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Some generalizations of Suzuki and Edelstein type theorems

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Abstract
We prove some generalizations of Suzuki's fixed point theorem and Edelstein's theorem.

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Introduction and preliminaries

Let (X,d) be a complete metric space and T be a selfmap of X. Then T is called a *contraction* if there exists $r \in [0,1)$ such that

$$d(Tx,Ty) \le rd(x,y)$$

for all $x,y \in X$.

The following famous theorem is referred to as the Banach contraction principle.

Theorem 1 (Banach [1]) *Let (X,d) be a complete metric space, and let T be a contraction on X. Then T has a unique fixed point.*

This theorem is a very forceful and simple, and it has become a classical tool in nonlinear analysis. It has many generalizations, see [2–19].

In 2008, Suzuki [20] introduced a new type of mapping and presented a generalization of the Banach contraction principle in which the completeness can also be characterized by the existence of a fixed point of these mappings.

Theorem 2 [20] *Let (X,d) be a complete metric space, and let T be a mapping on X. Define a nonincreasing function θ from $[0,1)$ onto $(1/2,1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)/r^2 & \text{if } (\sqrt{5}-1)/2 \le r \le 1/\sqrt{2}, \\ 1/(1+r) & \text{if } 1/\sqrt{2} \le r < 1. \end{cases} \tag{1}$$

Assume that there exists $r \in [0,1)$ such that $\theta(r)d(x,Tx) \le d(x,y)$ implies $d(Tx,Ty) \le rd(x,y)$ for all $x,y \in X$. Then there exists a unique fixed point z of T. Moreover, $\lim_n T^n x = z$ for all $x \in X$.

Its further outcomes by Altun and Erduran [21], Karapinar [22, 23], Kikkawa and Suzuki [24, 25], Moț and Petrușel [26], Dhompongsa and Yingtaweesittikul [27], Popescu [28, 29], Singh and Mishra [30–32] are important contributions to metric fixed point theory.

Popescu [28] introduced a new type of contractive operator and proved the following theorem.

Theorem 3 [28] *Let (X,d) be a complete metric space and $T: X \to X$ be a (s,r)-contractive single-valued operator*:

$$x, y \in X \quad \text{with } d(y, Tx) \le sd(y,x) \quad \text{implies} \quad d(Tx, Ty) \le rM_T(x,y),$$

where $r \in [0,1)$, $s > r$ and

$$M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Then T has a fixed point. Moreover, if $s \ge 1$, then T has a unique fixed point.

As a direct consequence of Theorem 3, we obtain the following result.

Theorem 4 *Let (X,d) be a complete metric space, and let T be a mapping on X. Assume that there exist $r \in [0,1)$ and $s > r$ such that*

$$d(y, Tx) \le sd(y,x) \quad \text{implies} \quad d(Tx,Ty) \le rd(x,y) \tag{2}$$

for all $x, y \in X$. Then there exists a fixed point z of T. Further, if $s \ge 1$, then there exists a unique fixed point of T.

The following theorem is a well-known result in fixed point theory.

Theorem 5 (Edelstein [33]) *Let (X,d) be a compact metric space, and let T be a mapping on X. Assume $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$ with $x \ne y$. Then T has a unique fixed point.*

Inspired by Theorem 2, Suzuki [34] proved a generalization of Edelstein's fixed point theorem (see also [35–38]).

Theorem 6 [34] *Let (X,d) be a compact metric space, and let T be a mapping on X. Assume that $(1/2)d(x,Tx) < d(x,y)$ implies $d(Tx,Ty) < d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point.*

In this paper, we prove generalizations of Theorem 2, Theorem 4, Theorem 5 and extend Theorem 6. The direction of our extension is new, very simple and inspired by Theorem 3.

Main results

We start this section by proving the following theorem.

Theorem 7 *Let (X,d) be a complete metric space, and let T be a mapping on X. Assume that there exist $r \in [0,1)$, $a \in [0,1]$, $b \in [0,1)$, $(a+b)r^2 + r \le 1$ if $r \in [1/2, 1/\sqrt{2})$, $a + (a+b)r \le$*

1 *if* $r \in [1/\sqrt{2}, 1)$ *such that*

$$ad(x, Tx) + bd(y, Tx) \leq d(y, x) \quad \textit{implies} \quad d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. *Then there exists a unique fixed point* z *of* T. *Moreover,* $\lim_n T^n x = z$ *for all* $x \in X$.

Proof Since $ad(x, Tx) + bd(Tx, Tx) = ad(x, Tx) \leq d(Tx, x)$ holds for every $x \in X$, by hypothesis, we get

$$d(Tx, T^2 x) \leq rd(x, Tx) \tag{3}$$

for all $x \in X$. We now fix $u \in X$ and define a sequence $\{u_n\} \in X$ by $u_n = T^n u$. Then (3) yields $d(u_n, u_{n+1}) \leq r^n d(u, Tu)$, so $\sum_{n=1}^{\infty} d(u_n, u_{n+1}) < \infty$. Hence $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $z \in X$. We next show that

$$d(Tx, z) \leq rd(x, z) \tag{4}$$

for all $x \in X$, $x \neq z$. Since $\lim_n d(u_n, Tu_n) = 0$, $\lim_n d(x, Tu_n) = \lim_n d(x, u_n) = d(x, z)$, there exists a positive integer ν such that $ad(u_n, Tu_n) + bd(x, Tu_n) \leq d(x, u_n)$ for all $n \geq \nu$. By hypothesis, we get $d(Tu_n, Tx) \leq rd(u_n, x)$. Letting n tend to ∞, we obtain $d(z, Tx) \leq rd(z, x)$. That is, we have shown (4).

Now we assume that $T^j z \neq z$ for every integer $j \geq 1$. Then (4) yields

$$d(T^{j+1} z, z) \leq r^j d(Tz, z) \tag{5}$$

for every integer $j \geq 1$. We consider the following three cases:

(a) $0 \leq r < 1/2$,
(b) $1/2 \leq r < 1/\sqrt{2}$,
(c) $1/\sqrt{2} \leq r < 1$.

In the case (a) we note that $2r < 1$. Then, by (3) and (5), we have

$$d(z, Tz) \leq d(z, T^2 z) + d(Tz, T^2 z) \leq rd(z, Tz) + rd(z, Tz) = 2rd(z, Tz) < d(z, Tz).$$

This is a contradiction.

In the case (b), we note that $2r^2 < 1$. If we assume $ad(T^2 z, T^3 z) + bd(z, T^3 z) > d(z, T^2 z)$, then we have, in view of (3) and (5),

$$\begin{aligned} d(z, Tz) &\leq d(z, T^2 z) + d(Tz, T^2 z) \\ &< ad(T^2 z, T^3 z) + bd(z, T^3 z) + d(Tz, T^2 z) \\ &\leq ar^2 d(z, Tz) + br^2 d(z, Tz) + rd(z, Tz) \\ &= [(a + b)r^2 + r] d(z, Tz) \\ &\leq d(z, Tz). \end{aligned}$$

This is a contradiction. Hence $ad(T^2z, T^3z) + bd(z, T^3z) \leq d(z, T^2z)$. By hypothesis and (5), we have

$$\begin{aligned} d(z, Tz) &\leq d\big(z, T^3z\big) + d\big(Tz, T^3z\big) \\ &\leq r^2 d(z, Tz) + rd\big(z, T^2z\big) \\ &\leq r^2 d(z, Tz) + r^2 d(z, Tz) \\ &= 2r^2 d(z, Tz) \\ &< d(z, Tz). \end{aligned}$$

This is also a contradiction.

In the case (c), we assume there exists an integer $\nu \geq 1$ such that

$$ad(u_n, u_{n+1}) + bd(z, u_{n+1}) > d(z, u_n)$$

for all $n \geq \nu$. Then

$$\begin{aligned} d(z, u_n) &< ad(u_n, u_{n+1}) + b\big[ad(u_{n+1}, u_{n+2}) + bd(z, u_{n+2})\big] \\ &\leq (a + abr)d(u_n, u_{n+1}) + b^2 d(z, u_{n+2}) \\ &< (a + abr)d(u_n, u_{n+1}) + b^2\big[ad(u_{n+2}, u_{n+3}) + bd(z, u_{n+3})\big] \\ &\leq \big(a + abr + ab^2r^2\big)d(u_n, u_{n+1}) + b^3 d(z, u_{n+3}). \end{aligned}$$

Continuing this process, we get

$$\begin{aligned} d(z, u_n) &< \big(a + abr + ab^2r^2 + \cdots + ab^{p-1}r^{p-1}\big)d(u_n, u_{n+1}) + b^p d(z, u_{n+p}) \\ &\leq a\frac{1 - (br)^p}{1 - br} d(u_n, u_{n+1}) + b^p d(z, u_{n+p}) \end{aligned}$$

for all $n \geq \nu$, $p \geq 1$. Letting p tend to ∞, we obtain

$$d(z, u_n) \leq \frac{a}{1 - br} d(u_n, u_{n+1})$$

for all $n \geq \nu$. Thus,

$$d(z, u_{n+1}) \leq \frac{a}{1 - br} d(u_{n+1}, u_{n+2}) \leq \frac{ar}{1 - br} d(u_n, u_{n+1})$$

for all $n \geq \nu$, so

$$\begin{aligned} d(u_n, u_{n+1}) &\leq d(z, u_n) + d(z, u_{n+1}) \\ &< \frac{a}{1 - br} d(u_n, u_{n+1}) + \frac{ar}{1 - br} d(u_n, u_{n+1}) \\ &= \frac{a + ar}{1 - br} d(u_n, u_{n+1}) \\ &\leq d(u_n, u_{n+1}) \end{aligned}$$

for all $n \geq \nu$. This is a contradiction. Hence there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that

$$ad(u_{n(k)}, u_{n(k)+1}) + bd(z, u_{n(k)+1}) \leq d(z, u_{n(k)})$$

for all $k \geq 1$. By hypothesis, we get $d(Tz, Tu_{n(k)}) \leq rd(z, u_{n(k)})$ for all $k \geq 1$. Letting k tend to ∞, we get $d(z, Tz) = 0$, that is, $z = Tz$. This is a contradiction.

Thus there exists an integer $j \geq 1$ such that $T^j z = z$. By (3) we get $d(z, Tz) = d(T^j z, T^{j+1} z) \leq r^j d(z, Tz)$, so $d(z, Tz) = 0$, that is, $Tz = z$.

Now we suppose that y is another fixed point of T, that is, $Ty = y$. Then

$$ad(y, Ty) + bd(z, Ty) = bd(z, y) \leq d(z, y),$$

so, by hypothesis, $d(y, z) = d(Ty, Tz) \leq rd(y, z)$. Hence $d(y, z) = 0$. This is a contradiction. □

Remark 1 For $r \in [0, 1/2)$, taking $a = 1$, $b = 0$, we obtain Suzuki's condition from Theorem 2. Moreover, from our condition and the triangle inequality, we get

$$ad(x, Tx) + b\big[d(x, Tx) - d(y, x)\big] \leq d(y, x),$$

that is,

$$\frac{a + b}{1 + b} d(x, Tx) \leq d(y, x).$$

If $r \in [1/\sqrt{2}, 1)$, we have

$$\frac{a + b}{1 + b} = \frac{1}{1 + r} = \theta(r),$$

hence our condition implies Suzuki's condition. We also note that if we take $a = (1 - r)/r^2$, $b = 0$ for $r \in [1/2, 1/\sqrt{2})$, we get Suzuki's condition. Therefore, our theorem generalizes, extends and complements Suzuki's theorem.

Example 1 Define a complete metric space X by $X = \{-1, 0, 1, 2\}$ and a mapping T on X by $Tx = 0$ if $x \in \{-1, 0, 1\}$ and $T2 = -1$. Then T satisfies our condition from Theorem 7 for every $r \in [0, 1/3) \cup [1/2, 1)$, but T does not satisfy Suzuki's condition from Theorem 2.

Proof Since $\theta(r)d(1, T1) \leq 1 = d(1, 2)$ for every $r \in [0, 1)$, and $d(T1, T2) = 1 = d(1, 2)$, T does not satisfy Suzuki's condition. If $r \in [1/2, (\sqrt{5} - 1)/2)$, we have $r^2 + r < 1$, so taking $a + b = (1 - r)/r^2$, we get $a + b > 1$. Hence $ad(1, T1) + bd(1, T2) = a + 2b > 1 = d(1, 2)$ and $ad(2, T2) + bd(2, T1) = 3a + 2b > 1 = d(1, 2)$. Now it is obvious that T satisfies our condition. If $r \in [(\sqrt{5} - 1)/2, 1)$, we take $b = 1/2$. We have two cases: $r \in [(\sqrt{5} - 1)/2, 1/\sqrt{2})$ and $r \in [1/\sqrt{2}, 1)$. In the first case we put $a = (2 - 2r - r^2)/(2r^2)$ and in the second $a = (2 - r)/(2 + 2r)$. We have $a + 2b = 1 + a > 1$ in both cases, so T satisfies our condition. If $r \in [0, 1/3)$ for $a = 1$, $b = 1/2$, it is obvious that T satisfies our condition. □

The following theorem is a generalization of Theorem 4.

Theorem 8 *Let (X,d) be a complete metric space, and let T be a mapping on X. Assume that there exist $r \in [0,1)$, $s > r$ such that*

$$\frac{s-r}{1+r}d(x,Tx)+d(y,Tx)\leq sd(y,x) \quad \textit{implies} \quad d(Tx,Ty)\leq rd(x,y)$$

for all $x, y \in X$. Then T has a unique fixed point. Moreover, if $s \geq 1$, then T has a unique fixed point.

Proof Let $u_1 \in X$ and the sequence u_n be defined by $u_{n+1} = Tu_n$. Since

$$0 = d(u_{n+1},Tu_n) \leq sd(u_{n+1},u_n) - \frac{s-r}{1+r}d(u_n,Tu_n),$$

we get from hypothesis $d(u_{n+1},u_{n+2}) \leq rd(u_{n+1},u_n)$ for all $n \geq 1$. Therefore, $d(u_{n+1},u_{n+2}) \leq r^n d(u_1,u_2)$ for all $n \geq 1$. Thus

$$\sum_{n=1}^{\infty} d(u_{n+1},u_n) \leq \sum_{n=1}^{\infty} r^{n-1}d(u_1,u_2) < \infty.$$

Hence $\{u_n\}$ is a Cauchy sequence. Since X is complete, $\{u_n\}$ converges to some point $z \in X$.

Now, we will show that there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that

$$d(z,Tu_{n(k)}) \leq sd(z,u_{n(k)}) - \frac{s-r}{1+r}d(u_{n(k)},Tu_{n(k)})$$

for all $k \geq 1$. Arguing by contradiction, we suppose that there exists a positive integer ν such that

$$d(z,Tu_n) > sd(z,u_n) - \frac{s-r}{1+r}d(u_n,Tu_n)$$

for all $n \geq \nu$. Then we have

$$\begin{aligned} d(z,u_{n+2}) &> sd(z,u_{n+1}) - \frac{s-r}{1+r}d(u_{n+1},u_{n+2}) \\ &> s^2 d(z,u_n) - s\cdot\frac{s-r}{1+r}d(u_n,u_{n+1}) - \frac{s-r}{1+r}d(u_{n+1},u_{n+2}) \\ &\geq s^2 d(z,u_n) - \frac{s-r}{1+r}\bigl[sd(u_n,u_{n+1}) + rd(u_n,u_{n+1})\bigr] \\ &= s^2 d(z,u_n) - \frac{s-r}{1+r}(s+r)d(u_n,u_{n+1}). \end{aligned}$$

By induction, we get for all $n \geq \nu$, $p \geq 1$ that

$$d(z,u_{n+p}) > s^p d(z,u_n) - \frac{s-r}{1+r}\bigl(s^{p-1} + s^{p-2}r + \cdots + r^{p-1}\bigr)d(u_n,u_{n+1}).$$

Then we have

$$\begin{aligned} d(z,u_{n+p}) &> s^p d(z,u_n) - \frac{s-r}{1+r}\cdot s^{p-1}\cdot\frac{1-(r/s)^p}{1-r/s}d(u_n,u_{n+1}) \\ &= s^p\left[d(z,u_n) - \frac{s-r}{1+r}\cdot\frac{1-(r/s)^p}{s-r}d(u_n,u_{n+1})\right]. \end{aligned}$$

Hence

$$s^p\left[d(z,u_n)-\frac{1-(r/s)^p}{1+r}d(u_n,u_{n+1})\right]<d(z,u_{n+p}). \tag{6}$$

On the other hand,

$$\begin{aligned}d(u_{n+p},u_n)&\le d(u_n,u_{n+1})+d(u_{n+1},u_{n+2})+\cdots+d(u_{n+p-1},u_{n+p})\\&\le\left(1+r+\cdots+r^{p-1}\right)d(u_n,u_{n+1})\\&=\frac{1-r^p}{1-r}d(u_n,u_{n+1}).\end{aligned}$$

Letting $p\to\infty$, we get for all $n\ge 1$ that $d(z,u_n)\le d(u_n,u_{n+1})/(1-r)$. Thus

$$d(z,u_{n+p})\le d(u_{n+p},u_{n+p+1})/(1-r)\le r^p d(u_n,u_{n+1})/(1-r). \tag{7}$$

By (6) and (7) we have for all $n\ge\nu$, $p\ge 1$ that

$$\frac{r^p}{1-r}d(u_n,u_{n+1})>s^p\left[d(z,u_n)-\frac{1-(r/s)^p}{1+r}d(u_n,u_{n+1})\right],$$

so

$$\frac{(r/s)^p}{1-r}d(u_n,u_{n+1})>d(z,u_n)-\frac{1-(r/s)^p}{1+r}d(u_n,u_{n+1}).$$

Taking the limit as $p\to\infty$, we obtain that $d(z,u_n)\le d(u_n,u_{n+1})/(1+r)$ for all $n\ge\nu$. Then we have

$$d(z,u_{n+1})\le d(u_{n+1},u_{n+2})/(1+r)\le rd(u_n,u_{n+1})/(1+r)$$

and

$$rd(u_n,u_{n+1})/(1+r)>sd(z,u_n)-(s-r)d(u_n,u_{n+1})/(1+r).$$

This implies $d(z,u_n)<d(u_n,u_{n+1})/(1+r)$ for all $n\ge\nu$. Thus,

$$d(u_n,u_{n+1})\le d(z,u_n)+d(z,u_{n+1})<d(u_n,u_{n+1})/(1+r)+rd(u_n,u_{n+1})/(1+r)=d(u_n,u_{n+1}).$$

This is a contradiction. Therefore there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that

$$d(z,Tu_{n(k)})\le sd(z,u_{n(k)})-\frac{s-r}{1+r}d(u_{n(k)},Tu_{n(k)})$$

for all $k\ge 1$. By hypothesis, we get $d(Tz,Tu_{n(k)})\le rd(z,u_{n(k)})$. Letting $k\to\infty$, we obtain $d(Tz,z)=0$, that is, $z=Tz$.

If $s\ge 1$, we assume that y is another fixed point of T. Then $d(z,Ty)=d(z,y)\le sd(z,y)-(s-r)d(y,Ty)/(1+r)=sd(z,y)$, so, by hypothesis, $d(z,y)=d(Tz,Ty)\le rd(z,y)$. Since $r<1$, this is a contradiction. □

Edelstein's theorem

The following theorem extends Theorem 6 and generalizes Theorem 5.

Theorem 9 *Let (X, d) be a compact metric space, and let T be a mapping on X. Assume that*

$$ad(x, Tx) + bd(y, Tx) < d(y, x) \quad \textit{implies} \quad d(Tx, Ty) < d(x, y) \tag{8}$$

for $x, y \in X$, where $a > 0$, $b > 0$, $2a + b < 1$. Then T has a unique fixed point.

Proof We put

$$\beta = \inf\{d(x, Tx) : x \in X\}$$

and choose a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(x_n, Tx_n) = \beta$. Since X is compact, without loss of generality, we may assume that $\{x_n\}$ and $\{Tx_n\}$ converge to some elements $v, w \in X$, respectively. We have

$$\lim_{n\to\infty} d(x_n, w) = \lim_{n\to\infty} d(Tx_n, v) = d(v, w) = \beta.$$

We shall show $\beta = 0$. Arguing by contradiction, we assume $\beta > 0$. Since

$$\lim_{n\to\infty} \bigl[ad(x_n, Tx_n) + bd(w, Tx_n)\bigr] = a\beta < \beta = \lim_{n\to\infty} d(w, x_n),$$

we can choose a positive integer ν such that

$$ad(x_n, Tx_n) + bd(w, Tx_n) < d(w, x_n)$$

for all $n \geq \nu$. By hypothesis, $d(Tw, Tx_n) < d(w, x_n)$ holds for $n \geq \nu$. This implies

$$d(w, Tw) = \lim_{n\to\infty} d(Tw, Tx_n) \leq \lim_{n\to\infty} d(w, x_n) = \beta.$$

From the definition of β, we obtain $d(w, Tw) = \beta$. Since $ad(w, Tw) + bd(Tw, Tw) < d(Tw, w)$, we have

$$d\bigl(Tw, T^2 w\bigr) < d(w, Tw) = \beta,$$

which contradicts the definition of β. Therefore we obtain $\beta = 0$. We have $\lim_{n\to\infty} d(x_n, w) = \lim_{n\to\infty} d(Tx_n, v) = \lim_{n\to\infty} d(Tx_n, x_n) = d(v, w) = 0$, so $v = w$. Thus, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = w$.

We next show that T has a fixed point. Arguing by contradiction, we assume that T does not have a fixed point. Since $ad(x_n, Tx_n) + bd(Tx_n, Tx_n) < d(Tx_n, x_n)$ for all $n \geq 1$, we get $d(T^2 x_n, Tx_n) < d(Tx_n, x_n)$, so $\lim_{n\to\infty} T^2 x_n = w$. By induction, we obtain that $d(T^p x_n, T^{p+1} x_n) < d(T^{p-1} x_n, T^p x_n) < \cdots < d(x_n, Tx_n)$ and $\lim_{n\to\infty} T^p x_n = w$ for all integers $p \geq 1$. If there exist an integer $p \geq 1$ and a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$ad\bigl(T^{p-1} x_{n(k)}, T^p x_{n(k)}\bigr) + bd\bigl(w, T^p x_{n(k)}\bigr) < d\bigl(w, T^{p-1} x_{n(k)}\bigr)$$

for all $k \geq 1$, by hypothesis we get $d(Tw, T^p x_{n(k)}) < d(w, T^{p-1}x_{n(k)})$. Taking the limit as $k \to \infty$, we obtain $d(w, Tw) = 0$, that is, $Tw = w$, which is a contradiction. Hence, we can assume that for every $m \geq 1$, there exists an integer $n(m) \geq 1$ such that

$$ad\bigl(T^{m-1}x_n, T^m x_n\bigr) + bd\bigl(w, T^m x_n\bigr) \geq d\bigl(w, T^{m-1}x_n\bigr) \tag{9}$$

for all $n \geq n(m)$. Since

$$\lim_{p\to\infty} \frac{pb^p}{1-b^p} = 0,$$

and

$$\frac{2a}{1-b} < 1,$$

we can choose p satisfying

$$\frac{pb^p}{1-b^p} + \frac{(p-1)b^{p-1}}{1-b^{p-1}} + \frac{2a}{1-b} < 1. \tag{10}$$

We put $\nu = \max\{n(1), n(2), \dots, n(p)\}$. Then by (9) we have

$$\begin{aligned}
d(w,x_n) &\leq ad(x_n, Tx_n) + bd(w, Tx_n) \\
&\leq ad(x_n, Tx_n) + b\bigl[ad\bigl(Tx_n, T^2x_n\bigr) + bd\bigl(w, T^2x_n\bigr)\bigr] \\
&= ad(x_n, Tx_n) + abd\bigl(Tx_n, T^2x_n\bigr) + b^2d\bigl(w, T^2x_n\bigr) \\
&\leq \cdots \\
&\leq ad(x_n, Tx_n) + abd\bigl(Tx_n, T^2x_n\bigr) + \cdots \\
&\quad + ab^{p-1}d\bigl(T^{p-2}x_n, T^{p-1}x_n\bigr) + b^p d\bigl(w, T^p x_n\bigr) \\
&\leq \bigl(a + ab + \cdots + ab^{p-1}\bigr)d(x_n, Tx_n) + b^p d\bigl(w, T^p x_n\bigr) \\
&\leq \bigl[a\bigl(1-b^p\bigr)/(1-b)\bigr]d(x_n, Tx_n) + b^p d\bigl(w, T^p x_n\bigr)
\end{aligned}$$

for all $n \geq \nu$. Since

$$\begin{aligned}
d\bigl(w, T^p x_n\bigr) &\leq d(w,x_n) + d(x_n, Tx_n) + \cdots + d\bigl(T^{p-1}x_n, T^p x_n\bigr) \\
&< d(w,x_n) + pd(x_n, Tx_n),
\end{aligned}$$

we get

$$d(w,x_n) < \bigl[a\bigl(1-b^p\bigr)/(1-b)\bigr]d(x_n, Tx_n) + b^p\bigl[d(w,x_n) + pd(x_n, Tx_n)\bigr],$$

so

$$d(w,x_n) < \left(\frac{a}{1-b} + \frac{pb^p}{1-b^p}\right)d(x_n, Tx_n) \tag{11}$$

for all $n \geq \nu$. Similarly, we can obtain

$$\begin{aligned} d(w, Tx_n) &< \left[\frac{a}{1-b} + \frac{(p-1)b^{p-1}}{1-b^{p-1}} \right] d\bigl(Tx_n, T^2x_n\bigr) \\ &< \left[\frac{a}{1-b} + \frac{(p-1)b^{p-1}}{1-b^{p-1}} \right] d(x_n, Tx_n) \end{aligned}$$

for all $n \geq \nu$. Using (11), we get

$$d(x_n, Tx_n) \leq d(w, x_n) + d(w, Tx_n) < \left[\frac{2a}{1-b} + \frac{pb^p}{1-b^p} + \frac{(p-1)b^{p-1}}{1-b^{p-1}} \right] d(x_n, Tx_n)$$

for all $n \geq \nu$. Thus, by (10), we obtain $d(x_n, Tx_n) < d(x_n, Tx_n)$, which is a contradiction. Therefore there exists $z \in X$ such that $Tz = z$. Fix $y \in X$ with $y \neq x$. Then since $ad(x, Tx) + bd(y, Tx) = bd(y, x) < d(y, x)$, we have $d(Ty, x) = d(Ty, Tx) < d(y, x)$ and hence y is not a fixed point of T. Therefore, the fixed point of T is unique. □

Remark 2 The proof of Theorem 9 is available for $a = 1/2$, $b = 0$. In this case we obtained Theorem 6. We do not know if Theorem 9 is still correct for $a = 0$, $b = 1$, or, more generally, for $2a + b = 1$. This is an open question.

Example 2 Define a complete metric space X by $X = \{A, B, C, D, E\}$ such that $d(A, B) = d(A, C) = d(B, D) = d(C, D) = 2$, $d(A, D) = d(B, C) = 3$, $d(A, E) = d(C, E) = 5/2$, $d(B, E) = d(D, E) = 1$ and a mapping T on X by $TA = B$, $TB = E$, $TC = D$, $TD = E$, $TE = E$. Then T satisfies our condition from Theorem 9 for $a = 1/8$, $b = 2/3$, but T does not satisfy Suzuki's condition from Theorem 6.

Proof We have $d(A, C) = 2 = d(TA, TC)$ and $(1/2)d(A, TA) = 1 < d(A, C) = 2$, so T does not satisfy Suzuki's condition from Theorem 6. Moreover, we have $ad(A, TA) + bd(C, TA) = ad(C, TC) + bd(A, TC) = 2a + 3b = 9/4 > d(A, C)$. It is now obvious that T satisfies our condition from Theorem 9. □

Competing interests
The author declares that they have no competing interests.

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