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ON THE ORDER OF EXPONENTIAL GROWTH OF THE SOLUTION OF THE LINEAR DIFFERENCE EQUATION WITH PERIODIC COEFFICIENT IN BANACH SPACE

H. ATTIA HUSSEIN

Department of Mathematics
Faculty of Science
Alexandria University
Alexandria, Egypt

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ABSTRACT. An equation of the form $y - A(t)y = f(t)$ is considered, where $\Delta y = \frac{y(t+\delta) - y(t)}{\delta}$, and the necessary and sufficient criteria for the exponential growth of the solution of this equation is obtained.

KEY WORDS AND PHRASES. Difference equations, solution of exponential growth.

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1. INTRODUCTION.

Let E be a complex Banach space. Denote by $\{ A(t) : t \geq o\}$ a family of linear bounded operators from E into itself. We assume that $A(t)$ is periodic and strongly continuous in $t \in [o, \infty)$.

Let $||\cdot||$ be the norm in E. Denote by E_α the set of all elements $f(t) \in E$ such that

$$\sup ||f(t)|| \exp(-\alpha t) < \infty.$$

2. RESULTS.

Let $\Delta y = \frac{y(t+\delta) - y(t)}{\delta}$, $\delta > o$, $y(t)$ be a solution of the difference equation

$$\Delta y - A(t)\, y = f(t) \quad , t \geq \delta \tag{2.1}$$

such that

$$y(t) = \Theta \quad , \quad o \leq t < \delta \tag{2.2}$$

where Θ is the zero of E.

Let us assume that $f \in E_\alpha$. The solution of equation (2.1) can be written in the form

$$y(t) = \delta \sum_{i=o}^{t-\delta} A(i)\, y(i) + \delta \sum_{i=o}^{t-\delta} f(i) \tag{2.3}$$

where $t = [n\delta]$, $[a]$ denotes the greatest positive integer $\leq a$ and δ is a positive integer.

Without loss of generality we suppose that $\delta = 1$.

Putting $t = 1, 2, \ldots, n$ in (2.3), one obtains

$$y(t) = \sum_{j=1}^{n-1} \prod_{i=n-1}^{j} (I + A(i))\, f(j-1) + f(t-1) \tag{2.4}$$

where I is the unit operator. Let w be the period of $A(t)$.

$$\left[\prod_{i=n-1}^{j} B(i) = B(n-1)\, B(n-2) \ldots B(j),\ j \leq n-1 \right]$$

Substituting $t = [S\, w]$ into equation (2.4), we obtain

$$y(t) = \sum_{r=1}^{s} \left[\prod_{k=w-1}^{o} [I + A(k)]^{s-r} \left\{ \sum_{j=1}^{w-1} f_j(r-1)w+j-1) + f((r-1)w + w-1) \right\} \right] \tag{2.5}$$

where

$f_1(\xi w) = A(w-1)\, A(w-2) \ldots A(1)\ f(\xi w)$

$f_2(\xi w+1) = A(w-1)\, A(w-2) \ldots A(2)\ f(\xi w+1)$

....

....

$f_{w-1}(\xi w+w-2) = A(w-1)\ f(\xi w + w-2).$

Setting $B = \prod_{k=w-1}^{o} [I + A(k)]$ in (2.5) we get

$$y(t) = \sum_{r=1}^{\frac{t}{w}} B^{\frac{t-rw}{w}} \left\{ \sum_{j=1}^{w-1} f_j\ ((r-1)w + j-1) + f((r-1)w+w-1) \right\}.$$

The last equation can be written in the form

$$y(t) = -\frac{1}{2\pi i} \oint_{\gamma} \sum_{r=1}^{\frac{t}{w}} \lambda^{\frac{t-rw}{w}} (B-\lambda I)^{-1} \left\{ \sum_{j=1}^{w-1} f_j((r-1)w+j-1) + f((r-1)\, w + w-1) \right\} \tag{2.6}$$

where γ is a contour which circumscribes all the specter of the operator B, [1].

It can be seen that if $f \in E_\alpha$, then $(B- \lambda\, I)^{-1} f \in E_\alpha$ for every $\lambda \epsilon \gamma$. From equation (2.6) we obtain a necessary and sufficient criterion for the exponential growth of the solution with an index β. Let σ_B denote the specter of the operator B. Assume that $\lambda_o \in \sigma_B$. Set $\alpha_o = \frac{1}{w} \ln |\lambda_o|$.
The following theorem holds:

<u>THEOREM</u>. If $f \in E_\alpha$, then the solution y of equation (2.1) belongs to E_β such that

$\beta = \alpha$, when $\alpha > \alpha_0$

$\beta > \alpha$, when $\alpha = \alpha_0$

$\beta = \alpha_0$, when $\alpha < \alpha_0$.

PROOF. To prove the sufficiency, we consider the following three cases:

(1) If $\alpha > \frac{1}{w} \ln|\lambda|$ then $y(t)$ defined by (2.6) belongs to E_α.

(2) If $\alpha > \frac{1}{w} \ln|\lambda|$ then from (2.6) we obtain

$$||y|| \le D \sum_{r=1}^{\frac{t}{w}} \exp\left(\frac{1}{w} \ln|\lambda| (t-rw)\right) \left\{ \sum_{j=1}^{w-1} ||f_j((r-1)w+j-1)|| + ||f((r-1)w + w-1)|| \right\}$$

$$< D_1 \exp(\alpha t).\frac{t}{w} + D_2 \exp(\alpha t).(w-2)\frac{t}{w}$$

$$< D' \exp(\alpha t).t$$

(where D, D_1, D_2 and D' are constants).

This means that $y \in E_\beta$ where $\beta > \alpha$.

(3) If $\alpha < \frac{1}{w} \ln|\lambda|$ and $||f|| \le c \exp(\alpha t)$, then from (2.6) we have

$$||y|| \le C_1 \exp\left(\frac{1}{w} \ln|\lambda|.t\right)$$

and $y \in E_{\frac{1}{w}\ln|\lambda|}$ $(\alpha < \frac{1}{w} \ln|\lambda|)$.

We now prove the necessity:

If λ_0 is an eigenvalue and x_0 is an eigenvector for the operator B such that $Bx_0 = \lambda_0 x_0$,

where x_0 is an element of Banach space such that $||x_0|| = 1$, by taking $f(t) = \exp(\alpha t).x_0$ equation (2.6) with

$(B - \lambda I)^{-1} x_0 = \dfrac{x_0}{\lambda_0 - \lambda}$ becomes

$$y(t) = \sum_{r=1}^{\frac{t}{w}} \exp\left(\frac{1}{w} \ln|\lambda_0| (t-rw)\right) \left\{ \sum_{j=1}^{w-1} f_j((r-1)w+j-1) + f((r-1)w + w-1) \right\}. \quad (2.7)$$

Multiplying the last equation by $\exp(-\alpha_0 t)$, where $\alpha_0 = \frac{1}{w} \ln|\lambda_0|$, we have

$$y(t)\exp(-\alpha_0 t) = \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \exp(-\alpha_0 wr) \left\{ \sum_{j=1}^{w-1} f_j((r-1)w+j-1) + f(r-1)w + w-1) \right\}$$

where $\theta = \arg \lambda$.

$$y(t)\exp(-\alpha_0 t) = \exp\left(\frac{i\theta t}{w} + w(1-\alpha)-1\right) \sum_{r-1}^{\frac{t}{w}} \exp((\alpha - \alpha_0)wr).x_0$$

$$+ \exp\left(\frac{i\theta t}{w}\right) \sum_{r-1}^{\frac{t}{w}} \sum_{j-1}^{w-1} \exp(-\alpha_0 wr) \; f_j((r-1)w+ j-1)$$

$$= \frac{\exp\left(\frac{i\theta t}{w} + (\alpha - \alpha_0)\right)}{\exp(\alpha - \alpha_0) - 1} \left[\exp(\alpha - \alpha_0)\, t - 1\right] x_0$$

$$+ \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \sum_{j=1}^{w-1} \exp(-\alpha_0 wr).\ f_j((r-1)w + j-1) \qquad (2.8)$$

Now for the last relation we have the following cases:

1) If $\alpha > \alpha_0$ then by using formula (2.8) we get

$$\lim_{t\to\infty} y(t) \exp(-\alpha_0 t) = \infty .$$

This means that $y \notin E_{\alpha_0}$ but $y \in E_\alpha$ $(\alpha > \alpha_0)$.

2) If $\alpha = \alpha_0$ then from (2.8)

$$y(t) \exp(-\alpha_0 t) = \exp\left(w(1-\alpha) - 1 + \frac{i\theta t}{w}\right)\left(\frac{t}{w} - 1\right) x_0$$

$$+ \exp\left(\frac{i\theta t}{w}\right) \sum_{r=1}^{\frac{t}{w}} \sum_{j=1}^{w-1} \exp(-\alpha_0 wr)\ f_j((r-1)w + j-1).$$

Using the last equation we get

$$\lim_{t\to\infty} y(t) \exp(-\alpha_0 t) = \infty .$$

This means that $y \in E\,\alpha$ but $y \in E_\beta$ $(\beta > \alpha)$.

3) If $\alpha = \alpha_0$ then from (2.8) we have $y \notin E_\alpha$ but $y \in E_{\alpha_0}$.

This completes the proof.

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