - 2},$$

where t_v is the image of a generator z_v of the center of the vertex group G_v under abelianization, and $p(A_v, t_v)$ is the characteristic polynomial of the automorphism $A_v = \phi(z_v)$ in the variable t_v. Observe that only the variables $t_1, \dots, t_n$ appear in these Alexander polynomials. Consequently, the Newton polytopes lie in $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset H^1(M;\mathbb{R})$. Since the Alexander polynomials factor, their Newton polytopes are Minkowski sums, for instance,

$$\mathcal{N}(\Delta) = \sum_{v\in\mathcal{V}(\Gamma_{\mathcal{A}})} \mathcal{N}\big[(t_v - 1)^{m_v-2}\big],$$

and similarly for $\mathcal{N}(\Delta^\phi)$.

Write $d_v = m_v - 2$. If $d_v > 0$ and $t_v = t_1^{q_1} \cdots t_n^{q_n}$, the Newton polytope $\mathcal{N}\big[(t_v-1)^{d_v}\big]$ is the convex hull of $\mathbf{0} = (0, \dots, 0)$ and $(d_v q_1, \dots, d_v q_n)$ in $\mathbb{R}^n$, a line segment. Thus, the Newton polytope $\mathcal{N}(\Delta)$ is a Minkowski sum of line segments, i.e., a zonotope. As such, it is determined by the matrix

$$Z = \begin{pmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_j \end{pmatrix}, \tag{18}$$

where j is the number of vertices $v \in \mathcal{V}(\Gamma_{\mathcal{A}})$ for which $d_v > 0$, and $\mathbf{q}_i = \begin{pmatrix} d_v q_1 & \cdots & d_v q_n \end{pmatrix}^\top$ if $t_v = t_1^{q_1} \cdots t_n^{q_n}$.

Now consider the Newton polytope of the twisted Alexander polynomial Δ^ϕ,

$$\mathcal{N}(\Delta^\phi) = \sum_{v\in\mathcal{V}(\Gamma_{\mathcal{A}})} \mathcal{N}\big[p(A_v, t_v)^{d_v}\big].$$

Since the characteristic polynomial $p(A_v, t_v)$ is monic of degree k, the Newton polytope $\mathcal{N}\big[p(A_v,t_v)^{d_v}\big]$ is the convex hull of $\mathbf{0}$ and $k \cdot (d_v q_1, \dots, d_v q_n)$ if $t_v = t_1^{q_1} \cdots t_n^{q_n}$. Hence, the Newton polytope $\mathcal{N}(\Delta^\phi)$ is the zonotope determined by the matrix $k \cdot Z$, which is clearly equivalent to $\mathcal{N}(\Delta)$. $\square$

The Alexander and Thurston norm balls arise in the context of Bieri-Neumann-Strebel (BNS) invariants of the group $G = \pi_1(M)$. Let

$$\mathbb{S}(G) = \big(H^1(G;\mathbb{R}) \setminus \{\mathbf{0}\}\big)/\mathbb{R}^+,$$

where $\mathbb{R}^+$ acts by scalar multiplication, and view points $[\xi]$ as equivalence classes of homomorphisms $G \to \mathbb{R}$. For $[\xi] \in \mathbb{S}(G)$, define a submonoid G_ξ of G by $G_\xi = \{g \in G \mid \xi(g) \geq 0\}$. If K is a group upon which G acts, with the commutator subgroup G' acting by inner automorphisms, the BNS invariant of G and K is the set $\Sigma_{G,K}$ of all elements $[\xi] \in \mathbb{S}(G)$ for which K is finitely generated over a finitely generated submonoid of G_ξ. The set $\Sigma_{G,K}$ is an open subset of the sphere $\mathbb{S}(G)$.

Let $K = G'$, with G acting by conjugation. When $G = \pi_1(M)$, where M is a compact, irreducible, orientable 3-manifold, Bieri, Neumann, and Strebel [3] show that the BNS invariant $\Sigma_{G,G'}$ is equal to the projection to $\mathbb{S}(G)$ of the interiors of the fibered faces of the Thurston norm ball $\mathbb{B}_T$.

Assume that $H_1(M)$ is torsion-free, and consider the maximal abelian cover M' of M, with fundamental group $\pi_1(M') = G'$. The first homology of M', $B = H_1(M') = G'/G''$, admits the structure of a module over $\mathbb{Z}[H]$, where $H = G/G'$, and is known as the Alexander invariant of M. Note that the Alexander polynomial $\Delta(G) = \Delta(M)$ is the order of the Alexander invariant. As shown by Dunfield [12], the BNS invariant $\Sigma_{G,B}$ is closely related to the Alexander polynomial.

Theorem 6.2 *Let $\mathcal{A}$ be an essential line arrangement in $\mathbb{CP}^2$, with boundary manifold M. Let G be the fundamental group of M, $B = G'/G''$ the Alexander invariant, and $\Delta = \operatorname{ord}(B)$ the Alexander polynomial. Then the BNS invariant $\Sigma_{G,B}$ is equal to the projection to $\mathbb{S}(G)$ of the interiors of the top-dimensional faces of the Alexander ball $\mathbb{B}_A$.*

Proof Write $\Delta = \sum c_i g_i$, where $c_i \neq 0$ and $g_i \in H = G/G'$. The Newton polytope $\mathcal{N}(\Delta)$ is the convex hull of the g_i in $H_1(M;\mathbb{R})$. Call a vertex g_i of $\mathcal{N}(\Delta)$ a "± 1 vertex" if the corresponding coefficient c_i is equal to ± 1. For an arbitrary compact, orientable 3-manifold M whose boundary, if any, is a union of tori, Dunfield [12] proves that the BNS invariant $\Sigma_{G,B}$ is given by the projection to $\mathbb{S}(G)$ of the interiors of the top-dimensional faces of $\mathbb{B}_A$ which correspond to ± 1 vertices of $\mathcal{N}(\Delta)$.

If M is the boundary manifold of a line arrangement $\mathcal{A} \subset \mathbb{CP}^2$, then, as shown in the proof of Theorem 6.1, the Newton polytope $\mathcal{N}(\Delta)$ of the Alexander polynomial is a zonotope. Since the factors $(t_v - 1)^{m_v - 2}$ of the Alexander polynomial Δ have leading coefficients and constant terms equal to ± 1, *every* vertex of the associated zonotope $\mathcal{N}(\Delta)$ is a ± 1 vertex. The result follows. $\square$

Let Δ be the Alexander polynomial of the boundary manifold of a line arrangement $\mathcal{A} \subset \mathbb{CP}^2$. Recall that the Newton polytope $\mathcal{N}(\Delta)$ is determined by the $n \times j$ integer matrix Z given in (18), where $|\mathcal{A}| = n + 1$ and j is the number of distinct factors in Δ. The matrix Z also determines a "secondary" arrangement $\mathcal{S} = \{H_i\}_{i=1}^{j}$ of j hyperplanes in $\mathbb{R}^n$, where H_i is the orthogonal complement of the i-th column of Z. The complement $\mathbb{R}^n \setminus \bigcup_{i=1}^{j} H_i$ of the real arrangement $\mathcal{S}$ is a disjoint union of connected open sets known as chambers. Let $\operatorname{ch}(\mathcal{S})$ be the set of chambers. The number of chambers may be calculated by a well known result of Zaslavsky [43]. If $P(\mathcal{S},t)$ is the Poincaré polynomial of (the lattice of) $\mathcal{S}$, then

$$|\mathrm{ch}(\mathcal{S})| = P(\mathcal{S},1).$$

The number of chambers of the arrangement $\mathcal{S}$ determined by the matrix Z is also known to be equal to the number of vertices of the zonotope $\mathcal{N}(\Delta)$ determined by Z, see [4]. Hence, we have the following corollary to Theorem 6.2.

Corollary 6.3 *The BNS invariant $\Sigma_{G,B}$ has $P(\mathcal{S},1)$ connected components.*

Example 6.4 Recall the Falk arrangements $\mathcal{F}_1$ and $\mathcal{F}_2$ from Example 5.3. Let G_i be the fundamental group of the boundary manifold of $\mathcal{F}_i$, B_i the corresponding Alexander invariant, etc. The Alexander polynomials $\Delta_i = \Delta(G_i)$ are recorded in (15). The zonotopes $\mathcal{N}(\Delta_1)$ and $\mathcal{N}(\Delta_2)$ are determined by the matrices

$$Z_1 = \begin{pmatrix} 2&0&0&0&0&2&0\\ 0&2&0&0&0&2&0\\ 0&0&2&0&0&2&2\\ 0&0&0&2&0&2&2\\ 0&0&0&0&2&2&2 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 2&0&0&0&0&1&0&1\\ 0&2&0&0&0&1&0&1\\ 0&0&2&0&0&1&1&0\\ 0&0&0&2&0&1&1&0\\ 0&0&0&0&3&1&1&1 \end{pmatrix}.$$

The Poincaré polynomials of the associated secondary arrangements $\mathcal{S}_1$ and $\mathcal{S}_2$ are

$$P(\mathcal{S}_1,t) = 1 + 7t + 21t^2 + 33t^3 + 27t^4 + 9t^5$$

and

$$P(\mathcal{S}_2,t) = 1 + 8t + 28t^2 + 51t^3 + 47t^4 + 17t^5.$$

Consequently, the BNS invariant Σ_{G_1,B_1} has $P(\mathcal{S}_1,1) = 98$ connected components, while Σ_{G_2,B_2} has $P(\mathcal{S}_2,1) = 152$ connected components.

7 Cohomology ring and holonomy Lie algebra

As shown in [7], the cohomology ring of the boundary manifold M of a hyperplane arrangement has a very special structure: it is the "double" of the cohomology ring of the complement. For a line arrangement, this structure leads to purely combinatorial descriptions of the skew 3-form encapsulating $H^*(M;\mathbb{Z})$, and of the holonomy Lie algebra of M.

7.1 The doubling construction

Let R be a coefficient ring; we will assume either $R = \mathbb{Z}$ or $R = \mathbb{F}$, a field of characteristic 0. Let $A = \bigoplus_{k=0}^{m} A^k$ be a graded, finite-dimensional algebra over R. Assume that A is graded-commutative, of finite type (i.e., each graded piece A^k is a finitely generated R-module), and connected (i.e., $A^0 = R$). Let $b_k = b_k(A)$ denote the rank of A^k.

Let $\bar{A} = \operatorname{Hom}_R(A, R)$ be the dual of the R-module A, with graded pieces $\bar{A}^k = \operatorname{Hom}_R(A^k, R)$. Then $\bar{A}$ is an A-bimodule, with left and right multiplication given by $(a \cdot f)(b) = f(ba)$ and $(f \cdot a)(b) = f(ab)$, respectively. Note that, if $a \in A^k$ and $f \in \bar{A}^j$, then $af, fa \in \bar{A}^{j-k}$.

Following [7], we define the *(graded) double* of A to be the graded R-algebra $\widehat{A}$ with underlying R-module structure the direct sum $A \oplus \bar{A}$, multiplication

$$(a, f) \cdot (b, g) = (ab, ag + fb), \tag{19}$$

for $a, b \in A$ and $f, g \in \bar{A}$, and grading

$$\widehat{A}^k = A^k \oplus \bar{A}^{2m-1-k}. \tag{20}$$

7.2 Poincaré duality

Let $A = \bigoplus_{k=0}^{m} A^k$ be a graded algebra as above. We say A is a *Poincaré duality* algebra (of formal dimension m) if the R-module A^m is free of rank 1 and, for each k, the pairing $A^k \otimes A^{m-k} \to A^m$ given by multiplication is non-singular. In particular, each graded piece A^k must be a free R-module.

Given a PD_m algebra A, fix a generator ω for A^m. We then have an alternating m-form, $\eta_A \colon A^1 \wedge \cdots \wedge A^1 \to R$, defined by

$$a_1 \cdots a_m = \eta_A(a_1, \dots, a_m) \cdot \omega. \tag{21}$$

If A is 3-dimensional, the full multiplicative structure of A can be recovered from the form η_A (and the generator $\omega \in A^3$).

The classical example of a Poincaré duality algebra is the rational cohomology ring, $H^*(M; \mathbb{Q})$, of an m-dimensional closed, orientable manifold M. As shown by Sullivan [37], any rational, alternating 3-form η can be realized as $\eta = \eta_{H^*(M;\mathbb{Q})}$, for some 3-manifold M.

Lemma 7.3 *Let $A = \bigoplus_{k=0}^{m} A^k$ be a graded, graded commutative, connected, finite-type algebra over $R = \mathbb{Z}$ or $\mathbb{F}$. Assume A is a free R-module, and $m > 1$. If $\widehat{A}$ is the graded double of A, then:*

(1) $\widehat{A}$ *is a Poincaré duality algebra over* R*, of formal dimension* $2m-1$*.*

(2) *If* $m>2$*, then* $\eta_{\widehat{A}}=0$*.*

(3) *If* $m=2$*, then for every* $a,b,c\in A^1$ *and* $f,g,h\in \bar{A}^2$*,*

$$\eta_{\widehat{A}}((a,f),(b,g),(c,h)) = f(bc)+g(ca)+h(ab).$$

Proof (1) The R-module $\widehat{A}^{2m-1}=\bar{A}^0$ is isomorphic to R via the map $f\mapsto f(1)$. Take $\omega=\bar{1}$ as generator of $\widehat{A}^{2m-1}$. The pairing $\widehat{A}^k\otimes\widehat{A}^{2m-1-k}\to\widehat{A}^{2m-1}$ is non-singular: its adjoint,

$$\widehat{A}^k\to \operatorname{Hom}_R(\widehat{A}^{2m-1-k},\widehat{A}^{2m-1}),\quad (a,f)\mapsto((b,g)\mapsto ag+fb),$$

is readily seen to be an isomorphism.

(2) If $m>2$, then $\widehat{A}^1=A^1$, and $\eta_{\widehat{A}}$ vanishes, since $A^{2m-1}=0$.

(3) If $m=2$, then $\widehat{A}^1=A^1\oplus\bar{A}^2$, and the expression for $\eta_{\widehat{A}}$ follows immediately from (19). □

7.4 The double of a 2-dimensional algebra

In view of the above Lemma, the most interesting case is when $m=2$, so let us analyze it in a bit more detail. Write $A=A^0\oplus A^1\oplus A^2$, and fix ordered bases, $\{\alpha_1,\dots,\alpha_{b_1}\}$ for A^1 and $\{\beta_1,\dots,\beta_{b_2}\}$ for A^2. The multiplication map, $\mu\colon A^1\otimes A^1\to A^2$, is then given by

$$\mu(\alpha_i,\alpha_j)=\sum_{k=1}^{b_2}\mu_{i,j,k}\,\beta_k, \tag{22}$$

for some integer coefficients $\mu_{i,j,k}$ satisfying $\mu_{j,i,k}=-\mu_{i,j,k}$.

Now consider the double

$$\widehat{A}=\widehat{A}^0\oplus\widehat{A}^1\oplus\widehat{A}^2\oplus\widehat{A}^3=A^0\oplus(A^1\oplus\bar{A}^2)\oplus(A^2\oplus\bar{A}^1)\oplus\bar{A}^0.$$

Pick dual bases $\{\bar{\alpha}_j\}_{1\le j\le b_1}$ for $\bar{A}^1$ and $\{\bar{\beta}_k\}_{1\le k\le b_2}$ for $\bar{A}^2$. The multiplication map $\hat{\mu}\colon\widehat{A}^1\otimes\widehat{A}^1\to\widehat{A}^2$ restricts to μ on $A^1\otimes A^1$, vanishes on $\bar{A}^2\otimes\bar{A}^2$, while on $A^1\otimes\bar{A}^2$, it is given by

$$\hat{\mu}(\alpha_j,\bar{\beta}_k)=\sum_{i=1}^{b_1}\mu_{i,j,k}\,\bar{\alpha}_i. \tag{23}$$

As a consequence, we see that the multiplication maps μ and $\hat{\mu}$ determine one another.

In the chosen basis for $\widehat{A}^1 = A^1 \oplus \bar{A}^2$, the form $\eta_{\widehat{A}} \in \bigwedge^3 \widehat{A}^1$ can be expressed as

$$\eta_{\widehat{A}} = \sum_{1\leq i<j\leq b_1} \sum_{k=1}^{b_2} \mu_{i,j,k}\, \alpha_i \wedge \alpha_j \wedge \bar{\beta}_k. \tag{24}$$

This shows again that the multiplication map $\hat{\mu}$ determines, and is determined by the 3-form $\eta_{\widehat{A}}$.

7.5 The cohomology ring of the boundary

Now let $\mathcal{A} = \{\ell_i\}_{i=0}^n$ be a line arrangement in $\mathbb{CP}^2$, with complement X, and let $A = H^*(X;\mathbb{Z})$ be the integral Orlik-Solomon algebra of $\mathcal{A}$. As is well known, $A = \bigoplus_{k=0}^2 A^k$ is torsion-free, and generated in degree 1 by classes $e_1, \dots, e_n$ dual to the meridians $x_1, \dots, x_n$ of the decone $\mathsf{d}\mathcal{A}$. Choosing a suitable basis $\{f_{i,k} \mid (i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})\}$ for A^2, the multiplication map $\mu\colon A^1 \wedge A^1 \to A^2$ is given on basis elements e_i, e_j with $i<j$ by:

$$\mu(e_i, e_j) = \begin{cases} f_{i,j} & \text{if } (i,j) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A}), \\ f_{k,j} - f_{k,i} & \text{if } \exists k \text{ such that } (k,i),(k,j) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A}), \\ 0 & \text{otherwise.} \end{cases} \tag{25}$$

The surjectivity of μ is manifest from this formula.

For $(i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})$, recall that $I(i,k) = \{j \mid \ell_j \supset \ell_i \cap \ell_k,\ 1 \leq j \leq n\}$. If $J \subset [n]$, write $e_J = \sum_{j\in J} e_j$. Using results from [7] and the above discussion, we obtain the following.

Theorem 7.6 *Let $\mathcal{A} = \{\ell_i\}_{i=0}^n$ be a line arrangement in $\mathbb{CP}^2$, with complement X and boundary manifold M. Then:*

1. *$H^*(M;\mathbb{Z})$ is the double of $H^*(X;\mathbb{Z})$.*
2. *$H^*(M;\mathbb{Z})$ is an integral Poincaré duality algebra of formal dimension 3.*
3. *$H^*(M;\mathbb{Z})$ is generated in degree 1 if and only if $\mathcal{A}$ is not a pencil.*
4. *$H^*(M;\mathbb{Z})$ determines (and is determined by) the 3-form $\eta_M := \eta_{H^*(M;\mathbb{Z})}$, given by*

$$\eta_M = \sum_{(i,k)\in\mathbf{nbc}_2(\mathsf{d}\mathcal{A})} e_{I(i,k)} \wedge e_k \wedge \bar{f}_{i,k}.$$

Proof (1) If $A = H^*(X;\mathbb{Z})$, then $\widehat{A} = H^*(M;\mathbb{Z})$, see [7, Theorem 4.2].

(2) This follows from Lemma 7.3, since A is torsion-free (alternatively, use Poincaré duality for the closed, orientable 3-manifold M).

(3) It is enough to show that the cup-product map $\hat{\mu}\colon \hat{A}^1 \otimes \hat{A}^1 \to \hat{A}^2$ is surjective if and only if $\mathcal{A}$ is not a pencil.

If $\mathcal{A}$ is a pencil, then $M = \sharp^n S^1 \times S^2$, and so $\hat{\mu} = 0$.

If $\mathcal{A}$ is not a pencil, each line ℓ_i with $1 \le i \le n$ must meet another line, say ℓ_j, also with $1 \le j \le n$. Then either $(i,j) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})$, in which case $\hat{\mu}(e_j, \bar{f}_{i,j}) = \bar{e}_i$, or there is an index $k \le j$ such that $(k,i) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})$, in which case $\hat{\mu}(e_k, \bar{f}_{k,i}) = -\bar{e}_i$. This shows $\bar{A}^1 \subset \operatorname{Im}(\hat{\mu})$. But we know $A^2 = \operatorname{Im}(\mu)$, and so $\hat{\mu}$ is surjective.

(4) This follows from formulas (24) and (25). ☐

Example 7.7 We illustrate part (4) of the above Theorem with some sample computations:

$$\eta_M = \begin{cases} 0 & \text{if } \mathcal{A} \text{ is a pencil,} \\ \left(\sum_{i=1}^n e_i\right) \cdot \sum_{j=2}^n e_j \bar{f}_{1,j} & \text{if } \mathcal{A} \text{ is a near-pencil,} \\ \sum_{1 \le i < j \le n} e_i e_j \bar{f}_{i,j} & \text{if } \mathcal{A} \text{ is a general position arrangement.} \end{cases}$$

Remark Let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2$ that is not a pencil. Then the fundamental group G of the boundary manifold M is a commutator-relators group, M is a $K(G,1)$-space, and $H_*(G;\mathbb{Z}) = H_*(M;\mathbb{Z})$ is torsion-free. In this situation, the cup-product structure on $H^*(G;\mathbb{Z}) = H^*(M;\mathbb{Z})$, and hence the 3-form η_M, may be computed directly from the commutator-relators presentation given in Proposition 3.7, see, for instance, [17, Thm. 2.3] and [30, Prop. 2.8]. Note, however, that the bases for the cohomology groups of G arising in this approach need not, in general, coincide with those obtained from the realization of $H^*(G;\mathbb{Z})$ as a double.

7.8 Holonomy Lie algebras

We now turn to a different object associated to a graded algebra A. As before, we will assume that A is graded-commutative, connected, and of finite-type, and that the ground ring R is either $\mathbb{Z}$ or $\mathbb{F}$, a field of characteristic 0. Denote by A_k the R-dual module $\bar{A}^k = \operatorname{Hom}_R(A^k, R)$; note that A_k is a free R-module of rank b_k.

The *holonomy Lie algebra* of A, denoted $\mathfrak{h}(A)$, is the quotient of the free Lie algebra $\mathrm{Lie}(A_1)$ by the ideal generated by the image of the comultiplication map, $\bar{\mu}\colon A_2 \to A_1 \wedge A_1 = \mathrm{Lie}_2(A_1)$. Picking generators $x_i = \bar{\alpha}_i$ for $A_1 = \bar{A}^1$, we obtain a finite presentation,

$$\mathfrak{h}(A) = \mathrm{Lie}(x_1, \dots, x_{b_1}) \Big/ \Big(\sum_{1 \le i < j \le b_1} \mu_{i,j,k}[x_i, x_j], \ \text{for } 1 \le k \le b_2 \Big). \tag{26}$$

Note that $\mathfrak{h}(A)$ inherits a natural grading from the free Lie algebra: all generators x_i are in degree 1, while all relations are homogeneous of degree 2.

Now let $\widehat{A}$ be the graded double of A. Using the description of the multiplication map $\hat{\mu}$ from §7.4, we obtain the following presentation for $\mathfrak{h}(\widehat{A})$, solely in terms of the multiplication map $\mu\colon A^1 \otimes A^1 \to A^2$, given by (22).

Lemma 7.9 *The holonomy Lie algebra of $\widehat{A}$ is the quotient of the free Lie algebra on degree 1 generators $\{x_i \mid 1 \le i \le b_1\}$ and $\{y_k \mid 1 \le k \le b_2\}$, modulo the Lie ideal generated by*

$$\begin{aligned} &\sum_{1 \le i < j \le b_1} \mu_{i,j,k}[x_i, x_j], && 1 \le k \le b_2, \\ &\sum_{1 \le j \le b_1} \sum_{1 \le k \le b_2} \mu_{i,j,k}[x_j, y_k], && 1 \le i \le b_1. \end{aligned}$$

Note that there is a canonical projection $\mathfrak{h}(\widehat{A}) \to \mathfrak{h}(A)$, sending $x_i \mapsto x_i$ and $y_k \mapsto 0$. The kernel of this projection contains $\mathrm{Lie}(y_1, \dots, y_{b_2})$, but in general the inclusion is strict.

7.10 The holonomy Lie algebra of the boundary

Let $\mathcal{A} = \{\ell_i\}_{i=0}^n$ be a line arrangement in $\mathbb{CP}^2$, with complement X. As shown by Kohno [26], the holonomy Lie algebra of $A = H^*(X;\mathbb{Z})$ has presentation

$$\mathfrak{h}(A) = \mathrm{Lie}(x_1, \dots, x_n) / \Big(\sum_{j \in I(i,k)} [x_j, x_k], \ \text{for } (i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A}) \Big). \tag{27}$$

From the preceding discussion, we find an explicit presentation for the holonomy Lie algebra of the boundary manifold of a line arrangement.

Proposition 7.11 *Let $\mathcal{A} = \{\ell_i\}_{i=0}^n$ be a line arrangement in $\mathbb{CP}^2$, with boundary manifold M. Then the holonomy Lie algebra of $\widehat{A} = H^*(M;\mathbb{Z})$ is the* *quotient of the free Lie algebra on degree 1 generators* $\{x_i \mid 1 \le i \le n\}$ *and* $\{y_{(i,k)} \mid (i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})\}$*, modulo the Lie ideal generated by*

$$\sum_{j\in I(i,k)} [x_j, x_k],$$

for $(i,k) \in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})$*, and*

$$\sum_{k\colon (i,k)\in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})} \sum_{j\in I(i,k)} [x_j, y_{(i,k)}] \; - \sum_{k\colon (k,i)\in \mathbf{nbc}_2(\mathsf{d}\mathcal{A})} \sum_{j\in I(k,i)} [x_j, y_{(k,i)}],$$

for $1 \le i \le n$*.*

Example 7.12 If $\mathcal{A}$ an arrangement of $n+1$ lines in general position, then the holonomy Lie algebra $\mathfrak{h}(\widehat{A})$ is the quotient of the free Lie algebra on generators x_i $(1 \le i \le n)$ and $y_{(i,j)}$ $(1 \le i < j \le n)$, modulo the Lie ideal generated by

$$\begin{array}{ll} [x_i, x_j], & 1 \le i < j \le n, \\ \sum_{j<i} [x_j, y_{(j,i)}] - \sum_{j>i} [x_j, y_{(i,j)}], & 1 \le i \le n. \end{array}$$

8 Cohomology jumping loci

In this section, we discuss the characteristic varieties and the resonance varieties of the boundary manifold of a line arrangement.

8.1 Characteristic varieties

Let X be a space having the homotopy type of a connected, finite-type CW-complex. For simplicity, we will assume that the fundamental group $G = \pi_1(X)$ has torsion-free abelianization $H_1(G) = \mathbb{Z}^n$. Consider the character torus $\operatorname{Hom}(G, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$. The *characteristic varieties* of X are the jumping loci for the cohomology of X, with coefficients in rank 1 local systems over $\mathbb{C}$:

$$V^k_d(X) = \{\phi \in \operatorname{Hom}(G, \mathbb{C}^*) \mid \dim H^k(X; \mathbb{C}_\phi) \ge d\}, \tag{28}$$

where $\mathbb{C}_\phi$ denotes the abelian group $\mathbb{C}$, with $\pi_1(X)$-module structure given by the representation $\phi\colon \pi_1(X) \to \mathbb{C}^*$. These loci are subvarieties of the algebraic torus $(\mathbb{C}^*)^n$; they depend only on the homotopy type of X, up to a monomial isomorphism of the character torus.

For a finitely presented group G (with torsion-free abelianization), set $V_d^k(G) := V_d^k(K(G,1))$. We will be only interested here in the degree 1 characteristic varieties. If $G = \pi_1(X)$ with X a space as above, then clearly $V_d^1(G) = V_d^1(X)$.

The varieties $V_d^1(G)$ can be computed algorithmically from a finite presentation of the group. If G has generators x_i and relations r_j, let $J_G = \big(\partial r_i/\partial x_j\big)$ be the corresponding Jacobian matrix of Fox derivatives. The abelianization J_G^{ab} is the *Alexander matrix* of G, with entries in $\Lambda = \mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, the coordinate ring of $(\mathbb{C}^*)^n$. Then:

$$V_d^1(G) \setminus \{1\} = V(E_d(J_G^{\mathrm{ab}})) \setminus \{1\}. \tag{29}$$

In other words, $V_d^1(G)$ consists of all those characters $\phi \in \mathrm{Hom}(G, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$ for which the evaluation of J_G^{ab} at ϕ has rank less than $n-d$ (plus, possibly, the identity 1).

8.2 Characteristic varieties of line arrangements

Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be a line arrangement in $\mathbb{CP}^2$. The characteristic varieties of the complement X are fairly well understood. It follows from foundational work of Arapura [1] that $V_d^1(X)$ is a union of subtori of the character torus $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*) = (\mathbb{C}^*)^n$, possibly translated by roots of unity. Moreover, components passing through 1 admit a completely combinatorial description. See [6, 28].

Turning to the characteristic varieties of the boundary manifold M, we have the following complete description of $V_1^1(M)$.

Theorem 8.3 *Let $\mathcal{A}$ be an essential line arrangement in $\mathbb{CP}^2$, and let G be the fundamental group of the boundary manifold M. Then*

$$V_1^1(G) = \bigcup_{v \in \mathcal{V}(\Gamma_{\mathcal{A}}), m_v \geq 3} \{t_v - 1 = 0\}.$$

Proof By Proposition 3.7, the group G admits a commutator-relators presentation, with equal number of generators and relations. So the Alexander matrix J_G^{ab} is a square matrix, which augments to zero. It follows that the characteristic variety $V_1^1(G)$ is the variety defined by the vanishing of the codimension 1 minors of J_G^{ab}. The ideal $I(G) = E_1(J_G^{\mathrm{ab}})$ of codimension 1 minors, the Alexander ideal, is given by $I(G) = \mathfrak{m}^2 \cdot (\Delta(G))$, where $\mathfrak{m}$ is the maximal ideal of $\mathbb{Z}H_1(G)$, see [31]. Consequently,

$$V_1^1(G) = \{\Delta(G) = 0\}. \tag{30}$$

On the other hand, we know from Theorem 5.2 that the Alexander polynomial of G is given by $\Delta(G) = \prod_{v\in\mathcal{V}(\Gamma_{\mathcal{A}})}(t_v - 1)^{m_v-2}$. The conclusion follows. $\square$

Note that $V_1^1(G)$ is the union of an arrangement of codimension 1 subtori in $\operatorname{Hom}(G,\mathbb{C}^*) = (\mathbb{C}^*)^n$, indexed by the vertices of the graph $\Gamma_{\mathcal{A}}$. We do not have an explicit description of the varieties $V_d^1(G)$, for $d > 1$.

8.4 Resonance varieties

Let A be a graded, graded-commutative, connected, finite-type algebra over $\mathbb{C}$. Since $a \cdot a = 0$ for each $a \in A^1$, multiplication by a defines a cochain complex

$$(A, a)\colon\quad 0 \longrightarrow A^0 \xrightarrow{\ a\ } A^1 \xrightarrow{\ a\ } A^2 \xrightarrow{\ a\ } \cdots. \tag{31}$$

The *resonance varieties* of A are the jumping loci for the cohomology of these complexes:

$$\mathcal{R}_d^k(A) = \{a \in A^1 \mid \dim H^k(A, a) \geq d\}, \tag{32}$$

for $k \geq 1$ and $1 \leq d \leq b_k(A)$. The sets $\mathcal{R}_d^k(A)$ are homogeneous algebraic subvarieties of the complex vector space $A^1 = \mathbb{C}^{b_1}$.

We will only be interested here in the degree 1 resonance varieties, $\mathcal{R}_d^1(A)$. Let $S = \operatorname{Sym}(A_1)$ be the symmetric algebra on the dual of A^1. If $\{x_1, \dots, x_{b_1}\}$ is the basis for A_1 dual to the basis $\{\alpha_1, \dots, \alpha_{b_1}\}$ for A^1, then S becomes identified with the polynomial ring $\mathbb{C}[x_1, \dots, x_{b_1}]$. Also, let $\mu\colon A^1 \otimes A^1 \to A^2$ is the multiplication map, given by (22). Then, as shown in [30] (generalizing a result from [6]):

$$\mathcal{R}_d^1(A) = V(E_d(\Theta)), \tag{33}$$

where $\Theta = \Theta_A$ is the $b_1 \times b_2$ matrix of linear forms over S, with entries

$$\Theta_{j,k} = \sum_{i=1}^{b_1} \mu_{i,j,k} x_i. \tag{34}$$

If X is a space having the homotopy type of a connected, finite-type CW-complex, define the resonance varieties of X to be those of $A = H^*(X;\mathbb{C})$. Similarly, if G is a finitely presented group, define the resonance varieties of G to be those of a $K(G,1)$ space. If $G = \pi_1(X)$, then $R_d^1(G) = R_d^1(X)$. Furthermore, if G is a commutator-relators group, then the matrix Θ above is (equivalent to) the "linearization" of the (transposed) Alexander matrix J_G^{ab}, see [30]. This suggests a relationship between $V_d^1(G)$ and $R_d^1(G)$. For more on this, see §9.4.

8.5 Resonance of line arrangements

Let $\mathcal{A} = \{\ell_i\}_{i=0}^n$ be an arrangement of lines in $\mathbb{CP}^2$, with complement X. The resonance varieties of the Orlik-Solomon algebra $A = H^*(X;\mathbb{C})$, first studied by Falk [16], are by now well understood. It follows from [6, 28] that $R^1_d(A)$ is the union of linear subspaces of $A^1 = \mathbb{C}^n$; these subspaces (completely determined by the underlying combinatorics) have dimension at least 2; and intersect only at 0.

Now let M be the boundary manifold, and $\widehat{A} = H^*(M;\mathbb{C})$ its cohomology ring. Recall that $\widehat{A}^1 = A^1 \oplus \bar{A}^2$, with basis $\{\alpha_i, \bar{\beta}_k\}$, where $1 \le i \le b_1 = n$ and $1 \le k \le b_2 = |\mathbf{nbc}_2(\mathsf{d}\mathcal{A})|$. Identify the ring $\widehat{S} = \operatorname{Sym}(\widehat{A}^1)$ with the polynomial ring in variables $\{x_i, y_k\}$. It follows from (23) that the matrix $\widehat{\Theta} = \Theta_{\widehat{A}}$ has the form

$$\widehat{\Theta} = \begin{pmatrix} \Phi & \Theta \\ -\Theta^\top & 0 \end{pmatrix}, \tag{35}$$

where Φ is the $b_1 \times b_1$ skew-symmetric matrix with entries $\Phi_{i,j} = \sum_{k=1}^{b_2} \mu_{i,j,k} y_k$. Using this fact, one can derive the following information about the resonance varieties of M. Write $\beta = 1 - b_1(A) + b_2(A)$ and $\mathcal{R}_d(\Phi) = V(E_d(\Phi))$.

Proposition 8.6 ([7]) *The resonance varieties of the doubled algebra $\widehat{A} = H^*(M;\mathbb{C})$ satisfy:*

(1) $\mathcal{R}^1_d(\widehat{A}) = \widehat{A}^1$ *for* $d \le \beta$.

(2) $\mathcal{R}^1_d(A) \times \bar{A}^2 \subseteq \mathcal{R}^1_{d+\beta}(\widehat{A})$.

(3) $\mathcal{R}_d(\Phi) \times \{0\} \subseteq \mathcal{R}^1_{d+b_2}(\widehat{A})$.

This allows us to give a complete characterization of the resonance variety $\mathcal{R}^1_1(G)$, for G a boundary manifold group.

Corollary 8.7 *Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be a line arrangement in $\mathbb{CP}^2$, $n \ge 2$, and $G = \pi_1(M)$. Then:*

$$\mathcal{R}^1_1(G) = \begin{cases} \mathbb{C}^n & \text{if } \mathcal{A} \text{ is a pencil,} \\ \mathbb{C}^{2(n-1)} & \text{if } \mathcal{A} \text{ is a near-pencil,} \\ \mathbb{C}^{b_1+b_2} & \text{otherwise.} \end{cases}$$

Proof If $\mathcal{A}$ is a pencil, then $G = F_n$, and so $\mathcal{R}^1_1(G) = \mathbb{C}^n$.

If $\mathcal{A}$ is a near-pencil, then $G = \mathbb{Z} \times \pi_1(\Sigma_{n-1})$ and a calculation yields $\mathcal{R}^1_1(G) = \mathbb{C}^{2(n-1)}$.

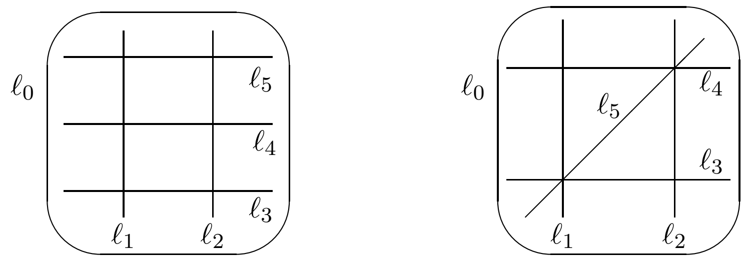

Figure 5: The product arrangement $\mathcal{A}$ and the braid arrangement $\mathcal{A}'$

If $\mathcal{A}$ is neither a pencil, nor a near-pencil, then $n \geq 3$, and a straightforward inductive argument shows that $\beta \geq 1$. Consequently, $\mathcal{R}^1_1(G) = H^1(G;\mathbb{C})$ by Proposition 8.6. ◻

8.8 A pair of arrangements

The arrangements $\mathcal{A}$ and $\mathcal{A}'$ depicted in Figure 5 have defining polynomials

$$\begin{aligned} Q(\mathcal{A}) &= x_0(x_1+x_0)(x_1-x_0)(x_2+x_0)x_2(x_2-x_0), \text{ and} \\ Q(\mathcal{A}') &= x_0(x_1+x_0)(x_1-x_0)(x_2+x_0)(x_2-x_0)(x_2-x_1). \end{aligned}$$

The respective boundary manifolds, M and M', share the same Poincaré polynomial, namely $P(t) = (1+t)(1+10t+t^2)$. Yet their cohomology rings, $\widehat{A}$ and $\widehat{A}'$, are not isomorphic—they are distinguished by their resonance varieties. Indeed, a computation with Macaulay 2 [19] reveals that

$$\mathcal{R}^1_7(\widehat{A}) = V(x_1, x_2, x_3, x_4, x_5,\, y_3y_5 - y_2y_6,\, y_3y_4 - y_1y_6,\, y_2y_4 - y_1y_5),$$

which is a variety of dimension 4, whereas

$$\begin{aligned} \mathcal{R}^1_7(\widehat{A}') = V(&x_1, x_2, x_3, x_4, x_5,\, y_2y_4 - y_1y_6,\, y_2y_5 - y_3y_6, \\ &y_3y_4 - y_4y_5 - y_3y_6 + y_4y_6,\, y_1y_5 - y_4y_5 - y_3y_6 + y_4y_6, \\ &y_1y_3 - y_2y_3 - y_4y_5 + y_1y_6 - y_3y_6 + y_4y_6), \end{aligned}$$

which is a variety of dimension 3.

9 Formality

In this section, we characterize those arrangements $\mathcal{A}$ for which the boundary manifold M is formal, in the sense of Sullivan [38]. It turns out that, with the exception of pencils and near-pencils, M is never formal.

9.1 Formal spaces and 1-formal groups

Let X be a space having the homotopy type of a connected, finite-type CW-complex. Roughly speaking, X is *formal*, if its rational homotopy type is completely determined by its rational cohomology ring. More precisely, X is formal if there is a zig-zag sequence of morphisms of commutative differential graded algebras connecting Sullivan's algebra of polynomial forms, $(A_{PL}(X,\mathbb{Q}),d)$, to $(H^*(X;\mathbb{Q}),0)$, and inducing isomorphisms in cohomology. Well known examples of formal spaces include spheres; simply-connected Eilenberg-MacLane spaces; compact, connected Lie groups and their classifying spaces; and compact Kähler manifolds. The formality property is preserved under wedges and products of spaces, and connected sums of manifolds.

A finitely presented group G is said to be 1*-formal*, in the sense of Quillen [36], if its Malcev Lie algebra (i.e., the Lie algebra of the prounipotent completion of G) is quadratic; see [35] for details. If X is a formal space, then $G=\pi_1(X)$ is a 1-formal group, as shown by Sullivan [38] and Morgan [33]. Complements of complex projective hypersurfaces are not necessarily formal, cf. [33]. Nevertheless, their fundamental groups are 1-formal, as shown by Kohno [26].

If X is the complement of a complex hyperplane arrangement, Brieskorn's calculation of the integral cohomology ring of X (see [34]) implies that X is (rationally) formal. However, the analogous property of $\mathbb{Z}_p$-formality does not necessarily hold, due to the presence of non-vanishing triple Massey products in $H^*(X;\mathbb{Z}_p)$, see Matei [29].

As mentioned above, our goal in this section is to decide, for a given line arrangement $\mathcal{A}$, whether the boundary manifold M is formal, and whether $G=\pi_1(M)$ is 1-formal. In our situation, Massey products in $H^*(G;\mathbb{Z})$ may be computed directly from the commutator-relators presentation given in Proposition 3.7, using the Fox calculus approach described in [17]. Yet determining whether such products vanish is quite difficult, as Massey products are only defined up to indeterminacy. So we turn to other, more manageable, obstructions to formality.

9.2 Associated graded Lie algebra

The lower central series of a group G is the sequence of normal subgroups $\{G_k\}_{k\geq 1}$, defined inductively by $G_1=G$, $G_2=G'$, and $G_{k+1}=[G_k,G]$. It is readily seen that the quotient groups, G_k/G_{k+1}, are abelian. Moreover, if G is finitely generated, so are all the LCS quotients. The *associated graded*

Lie algebra of G is the direct sum $\operatorname{gr}(G) = \bigoplus_{k\geq 1} G_k/G_{k+1}$, with Lie bracket induced by the group commutator, and grading given by bracket length.

If the group G is finitely presented, there is another graded Lie algebra attached to G, the (rational) holonomy Lie algebra, $\mathfrak{h}(G) := \mathfrak{h}(H^*(G;\mathbb{Q}))$. In fact, if X is any space having the homotopy type of a connected CW-complex with finite 2-skeleton, and if $G = \pi_1(X)$, then $\mathfrak{h}(G) = \mathfrak{h}(H^*(X;\mathbb{Q}))$, see [35]. Now suppose G is a 1-formal group. Then,

$$\operatorname{gr}(G)\otimes\mathbb{Q} \cong \mathfrak{h}(G), \tag{36}$$

as graded Lie algebras; see [36, 38]. In particular, the respective Hilbert series must be equal.

Returning to our situation, let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2$, with boundary manifold M. A finite presentation for the group $G = \pi_1(M)$ is given in Proposition 3.7. On the other hand, we know that $H^*(M;\mathbb{Q}) = \widehat{A}$, the double of the (rational) Orlik-Solomon algebra. Thus, $\mathfrak{h}(G) = \mathfrak{h}(\widehat{A})$, with presentation given in Proposition 7.11. Using these explicit presentations, one can compute, at least in principle, the Hilbert series of $\operatorname{gr}(G)\otimes\mathbb{Q}$ and $\mathfrak{h}(G)$.

Example 9.3 Let $\mathcal{A}$ be an arrangement of 4 lines in general position in $\mathbb{CP}^2$, and M its boundary manifold. A presentation for $G = \pi_1(M)$ is given in Example 3.11, while a presentation for $\mathfrak{h}(G)$ is given in Example 7.12. Direct computation shows that

$$\operatorname{Hilb}(\operatorname{gr}(G)\otimes\mathbb{Q}, t) = 6 + 9t + 36t^2 + 131t^3 + 528t^4 + \cdots,$$

whereas

$$\operatorname{Hilb}(\mathfrak{h}(G), t) = 6 + 9t + 36t^2 + 132t^3 + 534t^4 + \cdots.$$

Consequently, G is not 1-formal, and so M is not formal, either.

We can use the formality test (36) to show that several other boundary manifolds are not formal, but we do not know a general formula for the Hilbert series of the two graded Lie algebras attached to a boundary manifold group. Instead, we turn to another formality test.

9.4 The tangent cone formula

Let G be a finitely presented group, with $H_1(G)$ torsion-free. Consider the map $\exp\colon \operatorname{Hom}(G,\mathbb{C}) \to \operatorname{Hom}(G,\mathbb{C}^*)$, $\exp(f)(z) = e^{f(z)}$. Using this map, we may identify the tangent space at 1 to the torus $\mathrm{Hom}(G,\mathbb{C}^*)$ with the vector space $\mathrm{Hom}(G,\mathbb{C}) = H^1(G,\mathbb{C})$. Under this identification, the exponential map takes the resonance variety $R^1_d(G)$ to $V^1_d(G)$. Moreover, the tangent cone at 1 to $V^1_d(G)$ is contained in $R^1_d(G)$, see Libgober [27]. While this inclusion is in general strict, equality holds under a formality assumption.

Theorem 9.5 ([10]) *Suppose G is a 1-formal group. Then, for each $d \geq 1$, the exponential map induces a complex analytic isomorphism between the germ at 0 of $R^1_d(G)$ and the germ at 1 of $V^1_d(G)$. Consequently,*

$$\mathrm{TC}_1(V^1_d(G)) = R^1_d(G). \tag{37}$$

In particular, this "tangent cone formula" holds in the case when X is the complement of a complex hyperplane arrangement, and G is its fundamental group (see [6] for a direct approach in this situation).

9.6 Formality of boundary manifolds

We can now state the main result of this section, characterizing those line arrangements for which the boundary manifold is formal.

Theorem 9.7 *Let $\mathcal{A} = \{\ell_0, \dots, \ell_n\}$ be a line arrangement in $\mathbb{CP}^2$, with boundary manifold M. The following are equivalent:*

(1) *The boundary manifold M is formal.*

(2) *The group $G = \pi_1(M)$ is 1-formal.*

(3) *The tangent cone to $V^1_1(G)$ at the identity is equal to $\mathcal{R}^1_1(G)$.*

(4) *$\mathcal{A}$ is either a pencil or a near-pencil.*

Proof (1) $\Rightarrow$ (2) This follows from [36, 38].

(2) $\Rightarrow$ (3) This follows from [10].

(3) $\Rightarrow$ (4) Suppose $\mathcal{A}$ is neither a pencil nor a near-pencil. Then Corollary 8.7 implies that $\mathcal{R}^1_1(G) = H^1(G;\mathbb{C})$. On the other hand, Theorem 5.2 implies that $V^1_1(G)$ is a union of codimension 1 subtori in $\mathrm{Hom}(G,\mathbb{C}^*)$. Hence, the tangent cone $\mathrm{TC}_1(V^1_1(G))$ is the union of a hyperplane arrangement in $H^1(G;\mathbb{C})$; thus, it does not equal $\mathcal{R}^1_1(G)$.

(4) $\Rightarrow$ (1) If $\mathcal{A}$ is a pencil, then $M = \sharp^n S^1 \times S^2$. If $\mathcal{A}$ is a near-pencil, then $M = S^1 \times \Sigma_{n-1}$. In either case, M is built out of spheres by successive product and connected sum operations. Thus, M is formal. $\square$

Added in proof The structure of the Alexander polynomial of the boundary manifold M exhibited in Theorem 5.2 and Proposition 5.5 has recently been used in [11] to show that the fundamental group $G = \pi_1(M)$ is quasi-projective if and only if one of the equivalent conditions of Theorem 9.7 holds.

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