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ITERATED STIELTJES TRANSFORM OF GENERALIZED FUNCTIONS

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ABSTRACT. The generalized S_2-transform of a member of f of a certain space of generalized functions is defined as

$$F(x) = \left\langle f(t), \ K(x, t; \rho) \right\rangle, \quad \text{where}$$

$$K(x, t; \rho) = \int_0^\infty \frac{1}{(x+y)^\rho (y+t)^\rho} dy, \quad \rho > \tfrac{1}{2},$$

$$o < x < \infty \quad \text{and} \quad o < t < \infty .$$

An inversion theorem for the transform is established interpreting the convergence in the weak distributional sense.

KEY WORDS AND PHRASES. *Testing Function space, Dual space, Frechet space, Iterated Stieltjes Transform, Schwartz Distributions, Generalized Functions.*
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1. INTRODUCTION.

Recently, extensions of classical integral transformations to generalized functions have comprised an active and interesting area of research. In this paper an iterated Stieltjes transform (S_2-transform) is extended to generalized functions by so called direct approach, and an inversion formula is established. Note that, for simplicity, throughout the paper the transform of a function or generalized function is denoted by $F(x)$, and the actual transform is made evident by the context.

The Stieltjes transform of $\phi(t) \in L(o, \infty)$ is defined as

$$F(x) = \int_0^\infty \frac{\phi(t)}{(x+t)} dt, \quad x > o. \tag{1.1}$$

Widder [10, p. 325] studied various properties of the transform (1.1) and proved the inversion formula

$$\lim_{k \to \infty} L_{k,x}[F(x)] = \phi(x)$$

for almost all $x > o$, where the differential operator $L_{k,x}$ is defined as

$$L_{k,x}[F(x)] = \frac{(-1)^{k-1}}{k!(k-2)!}[x^{2k-1} F^{(k-1)}(x)]^{(k)} .$$

Benedetto [1], Zemanian [11, p. 244] and Pandey [7] have given the distributional extensions of the transform (1.1) following different approaches.

The transform (1.1) is generalized as

$$F(x) = \int_0^\infty \frac{\phi(t)}{(x+t)^\rho} dt \qquad \text{for } \rho > o \ (x > o) . \tag{1.2}$$

Pollard [8] defined the operator

$$L_{k,\,x;\,\rho}\,(F(x)) \;=\; \frac{(-1)^{k-1}\,2^{\rho-1}\,(2k-1)!\,\Gamma(\rho)}{k!\,(k-2)!\,\Gamma(2k+\rho-1)}\,[x^{2k+\rho-2}\,F(x)^{(k-1)}]^{(k)} \tag{1.3}$$

and proved the inversion formula

$$\lim_{k\to\infty} L_{k,\,x;\,\rho}\,(F(x)) \;=\; \phi(x) \qquad \text{for almost all } x > o\,.$$

Pathak [8] and Erdélyi [6] extended the transform (1.2) independently and differently to generalized functions.

The generalized iterated Stieltjes transform of $\phi(t) \in L\,(o,\infty)$ is defined as

$$F(x) \;=\; \int_0^\infty \frac{dy}{(x+y)^{\rho}} \int_0^\infty \frac{\phi(t)}{(y+t)^{\rho}}\,dt \quad \text{for} \quad \rho > \tfrac{1}{2}\ (x > o) \tag{1.4}$$

The change in order of integration in (1.4) leads to so called generalized S_2-transform as

$$F(x) \;=\; \int_0^\infty \phi(t)\;dt \int_0^\infty \frac{1}{(x+y)^{\rho}(y+t)^{\rho}}\,dy$$

$$\equiv \int_0^\infty K(x,\,t;\,\rho)\;\phi(t)\;dt \quad ; \quad \rho > \tfrac{1}{2}. \tag{1.4$'$}$$

Boas and Widder [2] have studied transforms (1.4) and (1.4)' in great detail for the case $\rho = 1$ and have given corresponding inversion formulas. Recently the author [5] considered the classical transforms (1.4) and (1.4)' for the general $\rho > \frac{1}{2}$ and proved the inversion formula

$$\lim_{k\to\infty} H_{k,\,x;\,\rho}\,(F(x)) \;=\; \phi(x) \qquad \text{for almost all } x > o \tag{1.5}$$

where $\quad H_{k,\,x;\,\rho} \equiv L_{k,\,x;\,\rho}\,(L_{k,\,x;\,\rho})$.

The distributional extension of (1.4)' for $\rho = 1$ was given by the author [4].

Our main objective in this paper is to extend the transform (1.4)' to generalized functions, and to prove the inversion formula (1.5) in the distributional sense.

The notation and the terminology of this work will follow that of [3] and [11]. I denotes the open interval (o,∞) and all testing functions herein are defined on I. Throughout the work x, t and y are variables over I. D(I) represents the space of infinity differentiable functions defined on I having compact support. The topology of D(I) is that which makes its dual the space D'(I) of the Schwartz distributions.

2. THE TESTING FUNCTION SPACE $S_\rho(I)$ AND ITS DUAL $S'_\rho(I)$.

We define $S_\rho(I)$ as the collection of all infinitely differentiable functions $\phi(t)$ defined on I such that

$$\gamma_k(\phi) \;=\; \sup_{o<t<\infty} \left|\, \xi(t) \quad \left(t\,\frac{d}{dt}\right)^k \phi(t) \,\right| < \infty$$

for each $k = o, 1, 2, \ldots$, where

$$\xi(t) \;=\; \begin{cases} t^{\rho} & , \quad o < t \le 1 \\ 1 & , \quad t > 1 \end{cases} \tag{2.1}$$

ρ being a fixed real number greater than $\frac{1}{2}$.

We assign to $S_\rho(I)$ the topology generated by semi-norms $\{\gamma_k\}$, thereby making it a countably multi-normed space. A sequence $\{\phi_n(x)\}$ where each $\phi_n(x) \in S_\rho(I)$ converges in $S_\rho(I)$ to $\phi(x)$ if $\gamma_k(\phi_n-\phi)$ tends to zero as $n \to \infty$ for each $k = o, 1, 2, \ldots$. A sequence $\{\phi_n(x)\}$ is said to be a Cauchy sequence if $\gamma_k(\phi_m-\phi_n)$ tends to zero as m and n approach to ∞ independently of each other for each $k = o, 1, 2, \ldots$. It can be seen that $S_\rho(I)$ is a Fréchet space, i.e., a complete countably multi-normed space. The dual space $S'_\rho(I)$ consists of all continuous linear functionals on $S_\rho(I)$ and is equipped with the usual weak topology.

It can be checked that the space $D(I)$ is contained in $S_\rho(I)$, and the topology of $D(I)$ is stronger than that induced on it by $S_\rho(I)$. Hence the restriction of any $f \in S'_\rho(I)$ to $D(I)$ is in $D'(I)$.

It can also be checked that if $f(t)$ is a function defined on I such that

$$\int_0^\infty \frac{|f(t)|}{\xi(t)}\, dt < \infty ,$$

then $f(t)$ generates a regular generalized function on $S_\rho(I)$ by

$$\left\langle f, \phi \right\rangle = \int_0^\infty f(t)\, \phi(t)\, dt; \quad \phi \in S_\rho(I) .$$

3. THE GENERALIZED S_2 TRANSFORM OF GENERALIZED FUNCTIONS.

Let $\displaystyle\int_0^\infty \frac{1}{(x + y)^\rho (y + t)^\rho}\, dy \equiv K(x, t; \rho)$, for $x, t > o$ $(\rho > \frac{1}{2})$.

The generalized S_2-transform of $f \in S'_\rho(I)$ is defined as a function $F(x)$ obtained by applying $f(t)$ on the kernel $K(x, t; \rho)$, i.e.,

$$F(x) = \left\langle f(t), K(x, t; \rho) \right\rangle , \quad x > o. \tag{3.1}$$

In order that (3.1) be meaningful, we need show that for a fixed $x > o$, $K(x, t; \rho)$ is a member of the testing function space $S_\rho(I)$. We prove this as following theorem.

THEOREM 1. For fixed $x > o$ and $\rho > \frac{1}{2}$, $K(x, t; \rho)$ belongs to $S_\rho(I)$.

PROOF. That $K(x, t; \rho)$ is infinitely differentiable function of t for $x > o$ and $\rho > \frac{1}{2}$, is obvious. Now we show that $\gamma_k(K(x, t; \rho))$ is finite for any $k = o, 1, 2, \ldots$

A simple computation shows that

$$(t\frac{d}{dt})^k[K(x, t; \rho)] = \int_0^\infty \frac{P_k(t, y; \rho)}{(x + y)^\rho (y + t)^{\rho+k}}\, dy \quad , \text{ where}$$

$P_k(t, y; \rho) = \Sigma_{i=o}^{k-1} C_i y^i t^{k-i}$, C_i being certain constants depending on ρ.

Now for any i such that $o \le i \le k-1$,

$$\frac{y^i t^{k-i}}{(y+t)^{\rho+k}} \le (\frac{y}{y+t})^i (\frac{t}{y+t})^{k-i} \frac{1}{(y+t)^\rho} < \frac{1}{(y+t)^\rho}$$

Therefore,

$$|(t\frac{d}{dt})^k[K(x, t; \rho)]| < A \int_0^\infty \frac{1}{(x+y)^\rho (y+t)^\rho}\, dy,$$

where $A = \Sigma_{i=o}^{k-1} |C_i|$.

$$\text{Now} \sup_{o < t < \infty} \int_0^\infty \frac{1}{(x + y)^\rho (y + t)^\rho}\, dy \le \sup_{o < t \le 1} \int_0^\infty \frac{1}{(x+y)^\rho (y+t)^\rho}\, dy$$

$$+ \sup_{1 < t < \infty} \int_0^\infty \frac{1}{(x+y)^\rho (y+t)^\rho}\, dy$$

$$= \sup_{o < t \le 1} [\int_0^1 + \int_0^\infty] \frac{1}{(x+y)^\rho (y+t)^\rho} dy$$

$$+ \sup_{1 < t < \infty} [\int_0^1 + \int_1^\infty] \frac{1}{(x+y)^\rho (y+t)^\rho} dy$$

$$< \sup_{o < t \le 1} [\frac{1}{x^\rho t^\rho} + \int_1^\infty \frac{1}{y^{2\rho}} dy] + \sup_{1 < t < \infty} [\frac{1}{x^\rho t^\rho} + \int_1^\infty \frac{1}{y^{2\rho}} dy]$$

Therefore,

$$\sup_{o < t < \infty} | \xi(t) \quad (t\frac{d}{dt})^k K(x, t; \rho)]| < A \sup_{o < t \le 1} t^\rho [\frac{1}{x^\rho t^\rho} + \frac{1}{2\rho-1}]$$

$$+ A \sup_{1 < t < \infty} [\frac{1}{x^\rho t^\rho} + \frac{1}{2\rho-1}]$$

$$< 2A [\frac{1}{x^\rho} + \frac{1}{2\rho-1}]$$

which is bounded for fixed $x > o$ and $\rho > \frac{1}{2}$.

This completes the proof of the theorem.

The next theorem gives the differentiability of the transform.

THEOREM 2. Let $f \in S'_\rho(I)$ and let $F(x)$ denote the generalized S_2-transform of f as defined by (3.1). Then $F(x)$ is infinitely differentiable function for $x > o$, and that

$$\overset{(k)}{F}(x) = \left\langle f(t), \frac{\partial^k}{\partial x^k} K(x, t; \rho) \right\rangle, \quad k = o, 1, 2, \ldots$$

The proof follows from the standard technique [11, p. 146], and is therefore omitted.

In order to prove an inversion theorem for the transform (3.1), we need prove following lemmas.

LEMMA 1. Let $H_{k,\, x;\, \rho}$ denote the second iterate of $L_{k,\, x;\, \rho}$ which is defined by (1.3). Then

$$H_{k,\, x;\, \rho} (\int_0^\infty \frac{1}{(x+y)^\rho (y+t)^\rho} dy) = [\frac{2^{\rho-1}(2k-1)!}{k!\,(k-2)!}]^2 x^{k+2\rho-3} t^k \int_0^\infty \frac{y^{2k-1}}{(x+y)^{2k+\rho-1} (y+t)^{2k+\rho-1}} dy$$

PROOF. In view of successive differentiation and simple computation it follows that

$$\Gamma(\rho)[x^{2k+\rho-2} \frac{d^{k-1}}{dx^{k-1}} (\frac{1}{(x+y)^\rho})]^{(k)} = (-1)^{k-1}\Gamma(2k+\rho-1) \frac{x^{k+\rho-2} y^k}{(x+y)^{2k+\rho-1}}, \quad \text{so that}$$

$$L_{k,\, x;\, \rho} (\frac{1}{(x+y)^\rho}) = \frac{2^{\rho-1}(2k-1)!}{k!\,(k-2)!} \frac{x^{k+\rho-2} y^k}{(x+y)^{2k+\rho-1}} \tag{3.2}$$

$$\equiv d_k \frac{x^{k+\rho-2} y^k}{(x+y)^{2k+\rho-1}}$$

Therefore,

$$L_{k,\, x;\, \rho} \int_0^\infty \frac{1}{(x+y)^\rho (y+t)^\rho} dy = d_k \int_0^\infty \frac{x^{k+\rho-2} y^k}{(x+y)^{2k+\rho-1} (y+t)^\rho} dy \tag{3.3}$$

Note that the differentiation under the integral sign in above integral is permissible.

Next we want to show that

$$L_{k,\ x;\ \rho} \int_0^\infty \frac{x^{k+\rho-2} y^k}{(x+y)^{2k+\rho-1}(y+t)^\rho}\, dy = \int_0^\infty \frac{x^{k+2\rho-3} y^{k+1-\rho}}{(x+y)^{2k+\rho-1}} L_{k,\ y;\ \rho} \left(\frac{1}{(y+t)^\rho}\right) dy, \quad (3.4)$$

The substitution $y = xu$ in the right hand side of the above equality leads to the integral

$$\int_0^\infty \frac{u^{k+1-\rho}}{(1+u)^{2k+\rho-1}} L_{k,\ xu;\ \rho} \left(\frac{1}{(xu+t)^\rho}\right) du$$

$$= \int_0^\infty \frac{u^{k+1-\rho}}{(1+u)^{2k+\rho-1}} u^{\rho-1} L_{k,\ x;\ \rho} \left(\frac{1}{(xu+t)^\rho}\right) du$$

$$= L_{k,\ x;\ \rho} \int_0^\infty \frac{u^k}{(1+u)^{2k+\rho-1}} \cdot \frac{1}{(xu+t)^\rho}\, du \quad , \quad (3.5)$$

as the differentiation under the integral sign is permissible. Taking the substitution $xu = y$ in the last integral, (3.5) equals to

$$L_{k,\ x;\ \rho} \int_0^\infty \frac{x^{k+\rho-2} y^k}{(x+y)^{2k+\rho-1}(y+t)^\rho}\, dy$$

which is the left hand side of (3.4), establishing the equality (3.4).

Now applying the result (3.2) in the right hand side of (3.4) we get,

$$L_{k,\ x;\ \rho} \left(L_{k,\ x;\ \rho} \int_0^\infty \frac{1}{(x+y)^\rho (y+t)^\rho}\, dy\right)$$

$$= d_k^2\ x^{k+2\rho-3}\ t^k \int_0^\infty \frac{y^{2k-1}}{(x+y)^{2k+\rho-1}(y+t)^{2k+\rho-1}}\, dy. \quad (3.6)$$

This completes the proof of the lemma.

LEMMA 2. If $n > m \geqq 1$, then

$$\int_0^\infty \frac{y^{m-1}}{(t+y)^n}\, dy = \frac{\Gamma(m)\,\Gamma(n-m)}{t^{n-m}\,\Gamma(n)}.$$

This is the familiar formula for the beta function.

LEMMA 3. Let the expression (3.6) be denoted by $F_k(t,\ x;\ \rho)$. Then

$$\int_0^\infty F_k(t,\ x;\ \rho)\, dx \to 1 \text{ as } k \to \infty$$

for all $t > o$.

PROOF. $\int_0^\infty F_k(t,\ x;\ \rho)\, dx = d_k^2 \int_0^\infty t^k x^{k+2\rho-3}\, dx \int_0^\infty \frac{y^{2k-1}}{(x+y)^{2k+\rho-1}(y+t)^{2k+\rho-1}}\, dy$

$$= d_k^2\ t^k \int_0^\infty \frac{y^{2k-1}}{(y+t)^{2k+\rho-1}}\, dy \int_0^\infty \frac{x^{k+2\rho-3}}{(x+y)^{2k+\rho-1}}\, dx\ ,$$

(By changing the order of integration which is obviously permissible).

$$= \left[\frac{2^{\rho-1}(2k-1)!}{k!\,(k-2)!}\right]^2 \frac{\Gamma(k+2\rho-2)\ \Gamma(k-\rho+1)\ \Gamma(k+\rho-1)\ \Gamma(k)}{[\Gamma(2k+\rho-1)]^2}\quad ,$$

(By Lemma 2, for $k > 2$).

Using Stirling's approximation formula, it follows that the last expression converges to 1 as $k \to \infty$.

LEMMA 4. Let $G_k(y) = \int_0^y F_k(1, x; \rho)\, dx$.

Then

$$\lim_{k \to \infty} G_k(y) = 0 \quad \text{for } 0 \leq y < 1, \tag{3.7}$$

and

$$\lim_{k \to \infty} G_k(y) = 1 \quad \text{for } y > 1. \tag{3.8}$$

PROOF. The proof follows along the lines of [2, Lemma 8.2], and is therefore omitted.

Now we prove the inversion theorem for the transform (3.1) which will be the main result of the paper.

THEOREM 3 (Inversion). Let $f \in S'_\rho(I)$ and let $F(x)$ be the generalized S_2 transform of f as defined by (3.1). Then for any $\phi \in D(I)$, $\lim_{k \to \infty} \left\langle H_{k,\, x;\, \rho} F(x), \phi(x) \right\rangle = \left\langle f, \phi \right\rangle$ where

$$H_{k,\, x;\, \rho}(F(x)) = L_{k,\, x;\, \rho}(L_{k,\, x;\, \rho}(F(x)).$$

$$\equiv \left[\frac{2^{\rho-1}(2k-1)!\,\Gamma(\rho)}{k!\,(k-2)!\,\Gamma(2k+\rho-1)}\right]^2 \left[x^{2k+\rho-2}\{x^{2k+\rho-2}F^{(k-1)}(x)\}^{(2k-1)}\right]^{(k)};$$

the differentiation herein referring to distributional differentiation.

PROOF. A simple computation shows that the operator $H_{k,\, x;\, \rho}$ can be represented as $x^{2\rho-2} P_{4k-2;\, \rho}(x\frac{d}{dx})$, where

$P_{4k-2;\, \rho}(x\frac{d}{dx}) = \Sigma_{i=0}^{4k-2} a_i(\rho)\, (x\frac{d}{dx})^i$, $a_i(\rho)$ being certain constants depending on ρ.

Now the theorem is proved by justifying the following steps:

$$\left\langle H_{k,\, x;\, \rho} F(x), \phi(x) \right\rangle = \left\langle x^{2\rho-2}[P(x\tfrac{d}{dx}) \; F(x), \phi(x) \right\rangle$$

$$= \int_0^\infty x^{2\rho-2}[P(x\tfrac{d}{dx}) \; F(x)]\, \phi(x)\, dx \tag{3.9}$$

$$= \int_0^\infty [P(x\tfrac{d}{dx}) \; F(x)]\, x^{2\rho-2}\, \phi(x)\, dx \tag{3.9$'$}$$

$$= \int_0^\infty F(x)\, [P(-x\tfrac{d}{dx} -1) \; x^{2\rho-2}\, \phi(x)]\, dx \tag{3.10}$$

$$= \int_0^\infty \left\langle f(t), K(x, t; \rho) \right\rangle [P(-x\tfrac{d}{dx} -1)\, x^{2\rho-2}\, \phi(x)]\, dx \tag{3.10$'$}$$

$$= \left\langle f(t), \int_0^\infty K(x, t; \rho)\, [P(-x\tfrac{d}{dx} -1) \; x^{2\rho-2}\, \phi(x)]\, dx \right\rangle \tag{3.11}$$

$$\to \left\langle f(t), \phi(t) \right\rangle, \quad \text{as } k \to \infty \; (P \equiv P_{4k-2}). \tag{3.12}$$

The step (3.9) is obvious in view of Theorem 2 and the fact that

$$x^{2\rho-2}\, P(x\tfrac{d}{dx})\, F(x)$$

generates a regular distribution in $D'(I)$. The step (3.9)' is actually the same as (3.9). The step (3.10) is obtained by applying integration by parts in (3.9)' successively and using the fact that the limit terms in the integral vanish. The step (3.10)' is the same as (3.10) in view of definition of $F(x)$. That (3.10)' equals to

(3.11) can be proved by the technique of Riemann sums [11, Lemma 5.6.2, p. 148]. In order to show that (3.11) → (3.12) as k → ∞, we need to prove that for any non-negative inteter n

$$\xi(t)\ \left(t\frac{d}{dt}\right)^n\left[\int_0^\infty K(x,\ t;\ \rho)\ P\left(-x\frac{d}{dx}-1\right)\left(x^{2\rho-2}\ \phi(x)\right)\ dx - \phi(t)\right]$$

→ o as k → ∞ uniformly for all t > o, where $\xi(t)$ is defined by (2.1).

Now,

$$\begin{aligned}
&\left(t\frac{d}{dt}\right)\int_0^\infty K(x,\ t;\ \rho)\ P\left(-x\frac{d}{dx}-1\right)\left(x^{2\rho-2}\ \phi(x)\right)\ dx \qquad (3.13)\\
&= \int_0^\infty \left(t\frac{d}{dt}\right)[K(x,\ t;\ \rho)]\ P\left(-x\frac{d}{dx}-1\right)\left(x^{2\rho-2}\ \phi(x)\right)\ dx\\
&= \int_0^\infty \left(-x\frac{d}{dx}+1-2\rho\right)[K(x,\ t;\ \rho)]\ P\left(-x\frac{d}{dx}-1\right)\left(x^{2\rho-2}\ \phi(x)\right)\ dx\\
&= \int_0^\infty K(x,\ t;\ \rho)\ \left(x\frac{d}{dx}+2-2\rho\right)\left[P\left(-x\frac{d}{dx}-1\right)\left(x^{2\rho-2}\ \phi(x)\right)\right]\ dx\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(By integration by parts)}\\
&= \int_0^\infty K(x,\ t;\ \rho)\ P\left(-x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}+2-2\rho\right)\left(x^{2\rho-2}\ \phi(x)\right)\ dx\\
&= \int_0^\infty K(x,\ t;\ \rho)\ P\left(-x\frac{d}{dx}-1\right)\left[x^{2\rho-2}\ \left(x\frac{d}{dx}\right)\phi(x)\right]\ dx.
\end{aligned}$$

Hence applying $\left(t\frac{d}{dt}\right)$ successively on the integral in (3.13), we get for any non-negative integer n,

$$\begin{aligned}
&\left(t\frac{d}{dt}\right)^n\int_0^\infty K(x,\ t;\ \rho)\ P\left(-x\frac{d}{dx}-1\right)\left(x^{2\rho-2}\ \phi(x)\right)\ dx\\
&= \int_0^\infty K(x,\ t;\ \rho)\ P\left(-x\frac{d}{dx}-1\right)\left[x^{2\rho-2}\ \left(x\frac{d}{dx}\right)^n\phi(x)\right]\ dx\\
&= \int_0^\infty P\left(x\frac{d}{dx}\right)[K(x,\ t;\ \rho)]\ x^{2\rho-2}\ \left(x\frac{d}{dx}\right)^n\phi(x)]\ dx\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(By integration by parts)}\\
&= \int_0^\infty x^{2\rho-2}\ P\left(x\frac{d}{dx}\right)[K(x,\ t;\ \rho)]\left(x\frac{d}{dx}\right)^n\phi(x)\ dx\\
&= \int_0^\infty H_{k,\ x;\ \rho}\ [K(x,\ t;\ \rho)]\left(x\frac{d}{dx}\right)^n\phi(x)\ dx\\
&= \int_0^\infty F_k(t,\ x;\ \rho)\left(x\frac{d}{dx}\right)^n\phi(x)\ dx \qquad \text{(By Lemma 1)}
\end{aligned}$$

Hence by Lemma 3, as k → ∞, it follows that

$$\begin{aligned}
&\left(t\frac{d}{dt}\right)^n\left[\int_0^\infty K(x,\ t;\ \rho)\ P\left(-x\frac{d}{dx}-1\right)\left(x^{2\rho-2}\ \phi(x)\right)\ dx - \phi(t)\right]\\
&= \int_0^\infty F_k(t,\ x;\ \rho)\ \left[\left(x\frac{d}{dx}\right)^n\phi(x) - \left(t\frac{d}{dt}\right)^n\phi(t)\right]\ dx.
\end{aligned}$$

Denote $\left(x\frac{d}{dx}\right)^n\phi(x)$ by $\psi(x)\in D(I)$. Now it suffices to prove that

$$\xi(t)\int_0^\infty F_k(t,\ x;\ \rho)\ [\psi(x)-\psi(t)]\ dx \qquad (3.14)$$

converges to zero as k → ∞ uniformly for all t > o.

From the definition of $F_k(t,\ x;\ \rho)$ as given by (3.6), it is easy to check that $F_k(t,\ x;\ \rho)$ is a homogeneous function of t and x of degree-1. The homogeneity of $F_k(t,\ x;\ \rho)$ and the substitution x = ty in the integral in (3.14) lead to

$$\xi(t)\ \int_0^\infty F_k(1,\ x;\ \rho)\ [\psi(xt)-\psi(t)]\ dx. \qquad (3.15)$$

Now breaking the integration in (3.15) into the intervals $(0,\ 1-\eta)$, $(1-\eta,\ 1+\eta)$ and

$(1+\eta, \infty)$, where η is a fixed positive number less than $\frac{1}{2}$, one can easily prove in view of Lemma 4 and [10, Lemma 5, p. 287], that (3.15) approaches to zero as $k \to \infty$ uniformly for all $t > o$.

This completes the proof of the theorem.

REMARK. In the present work, we have developed the entire theory for the transform defined by (1.4)'. The extension of the transform defined by (1.4) to generalized functions is still open. The difficulty faced in such an extension has been indicated in [4, p. 384] for the case $\rho = 1$.

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