Research Article

Existence of Random Attractor for Stochastic Fractional Long-Short Wave Equations with Periodic Boundary Condition

Na Liu[1] and Jie Xin[1,2]

[1]*School of Mathematics and Statistics Science, Ludong University, Yantai City 264025, China*
[2]*School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China*

Correspondence should be addressed to Jie Xin; fdxinjie@sina.com

We consider the asymptotic behaviors of stochastic fractional long-short equations driven by a random force. Under a priori estimates in the sense of expectation, using Galerkin approximation by the stopping time and the Borel-Cantelli lemma, we prove the existence and uniqueness of solutions. Then a global random attractor and the existence of a stationary measure are obtained via the Birkhoff ergodic theorem and the Chebyshev inequality.

1. Introduction

Long-short wave resonance equations arise in the study of the interaction of the surface waves with both gravity and capillary modes presence and also in the analysis of internal waves, as well as Rossby wave in [1]. In the plasma physics, they describe the resonance of the high-frequency electron plasma oscillation and associated low-frequency ion density perturbation in [2]. Long wave and short wave equations with periodic boundary condition have attracted considerable attention as a result of their rich physical and mathematical properties. Guo [3, 4] proved the existence of global solutions for long-short wave equations and generalized long-short wave equations. The existence of global attractor was studied in [5–9].

The stochastic partial differential equation (SPDE) is a kind of partial differential equation with random term and random coefficients, which come from the random environmental effects and the errors of measurement. SPDE is used to describe better complex phenomenon, for example, quantum field theory, statistical mechanics, and financial mathematics; see [10–12] and so on. In [13–19], the authors obtained the existence and uniqueness of the solution and of attractors for SPDEs.

In this paper, we consider that the random environmental effects and the errors of measurement are included into the model of fractional long-short wave equations. More specifically, we study the following equations:

$$iu_t - (-\Delta)^{\alpha} u - nu + i\delta u = f + \dot{W}_1, \tag{1}$$

$$n_t + \beta n + |u|^2_x = g + \dot{W}_2 \tag{2}$$

with the initial condition

$$\begin{aligned} u(x,0) &= u_0(x), \\ n(x,0) &= n_0(x), \end{aligned} \tag{3}$$

and periodic boundary condition

$$\begin{aligned} u(x,t) &= u(x + 2\pi, t), \\ n(x,t) &= n(x + 2\pi, t) = 0, \end{aligned} \tag{4}$$

where $x \in D = [0, 2\pi]$ in $\mathbb{R}^1$; $t \geq \tau \in \mathbb{R}_+$; $\delta, \beta, \alpha > 0$. The unknown complex valued function $u(x,t)$ is short surface wave packet and the unknown real valued function $n(x,t)$ is a long interfacial wave. W_1 and W_2 are independent $L^2(D)$ value Wiener processes which are from the errors of measurement or the random environmental effects and can be seen in detail in the next section.

Since the solution $u(x,t)$, if it exists, is a 2π-periodic function, we have the Fourier expansion

$$u(x,t) = \sum_{k \in \mathbb{Z}} a_k(t) e^{ikx}, \tag{5}$$

where $a_k(t) : \mathbb{R}^+ \to \mathbb{R}, \ x \in \mathbb{R}$.

Hence,

$$\partial_x u = \sum_{k \in \mathbb{Z}} ik a_k(t) e^{ikx}, \tag{6}$$

and $(-\triangle)^\alpha u$ is defined by

$$(-\triangle)^\alpha u = \sum_{k \in \mathbb{Z}} |k|^{2\alpha} a_k(t) e^{ikx}. \tag{7}$$

Since

$$\left| e^{irx} \right|^2 = 1,$$
$$\int_D e^{irx} dx = 0, \tag{8}$$
$$\forall r \in \mathbb{R},$$

the following definitions make sense. Let

$$A = \left\{ u \mid u = \sum_{k \in \mathbb{Z}} a_k(t) e^{ikx}, \ \sum_{k \in \mathbb{Z}} |k|^{4\alpha} a_k^2(t) \right.$$
$$\left. < \infty, \ \sum_{k \in \mathbb{Z}} |a_k(t)|^2 < \infty \right\}, \tag{9}$$

and let $H^{2\alpha}$ be a complete distance space of the set A under the norm

$$\|u\|_{H^{2\alpha}} = \left(2\pi \sum_{k \in \mathbb{Z}} |k|^{4\alpha} a_k^2(t) \right)^{1/2}$$
$$+ \left(2\pi \sum_{k \in \mathbb{Z}} |a_k(t)|^2 \right)^{1/2}. \tag{10}$$

It is easy to get it to be a Banach space, and that $\forall u \in H^{2\alpha}, \ u$ is space-periodic with the period 2π and its 2α order derivatives are in $L^2_{\text{per}}(D)$. And $\forall u, v \in H^\alpha$,

$$\frac{d}{dt} uv = u \frac{d}{dt} v + v \frac{d}{dt} u,$$
$$((-\triangle)^\alpha u, v) = ((-\triangle)^{\alpha_1} u, (-\triangle)^{\alpha_2} v), \tag{11}$$

when $\alpha_1 + \alpha_2 = \alpha, \ 0 \le \alpha_1, \alpha_2 \le \alpha$. $H^{2\alpha}$ is a Hilbert space with the inner product

$$(u, v)_{H^{2\alpha}} = ((-\triangle)^\alpha u, (-\triangle)^\alpha v). \tag{12}$$

The rest of this paper is arranged as follows. In Section 2, we present some preliminaries results. In Section 3, we give a series of time uniform a priori estimates in different energy spaces which will be used to prove our main results; see [20]. In Section 4, we show the existence and uniqueness of solutions for (1)-(2). In Section 5, the random weak attractor and the stationary measure are constructed.

2. Preliminary

In this paper, we outline the variational framework for studying problems (1)-(2) and list some preliminary lemmas which will be used later.

We define a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \ge 0}, \mathbb{P})$. $\mathbb{E}$ denotes the expectation operator with respect to $\mathbb{P}$. Stochastic terms $W_1(t)$ and $W_2(t)$ are defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \ge 0}, \mathbb{P})$ by

$$W_1(t) = q_1(x) \omega_1(t),$$
$$W_2(t) = q_2(x) \omega_2(t), \tag{13}$$

where $\omega_1(t)$ is a standard complex valued Wiener process, $\omega_2(t)$ is a standard real valued Winer process independent of $\omega_1(t)$, and $q_1(x), q_2(x)$ are sufficiently smooth functions in a manner. The different inner product spaces for the solution (u, n) of (1)-(2) are defined as

$$V_0 = H_0^\alpha(D) \times L^2(D),$$
$$V_1 = \left(H^{2\alpha}(D) \cap H_0^\alpha(D) \right) \times H_0^1(D),$$
$$V_2 = \left\{ \varphi \in H^{3\alpha}(D) \cap H_0^\alpha(D) : (-\triangle)^\alpha \varphi \in H_0^\alpha(D) \right\}$$
$$\times \left(H^2(D) \cap H_0^1(D) \right). \tag{14}$$

Endow each $V_i \ (i = 0, 1, 2)$ with the usual norm, and satisfy $V_2 \subset V_1 \subset V_0$ with compact embedding.

Let $(\mathscr{X}, \| \cdot \|_{\mathscr{X}}) \subset (\mathscr{Y}, \| \cdot \|_{\mathscr{Y}}) \subset (\mathscr{Z}, \| \cdot \|_{\mathscr{Z}})$ be three Banach reflective spaces which satisfy $\mathscr{X} \subset \mathscr{Y}$ with compact and dense embedding. The Banach space can be defined as follows:

$$\mathscr{G} = \left\{ v : v \in L^2(0, T; \mathscr{X}), \frac{dv}{dt} \in L^2(0, T; \mathscr{Z}) \right\} \tag{15}$$

endowed with the natural norm

$$\|v\|_{\mathscr{G}}^2 = \int_0^T \|v\| \mathscr{X} \, ds + \int_0^T \left\| \frac{dv}{dt} \right\|^2 \mathscr{Z} \, ds, \quad v \in \mathscr{G}. \tag{16}$$

We obtain the following lemma with regard to compactness result by [21].

Lemma 1. *If K is bounded in $\mathscr{G}$, then K is precompact in $L^2(0, T; \mathscr{Y})$.*

In order to get maximal estimates on stochastic integrals, we need another lemma. U and H are Hilbert spaces which are separable and W is a Q-Wiener process on U_0 with $U_0 = Q^{1/2}$. Let $L_2^0 = L_2^0(U_0, H)$ be the space of Hilbert-Schmidt operators from U_0 to H. For such operators, we obtain the following lemma by [22].

Lemma 2. *For any* $r \geq 1$ *and any* L_2^0-*valued predictable process* $\Phi(t)$, $t \in [0, T]$, *we get*

$$\mathbb{E} \int_0^T \Phi(s)\, dW(s) = 0, \tag{17}$$

$$\mathbb{E}\left(\sup \left| \int_0^s \Phi(\sigma)\, dW(\sigma) \right|^{2r} \right)$$

$$\leq c_r \sup \mathbb{E}\left(\left| \int_0^s \Phi(\sigma)\, dW(\sigma) \right|^{2r} \right) \tag{18}$$

$$\leq C_r \mathbb{E}\left(\int_0^t \|\Phi(s)\|_{L_2^0}^2 \right),$$

where c_r *and* C_r *are some positive constants dependent on* r.

3. Uniform A Priori Estimates in Time

3.1. A Priori Estimates in V_0

Lemma 3. *Provided that* $u_0, f, q_1 \in L^2(D)$, *then for any* $T > 0$ *and* $p \geq 1$, *we obtain* $u \in L^{2p}(\Omega; L^\infty([0,T]; L^2(D))) \cap L^\infty([0,\infty); L^{2p}(\Omega; L^2(D)))$.

Proof. Taking the inner product of (1) with $2u$, we get

$$\left(iu_t - (-\Delta)^\alpha u - nu + i\delta u, 2u\right) = \left(f + \dot{W}_1, 2u\right). \tag{19}$$

Taking the imaginary part of (19) and applying the Itô formula to $\|u\|^2$, one gets

$$\frac{d}{dt}\|u\|^2 = -2\delta\|u\|^2 + 2\,\mathrm{Im} \int_D f\bar{u}\, dx$$
$$+ 2\,\mathrm{Im} \int_D \bar{u}\dot{W}_1 dx + \|q_1\|^2. \tag{20}$$

By (20), using Hölder's and Young's inequalities, we can obtain

$$\frac{d}{dt}\|u\|^2 + \delta\|u\|^2 \leq \frac{1}{\delta}\|f\|^2 + \|q_1\|^2 + 2\,\mathrm{Im} \int_D \bar{u}\dot{W}_1 dx. \tag{21}$$

On the one hand, multiplying by $e^{\delta t}$ and integrating from 0 to t on both sides of (21), we get

$$\mathbb{E}\|u\|^2 \leq e^{-\delta t}\mathbb{E}\|u_0\|^2 + \frac{1}{\delta^2}\|f\|^2 + \frac{1}{\delta}\|q_1\|^2 \leq C,$$
$$t > 0, \tag{22}$$

where C is independent of T.

On the other hand, integrating from 0 to t and taking the supremum and the expectation on both sides of (21), we can obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} \|u\|^2 \leq \mathbb{E}\|u_0\|^2 + \left(\frac{1}{\delta}\|f\|^2 + \|q_1\|^2\right) T$$
$$+ \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \mathrm{Im} \int_D \bar{u}\dot{W}_1 dx\, ds \right|^2 + 1. \tag{23}$$

By Lemma 2, for any positive constant C, we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \mathrm{Im} \int_D \bar{u}\dot{W}_1 dx\, ds \right|^2 \leq C\|q_1\|^2 \mathbb{E} \int_0^T \|u\|^2 ds. \tag{24}$$

From (24), for any $T > 0$, there exists a positive constant C_T such that

$$\mathbb{E} \sup_{0 \leq t \leq T} \|u\|^2 \leq \mathbb{E}\|u_0\|^2 + \left(\frac{1}{\delta}\|f\|^2 + \|q_1\|^2\right) T$$
$$+ C\|q_1\|^2 \mathbb{E} \int_0^T \|u\|^2 ds + 1 \tag{25}$$
$$\leq C_T \left(\mathbb{E}\|u_0\|^4 + \|f\|^4 + \|q_1\|^4 + 1\right).$$

On the basis of the above estimates, we can further give an estimate of $\|u(t)\|_0^{2p}$ for any $p \geq 1$. Applying the Itô formula and Hölder's inequality, we obtain

$$\frac{d}{dt}\|u\|^{2p} \leq -\frac{\delta p}{2}\|u\|^{2p} + c\left(\|f\|^{2p} + \|q_1\|^{2p}\right)$$
$$+ 2p\|u\|^{2(p-1)}\,\mathrm{Im} \int_D \bar{u}\dot{W}_1 dx. \tag{26}$$

On the one hand, multiplying by $e^{(\delta p/2)t}$, integrating from 0 to t, and taking the expectation on both sides of (26), we can have

$$\mathbb{E}\|u\|^{2p} \leq e^{-(\delta p/2)t}\mathbb{E}\|u_0\|^{2p} + c\left(\|f\|^{2p} + \|q_1\|^{2p}\right) \leq C,$$
$$t > 0, \tag{27}$$

where C is independent of T.

On the other hand, by (26), we have

$$\frac{d}{dt}\|u\|^{2p} \leq c\left(\|f\|^{2p} + \|q_1\|^{2p}\right)$$
$$+ 2p\|u\|^{2p-1}\,\mathrm{Im} \int_D \bar{u}\dot{W}_1 dx. \tag{28}$$

Integrating from 0 to t and taking the supremum and the expectation on both sides of (28), we get

$$\mathbb{E} \sup_{0 \leq t \leq T} \|u\|^{2p}$$
$$\leq \mathbb{E}\|u(0)\|^{2p} + c\left(\|f\|^{2p} + \|q_1\|^{2p}\right) T + 1 \tag{29}$$
$$+ \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t p\|u\|^{2(p-1)}\,\mathrm{Im} \int_D \bar{u}\dot{W}_1 dx\, ds \right|^2.$$

By Lemma 2, for any positive constant C, we have

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t p\|u\|^{2(p-1)}\,\mathrm{Im} \int_D \bar{u}\dot{W}_1 dx\, ds \right|^2$$
$$\leq Cp^2\|q_1\|^2 \mathbb{E} \int_0^T \|u\|^{4p-2} ds, \tag{30}$$

where $2p - 1 \geq 1$. Then inserting (30) into (29), and by (27), for any $T > 0$, we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} \|u\|^{2p} \leq C_T (u_0, f, q_1), \tag{31}$$

where $C_T(u_0, f, q_1)$ is a positive constant depending on u_0, f, q_1, and T.   □

Lemma 4. *Provided that $(u_0, n_0) \in V_0$, $q_1 \in H_0^\alpha(D)$, $q_2 \in L^2(D)$, $f, g \in L^2(D)$, then for any $T > 0$ and $p \geq 1$, we have $(u, n) \in L^\infty((0, \infty); L^{2p}(\Omega; V_0)) \cap L^{2p}(\Omega; L^\infty(0, T; V_0))$.*

Proof. Taking the inner product of (2) with $2n$, we have

$$\left(n_t + \beta n + |u|_x^2, 2n \right) = \left(g + \dot{W}_2, 2n \right). \tag{32}$$

Applying the Itô formula to $\|n\|^2$, since $n_t = -|u|_x^2 - \beta n + g + \dot{W}_2$, we get

$$\frac{d}{dt} \|n\|^2 - \|q_2\|^2 + 2 \int_D n |u|_x^2 \, dx + 2\beta \|n\|^2$$
$$- 2 \int_D gn \, dx - 2 \int_D n \dot{W}_2 \, dx = 0. \tag{33}$$

From (1), we get that

$$\int_D n |u|_x^2 \, dx = 2 \operatorname{Re} \int_D \overline{u}_x (un) \, dx$$

$$= -2 \operatorname{Re} \int_D \overline{u}_x \left(-iu_t + (-\Delta)^\alpha u - i\delta u + f - \dot{W}_1 \right) dx \tag{34}$$

$$= \int_D i\delta (u_x \overline{u} - \overline{u}_x u) \, dx - 2 \operatorname{Re} \int_D u \overline{f}_x \, dx$$

$$- 2 \operatorname{Re} \int_D u \overline{\dot{W}}_{1x} \, dx.$$

Notice that

$$\frac{d}{dt} \int_D i (u_x \overline{u} - u \overline{u}_x) \, dx$$

$$= 2 \int_D i (u_x \overline{u}_t - u_t \overline{u}_x) \, dx + 2 \operatorname{Im} \int_D q_1 \overline{q}_{1x} \, dx. \tag{35}$$

From (33), (34), and (35), we can have

$$\frac{d}{dt} \left(\|n\|^2 + \int_D i (u_x \overline{u} - u \overline{u}_x) \, dx \right) + 2\beta \|n\|^2$$

$$= -2 \int_D i\delta (u_x \overline{u} - \overline{u}_x u) \, dx + 4 \operatorname{Re} \int_D u \overline{f}_x \, dx$$

$$+ 2 \operatorname{Im} \int_D q_1 \overline{q}_{1x} \, dx + \|q_2\|^2 + 4 \operatorname{Re} \int_D u \overline{\dot{W}}_{1x} \, dx \tag{36}$$

$$+ 2 \int_D gn \, dx + 2 \int_D n \dot{W}_2 \, dx.$$

Applying the Itô formula to $\|u_x\|^2$, and taking the inner product of (1) with $(u_t + \delta u)$, we obtain

$$(iu_t - (-\Delta)^\alpha u - nu + i\delta u, (u_t + \delta u))$$

$$= 2 \left(f + \dot{W}_1, (u_t + \delta u) \right). \tag{37}$$

Taking the real part of (37), we get

$$\frac{d}{dt} \left\| (-\Delta)^{\alpha/2} u \right\|^2 + 2\delta \left\| (-\Delta)^{\alpha/2} u \right\|^2 + 2 \operatorname{Re} \int_D n u \overline{u}_t \, dx$$

$$+ 2\delta \int_D n |u|^2 \, dx + 2 \operatorname{Re} \int_D (\overline{u}_t + \delta \overline{u}) \dot{W}_1 \, dx \tag{38}$$

$$+ 2 \operatorname{Re} \int_D f \overline{u}_t \, dx + 2\delta \operatorname{Re} \int_D f \overline{u} \, dx = \|q_{1x}\|^2.$$

It is easy to check that

$$\frac{d}{dt} \int_D n |u|^2 \, dx = \int_D n_t |u|^2 \, dx + 2 \operatorname{Re} \int_D n u_t \overline{u} \, dx$$

$$+ 2 \int_D q_2 \operatorname{Im} (q_1 \overline{u}) \, dx. \tag{39}$$

Otherwise, by (2), we can have $n_t = -|u|_x^2 - \beta n + g + \dot{W}_2$. So

$$\int_D n_t |u|^2 \, dx = \int_D \left(-|u|_x^2 - \beta n + g + \dot{W}_2 \right) |u|^2 \, dx$$

$$= -\beta \int_D n |u|^2 \, dx + \int_D g |u|^2 \, dx + \int_D \dot{W}_2 |u|^2 \, dx,$$

$$- 2 \operatorname{Re} \int_D (\overline{u}_t + \delta \overline{u}) \dot{W}_1 \, dx$$

$$= 2 \operatorname{Im} \int_D (i\overline{u}_t + i\delta \overline{u}) \dot{W}_1 \, dx \tag{40}$$

$$= 2 \operatorname{Im} \int_D \left(\overline{f} + \overline{\dot{W}}_1 + (-\Delta)^\alpha \overline{u} + n\overline{u} \right) \dot{W}_1 \, dx$$

$$= 2 \operatorname{Im} \int_D (-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2} \dot{W}_1 \, dx$$

$$+ 2 \operatorname{Im} \int_D \left(n\overline{u} + \overline{f} \right) \dot{W}_1 \, dx.$$

From (2), we can obtain

$$\frac{d}{dt} \left(\left\| (-\Delta)^{\alpha/2} u \right\|^2 + 2 \operatorname{Re} \int_D f \overline{u} \, dx + \int_D n |u|^2 \, dx \right)$$

$$+ 2\delta \left\| (-\Delta)^{\alpha/2} u \right\|^2$$

$$= -2\delta \int_D n |u|^2 \, dx - 2\delta \operatorname{Re} \int_D f \overline{u} \, dx + \|q_{1x}\|^2$$

$$+ 2 \operatorname{Im} \int_D (-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2} \dot{W}_1 \, dx \tag{41}$$

$$+ 2 \operatorname{Im} \int_D \left(n\overline{u} + \overline{f} \right) \dot{W}_1 \, dx + 2 \int_D q_2 \operatorname{Im} (q_1 \overline{u}) \, dx$$

$$- \beta \int_D n |u|^2 \, dx + \int_D g |u|^2 \, dx + \int_D \dot{W}_2 |u|^2 \, dx.$$

Using Hölder's inequality, the Gagliardo-Nirenberg inequality, and Young's inequality, we can estimate each term. Now, let $H_0 = \|(-\Delta)^{\alpha/2} u\|^2 + 2 \operatorname{Re} \int_D f \overline{u} \, dx + \int_D n |u|^2 \, dx + \|n\|^2 +$

$\int_D i(u_x \bar{u} - u\bar{u}_x)dx$ and take $\eta = \min\{\delta, \beta\}$. Putting (36) and (41) into the result, we get

$$
\begin{aligned}
\frac{d}{dt}H_0 + \eta H_0 &\leq c\left(f, g, q_1, q_2\right) + c\|u\|^6 \\
&\quad + \int_D |u|^2 \dot{W}_2 dx + 2\int_D n\dot{W}_2 dx \\
&\quad - 2\,\mathrm{Im}\int_D (-\Delta)^{\alpha/2} u\,(-\Delta)^{\alpha/2}\dot{W}_1 dx \quad (42) \\
&\quad - 2\,\mathrm{Im}\int_D \left(n\bar{u} + \bar{f}\right)\dot{W}_1 dx \\
&\quad + 4\,\mathrm{Re}\int_D u\overline{\dot{W}}_{1x}dx.
\end{aligned}
$$

Multiplying by $e^{\eta t}$, integrating from 0 to t, and taking expectation on both sides of (42) yield

$$
\begin{aligned}
\mathbb{E}H_0 &\leq e^{-\eta t}\mathbb{E}H_0(0) + c\left(f, g, q_1, q_2\right) \\
&\quad + c\mathbb{E}\int_0^t e^{-\eta(t-s)}\|u\|^6 ds.
\end{aligned} \quad (43)
$$

From (27), we can estimate (43) and obtain

$$
\begin{aligned}
\mathbb{E}H_0(t) &\leq e^{-\beta t}\mathbb{E}H_0(0) + c\left(f, g, q_1, q_2, u_0\right) \\
&\leq c\left(f, g, q_1, q_2, u_0\right) \leq C, \quad t > 0,
\end{aligned} \quad (44)
$$

where C is independent of T.

Since

$$
\begin{aligned}
H_0(t) &\geq \frac{1}{2}\left(\left\|(-\Delta)^{\alpha/2} u\right\|^2 + \|n\|^2\right) \\
&\quad - c\left(\|u\|^2 + \|g\|^2 + \|u\|^6\right),
\end{aligned} \quad (45)
$$

for any $t > 0$, we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\left\|(-\Delta)^{\alpha/2} u\right\|^2 + \|n\|^2\right) \\
&\leq c\mathbb{E}\left(\|u\|^2 + \|g\|^2 + \|u\|^6\right) + c\mathbb{E}H_0(t) \leq C,
\end{aligned} \quad (46)
$$

where C is independent of T.

Besides, we estimate $H_0^p(t)$ for $p \geq 1$. Firstly, applying the Itô formula to $H_0^p(t)$, we have

$$
\begin{aligned}
\frac{d}{dt}H_0^p(t) &\leq -\frac{\eta p}{2}H_0^p(t) + c\left(\|u\|^{6p} + c\right) + pH_0^{p-1}(t) \\
&\quad \cdot \left(\int_D |u|^2 \dot{W}_2 dx + 2\int_D n\dot{W}_2 dx\right) + pH_0^{p-1}(t) \\
&\quad \cdot \left(-2\,\mathrm{Im}\int_D (-\Delta)^{\alpha/2} u\,(-\Delta)^{\alpha/2}\dot{W}_1 dx \right. \\
&\quad \left. - 2\,\mathrm{Im}\int \left(n\bar{u} + \bar{f}\right)\dot{W}_1 dx + 4\,\mathrm{Re}\int_D u\overline{\dot{W}}_{1x}dx\right).
\end{aligned} \quad (47)
$$

Multiplying by $e^{(\eta p/2)t}$, integrating from 0 to t, and taking expectation on both sides of (47) yield

$$
\begin{aligned}
\mathbb{E}H_0^p(t) &\leq e^{(-\eta p/2)t}\mathbb{E}H_0^p(0) + c \\
&\quad + c\mathbb{E}\int_0^t e^{(-\eta p/2)(t-s)}\|u\|^{6p} ds.
\end{aligned} \quad (48)
$$

From (27) and (48), we obtain

$$
\mathbb{E}\left(H_0^p(t)\right) \leq e^{(\eta p/2)t}\mathbb{E}H_0^p(0) + c \leq C, \quad t > 0, \quad (49)
$$

where C is independent of T.

Consequently, from (45), we obtain

$$
\mathbb{E}\left(\left\|(-\Delta)^{\alpha/2} u\right\|^2 + \|n\|^2\right) \leq C, \quad t > 0. \quad (50)
$$

On the one side, integrating from 0 to t on both sides of (42), we can deduce

$$
\begin{aligned}
H_0(t) &\leq H_0(0) + c\left(f, g, q_1, q_2\right)t + c\int_0^t \|u\|^6 ds \\
&\quad + \int_0^t \left(\int_D |u|^2 \dot{W}_2 dx + 2\int_D n\dot{W}_2 dx\right)ds \\
&\quad + \int_0^t \left(-2\,\mathrm{Im}\int_D (-\Delta)^{\alpha/2} u\,(-\Delta)^{\alpha/2}\dot{W}_1 dx\right. \\
&\quad - 2\,\mathrm{Im}\int_D \left(n\bar{u} + \bar{f}\right)\dot{W}_1 dx \\
&\quad \left. + 4\,\mathrm{Re}\int_D u\overline{\dot{W}}_{1x}dx\right)ds.
\end{aligned} \quad (51)
$$

Taking the supremum and expectation on both sides of (51) yields

$$
\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T} H_0(t) \leq \mathbb{E}H_0(0) + c\left(f, g, q_1, q_2\right)T + c\mathbb{E} \\
&\quad \cdot \sup_{0\leq t\leq T}\int_0^t \|u\|^6 ds + \mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t \int_D |u|^2 \dot{W}_2 dx\,ds\right|^2 \\
&\quad + \mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t \int_D n\dot{W}_2 dx\,ds\right|^2 + \mathbb{E} \\
&\quad \cdot \sup_{0\leq t\leq T}\left|\int_0^t \mathrm{Im}\int_D \mathrm{Im}\int_D (-\Delta)^{\alpha/2} u\,(-\Delta)^{\alpha/2}\dot{W}_1 dx\,ds\right|^2 \\
&\quad + \mathbb{E}\sup_{0\leq t\leq T}\left|\int_0^t \mathrm{Im}\int_D \left(n\bar{u} + \bar{f}\right)\dot{W}_1 dx\,ds\right|^2 + \mathbb{E} \\
&\quad \cdot \sup_{0\leq t\leq T}\left|\int_0^t \mathrm{Re}\int_D u\dot{W}_{1x}dx\,ds\right|^2.
\end{aligned} \quad (52)
$$

And then by estimating each term of the right hand side of (52), we obtain

$$
\mathbb{E}\sup_{0\leq t\leq T} H_0(t) \leq C_T\left(E_0, f, g, q_1, q_2\right). \quad (53)
$$

On the other side, for $H_0^p(t)$ ($p \geq 1$), integrating from 0 to t and taking the supremum and the expectation on both sides of (47) yield

$$\mathbb{E} \sup_{0 \leq t \leq T} H_0^p(t) \leq \mathbb{E} H_0^p(0) + c(q_1, q_2) T + c\mathbb{E}$$

$$\cdot \sup_{0 \leq t \leq T} \int_0^t \|u\|^{6p} \, ds + c + p^2 \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t H_0^{p-1}(s) \right.$$

$$\cdot \int |u|^2 \, \dot{W}_2 dx \, ds \Big|^2 + p^2 \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t 2H_0^{p-1}(s) \right.$$

$$\cdot \int n\dot{W}_2 dx \, ds \Big|^2 + p^2 \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t H_0^{p-1}(s) \, 4 \right. \tag{54}$$

$$\cdot \operatorname{Re} \int_D (-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2} \dot{W}_1 dx \, ds \Big|^2 + p^2 \mathbb{E}$$

$$\cdot \sup_{0 \leq t \leq T} \left| \int_0^t 2H_0^{p-1}(s) \operatorname{Im} \int_D u\overline{W}_{1x} dx \, ds \right|^2 + p^2 \mathbb{E}$$

$$\cdot \sup_{0 \leq t \leq T} \left| \int_0^t 2H_0^{p-1}(s) \operatorname{Im} \int (n\overline{u} + \overline{f}) \dot{W}_1 dx \, ds \right|^2.$$

Now we estimate each term of (54). For the third term on the right hand of (54),

$$c\mathbb{E} \sup_{0 \leq t \leq T} \int_0^t \|u\|^{6p} \, ds \leq c\mathbb{E} \int_0^T \|u\|^{6p} \, ds \leq C_T. \tag{55}$$

For the fifth and sixth terms on the right hand of (54), we have

$$p^2 \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t H_0^{p-1}(s) \int |u|^2 \, \dot{W}_2 dx \, ds \right|^2 + p^2 \mathbb{E}$$

$$\cdot \sup_{0 \leq t \leq T} \left| \int_0^t 2H_0^{p-1}(s) \int n\dot{W}_2 dx \, ds \right|^2 \leq c \|q_2\|^2 \tag{56}$$

$$\cdot \mathbb{E} \int_0^T \left(H_0^{2p}(s) + \|u\|^{2p} + \|u\|^{6p} + \|f\|^{2p} \right) ds$$

$$\leq C_T.$$

For the seventh and eight terms on the right hand of (54), using a similar method, we estimate

$$p^2 \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t H_0^{p-1}(s) \, 4 \right.$$

$$\cdot \operatorname{Re} \int_D (-\Delta)^{\alpha/2} u (-\Delta)^{\alpha/2} \dot{W}_1 dx \, ds \Big|^2 + p^2 \mathbb{E}$$

$$\cdot \sup_{0 \leq t \leq T} \left| \int_0^t 2H_0^{p-1}(s) \operatorname{Im} \int_D u\overline{W}_{1x} dx \, ds \right|^2 \tag{57}$$

$$\leq c \left\| (-\Delta)^{\alpha/2} q_1 \right\|^2 \mathbb{E} \int_0^T \left(H_0^{2p}(s) + \|u\|^{2p} + \|u\|^{6p} \right.$$

$$+ \|f\|^{2p} \right) ds \leq C_T.$$

For the last term on the right hand of (54), we estimate

$$p^2 \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t 2H_0^{p-1}(s) \operatorname{Im} \int (n\overline{u} + \overline{f}) \dot{W}_1 dx \, ds \right|^2$$

$$\leq c \left\| (-\Delta)^{\alpha/2} q_1 \right\|^2 \tag{58}$$

$$\cdot \mathbb{E} \int_0^T \left(H_0^{2p}(s) + \|u\|^{2p} + \|u\|^{6p} + \|f\|^{2p} \right) ds$$

$$\leq C_T.$$

By the above estimates, we clearly obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} H_0^p(t) \leq \mathbb{E} H_0^p(0) + C_T(q_1, q_2, E_0, n_1, n_0) \tag{59}$$

$$< C_T(q_1, q_2, E_0, n_1, n_0).$$

In addition, by (46), we can get

$$\mathbb{E} \sup_{0 \leq t \leq T} \left(\left\| (-\Delta)^{\alpha/2} u \right\|^{2p} + \|n\|^{2p} \right)$$

$$\leq c\mathbb{E} \sup_{0 \leq t \leq T} H_0^p(t) \tag{60}$$

$$+ c\mathbb{E} \sup_{0 \leq t \leq T} \left(\|u\|^{2p} + \|u\|^{6p} + \|f\|^{2p} \right) \leq C_T.$$

Then we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T} \|(u, n)\|_{V_0}^{2p} \leq C_T(f, g, q_1, q_2, u_0, n_0). \tag{61}$$

Then the proof is complete. □

3.2. A Priori Estimates in V_1

Lemma 5. *Provided that $f, g \in H^1(D)$, $(u_0, n_0) \in V_1$, $q_1 \in H^{2\alpha}$, $q_2 \in H^1(D)$, then for any $T > 0$ and $p \geq 1$, we have $(n, u) \in L^{\infty}((0, \infty); L^{2p}(\Omega; V_1)) \cap L^{2p}(\Omega; L^{\infty}(0, T; V_1))$.*

Proof. Taking the inner product of (1) with $(-\Delta)^{\alpha} u_t$, we get

$$(iu_t - (-\Delta)^{\alpha} u - nu + i\delta u, (-\Delta)^{\alpha} u_t)$$

$$= (f + \dot{W}_1, (-\Delta)^{\alpha} u_t). \tag{62}$$

Applying the Itô formula to $\|(-\Delta)^{\alpha} u\|^2$, we have

$$\frac{d}{dt} \left\| (-\Delta)^{\alpha} u \right\|^2 = 2 \operatorname{Re} \int_D (-\Delta)^{\alpha} u (-\Delta)^{\alpha} \overline{u}_t dx \tag{63}$$

$$+ \left\| (-\Delta)^{\alpha} q_1 \right\|^2.$$

Since $u_t = i(-(-\Delta)^{\alpha} u - nu + i\delta u - f - \dot{W}_1)$ and

$$\frac{d}{dt} \operatorname{Re} \int nu (-\Delta)^{\alpha} \overline{u} \, dx = \operatorname{Re} \int_D \left(n_t u (-\Delta)^{\alpha} \overline{u} \right.$$

$$+ nu_t (-\Delta)^{\alpha} \overline{u} + nu (-\Delta)^{\alpha} \overline{u}_t + q_2 (-\Delta)^{\alpha} \overline{q}_1 \tag{64}$$

$$+ i (q_2 (-\Delta)^{\alpha} \overline{q}_1 u - q_2 q_1 (-\Delta)^{\alpha} \overline{u})) dx,$$

we obtain

$$
\frac{d}{dt}\left(\left\|(-\Delta)^\alpha u\right\|^2\right.
$$

$$
- 2\operatorname{Re}\int_D nu\,(-\Delta)^\alpha \overline{u}\,dx + 2\operatorname{Re}\int_D f\,(-\Delta)^\alpha \overline{u}\,dx\right)
$$

$$
+ 2\alpha\left\|(-\Delta)^\alpha u\right\|^2 = \left\|(-\Delta)^\alpha q_1\right\|^2 + 2\left\|q_{1x}\right\|^2 + 2
$$

$$
\cdot \operatorname{Re}\int_D i\left(q_2 q_1\,(-\Delta)^\alpha \overline{u} - q_2\,(-\Delta)^\alpha \overline{q}_1 u\right)dx - 2
$$

$$
\cdot \operatorname{Re}\int_D nu_t\,(-\Delta)^\alpha u\,dx + 2\delta\operatorname{Re}\int_D nu\,(-\Delta)^\alpha \overline{u}\,dx
$$

$$
- 2\delta\operatorname{Re}\int_D f\,(-\Delta)^\alpha u\,dx - 2
$$

$$
\cdot \operatorname{Re}\int_D \dot{W}_1\left((-\Delta)^\alpha \overline{u}_t + \delta\,(-\Delta)^\alpha \overline{u}\right)dx - 2
$$

$$
\cdot \operatorname{Re}\int_D n_t u\,(-\Delta)^\alpha \overline{u}\,dx.
$$

(65)

According to the above inequality, we can estimate each term on the right hand side of (65) using Höder's inequality, the Gagliardo-Nirenberg, and Young's inequality. So, by (65), we get

$$
\frac{d}{dt}\left(\left\|(-\Delta)^\alpha u\right\|^2 - 2\operatorname{Re}\int_D nu\,(-\Delta)^\alpha \overline{u}\,dx + 2\right.
$$

$$
\cdot \operatorname{Re}\int_D f\,(-\Delta)^\alpha \overline{u}\,dx\right) + 2\alpha\left\|(-\Delta)^\alpha u\right\|^2
$$

$$
\leq \frac{\delta}{2}\left\|(-\Delta)^\alpha u\right\|^2 + \frac{\beta}{2}\left\|n_x\right\|^2 + c\left(\left\|q_2\right\|_{H_1}^4 + \left\|q_1\right\|_{H^{2\alpha}}^2\right.
$$

$$
+ \left\|g\right\|^4 + \left\|f\right\|^8 + \left\|u\right\|_{H_1}^8 + \left\|n\right\|^{12}\right) - 2
$$

$$
\cdot \operatorname{Re}\int_D \dot{W}_1\,(-\Delta)^\alpha \overline{u}_t y - 2\operatorname{Re}\int_D u\,(-\Delta)^\alpha \overline{u}\dot{W}_2 dx
$$

$$
+ 2\operatorname{Re}\int_D in\,(-\Delta)^\alpha \overline{u}\dot{W}_1 dx.
$$

(66)

Otherwise, applying the Itô formula to $\left\|n_x\right\|^2$, we have

$$
\frac{d}{dt}\left\|n_x\right\|^2 = 2\int_D n_x n_{xt} dx + \left\|q_{2x}\right\|^2.
$$

(67)

Since $n_t = g + \dot{W}_2 - \beta n - |u|_x^2$ and

$$
\frac{d}{dt}i\int_D \left(u_x\,(-\Delta)^\alpha \overline{u} - (-\Delta)^\alpha u\overline{u}_x\right)dx
$$

$$
= 2i\int_D \left(u_{xt}\,(-\Delta)^\alpha \overline{u} - \overline{u}_{xt}\,(-\Delta)^\alpha u\right)dx
$$

$$
+ 2i\operatorname{Re}\int_D q_{1x}\,(-\Delta)^\alpha \overline{q}_2 dx,
$$

(68)

we obtain

$$
\frac{d}{dt}\left(\left\|n_x\right\|^2 + i\int_D \left(u_x\,(-\Delta)^\alpha \overline{u} - (-\Delta)^\alpha u\overline{u}_x\right)dx\right)
$$

$$
+ 2\beta\left\|n_x\right\|^2
$$

$$
\leq \frac{\delta}{2}\left\|(-\Delta)^\alpha u\right\|^2 + \frac{\beta}{2}\left\|n_x\right\|^2
$$

$$
+ c\left(\left\|g_x\right\|^2 + \left\|f_x\right\|^2 + \left\|q_1\right\|_{H^{2\alpha}}^2 + \left\|q_{2x}\right\|^2\right)
$$

$$
+ 2\int_D n_x\dot{W}_{2x} dx + 4\operatorname{Re}\int_D (-\Delta)^\alpha \overline{u}\dot{W}_{1x} dx.
$$

(69)

Thus, by (66) and (69), taking $\eta = \min\{\delta, \beta\}$, and letting

$$
H_1\,(t) = \left\|(-\Delta)^\alpha u\right\|^2 + \left\|n_x\right\|^2 - 2\operatorname{Re}\int_D nu\,(-\Delta)^\alpha \overline{u}\,dx
$$

$$
+ 2\operatorname{Re}\int_D f\,(-\Delta)^\alpha \overline{u}\,dx
$$

$$
+ i\int_D \left(u_x\,(-\Delta)^\alpha \overline{u} - (-\Delta)^\alpha u\overline{u}_x\right)dx,
$$

(70)

we get

$$
\frac{d}{dt}H_1\,(t) + \eta H_1\,(t) \leq c\left(\left\|u\right\|_{H^1}^8 + \left\|n\right\|^{12} + \left\|q_2\right\|_{H_1}^4\right.
$$

$$
+ \left\|q_1\right\|_{H^{2\alpha}}^4 + \left\|g\right\|_{H^1}^4 + \left\|f\right\|_{H^1}^8 + 1\right) - 2
$$

$$
\cdot \operatorname{Re}\int_D \dot{W}_1\,(-\Delta)^\alpha \overline{u}_t y - 2\operatorname{Re}\int_D u\,(-\Delta)^\alpha \overline{u}\dot{W}_2 dx
$$

$$
+ 2\operatorname{Re}\int_D in\,(-\Delta)^\alpha \overline{u}\dot{W}_1 dx + 2\int_D n_x\dot{W}_{2x} dx + 4
$$

$$
\cdot \operatorname{Re}\int_D (-\Delta)^\alpha \overline{u}\dot{W}_{1x} dx.
$$

(71)

Multiplying by $e^{\eta t}$, integrating from 0 to t, and taking expecting on both sides of (71), by (46), we obtain

$$
\mathbb{E}H_1\,(t) \leq e^{-\beta t}\mathbb{E}H_1\,(0) + c\left(f, g, q_1, q_2, u_0\right)
$$

$$
\leq c\left(f, g, q_1, q_2, u_0\right) \leq C, \quad t > 0,
$$

(72)

where C is independent of T. Since

$$
H_1\,(t) \geq \frac{1}{2}\left(\left\|(-\Delta)^\alpha u\right\|^2 + \left\|n_x\right\|^2\right)
$$

$$
- c\left(\left\|u_x\right\|^8 + \left\|n\right\|^4 + 1 + \left\|f\right\|^2\right),
$$

(73)

for any $t > 0$, we obtain

$$
\mathbb{E}\left(\left\|(-\Delta)^\alpha u\right\|^2 + \left\|n_x\right\|^2\right)
$$

$$
\leq c\mathbb{E}\left(\left\|u_x\right\|^8 + \left\|n\right\|^4 + 1 + \left\|f\right\|^2\right) + c\mathbb{E}H_1\,(t) \leq C.
$$

(74)

where C is independent of T.

Next step, we estimate $H_1^p(t)$ for $p \geq 1$. On the one hand, applying the Itô formula to $H_1^p(t)$, we obtain

$$
\begin{aligned}
&\mathbb{E} H_1^p(t) \\
&\leq e^{(-\eta t/2)t} \mathbb{E} H_1^p(0) + c \\
&\quad + c\mathbb{E} \int_0^t e^{(-\eta t/2)(t-s)} \left(\left\| (-\Delta)^\alpha u \right\|^{8p} + \|n\|^{12p} + 1 \right) ds.
\end{aligned}
\tag{75}
$$

By (45), we obtain

$$
\mathbb{E} H_0^p(t) \leq e^{-(\eta p/2)t} \mathbb{E} H_0^p(0) + c \leq C, \quad t > 0, \tag{76}
$$

where C is independent of T. Thus, from (73), for any $t > 0$, we get

$$
\mathbb{E} \left(\left\| (-\Delta)^\alpha u \right\|^{2p} + \|n_x\|^{2p} \right) \leq C. \tag{77}
$$

Integrating from 0 to t and taking the supremum and expectation on both sides of (71), as with the estimates in Lemma 4 for each term, we deduce

$$
\mathbb{E} \sup_{0 \leq t \leq T} H_1(t) \leq C_T(f, g, q_1, q_2, u_0). \tag{78}
$$

On the other hand, for $H_0^p(t)$ ($p \geq 1$), integrating from 0 to t and taking the supremum and the expectation on both sides of (71), and estimating each term, we get

$$
\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} H_1^p(t) &\leq \mathbb{E} H_1^p(0) + C_T(f, g, q_1, q_2, u_0, n_0) \\
&\leq C_T(f, g, q_1, q_2, u_0, n_0).
\end{aligned}
\tag{79}
$$

Therefore, from (73), it is inferred that for $p \geq 1$

$$
\begin{aligned}
\mathbb{E} \sup_{0 \leq t \leq T} &\left(\left\| (-\Delta)^\alpha u \right\|^{2p} + \|n_x\|^{2p} \right) \\
&\leq C_T(f, g, q_1, q_2, u_0, n_0).
\end{aligned}
\tag{80}
$$

So then, we have

$$
\mathbb{E} \sup_{0 \leq t \leq T} \|(u, n)\|_{V_1}^{2p} \leq C_T(f, g, q_1, q_2, u_0, n_0), \tag{81}
$$

and the proof is complete. □

3.3. A Priori Estimates in V_2. By using the similar method and idea as Sections 3.1 and 3.2, we can achieve a priori estimates in V_2. For simplicity, we only provide the idea of the proof. Using (1)-(2) and applying the Itô formula to $\|(-\Delta)^\alpha n\|^2$ and $\|(-\Delta)^\alpha u_x\|^2$, respectively, we can get some inequalities by Höder's inequality, the Gagliardo-Nirenberg, and Young's inequality. After that, taking the supremum and expectation for inequalities and estimating on $\|(-\Delta)^\alpha n\|^2$ and $\|(-\Delta)^\alpha u_x\|^2$ by Gronwall-type, we can deduce the following lemma.

Lemma 6. *Provided that $(u_0, n_0) \in V_2$, $q_2 \in H^2(D)$, $q_1 \in H^{3\alpha}(D)$, and $f, g \in H^2(D)$, then for any $T > 0$ and $p \geq 1$, we have $(u, n) \in L^\infty(0, \infty; L^{2p}(\Omega; V_2)) \cap L^{2p}(\Omega; L^\infty(0, T; V_2))$.*

4. Proofs of Theorems 7 and 8

Based on the prior estimate, we acquire the existence and uniqueness of a solution for the stochastic fractional long-short wave equations (1)-(2) in spaces V_1.

Theorem 7. *If $(u_0, n_0) \in V_1$, $q_2 \in H^1(D)$, $q_1 \in H^{2\alpha}(D)$, and $f, g \in H^1(D)$, then there exists a unique solution $(u, n) \in (\mathbb{R}^+; V_1)$, almost surely satisfying (1)-(2). In addition, (u, n) is continuous from $\mathbb{R}^+$ to V_1.*

Proof. First, we know that $(n_0, u_0) \in V_1$. Let $\{e_i(x)\}_{i=1}^\infty$ be an orthonormal basis of eigenvectors of the Laplace operator on D, which is an orthonormal basis of $L^2(D)$. Consider P^k as the projection from $L^2(D)$ onto the space spanned by $\{e_i : i = 1, 2, \ldots, k\}$. Then the approximation solution (u^k, n^k) solves the approximation problem

$$
\begin{aligned}
iu_t^k - (-\Delta)^\alpha u^k - P^k\left(n^k u^k\right) + i\delta u^k &= \dot{W}_1^k + f^k, \\
n_t^k + P^k\left(\left|u^k\right|^2\right)_x + \beta n^k &= g^k + \dot{W}_2^k,
\end{aligned}
\tag{82}
$$

where P^k is the projector onto the first k vectors e_i, $\dot{W}_1^k = P^k \dot{W}_1$, $\dot{W}_2^k = P^k \dot{W}_2$, and P^k commutes with the operator $(-\Delta)^\alpha$. We will treat the above equations pathwise by introducing the following random processes solving

$$
\begin{aligned}
i\xi_t^k - (-\Delta)^\alpha \xi^k + i\delta \xi^k &= \dot{W}_1^k, \\
\zeta^k + \beta \zeta^k &= \dot{W}_2^k,
\end{aligned}
\tag{83}
$$

with periodic boundary conditions and initial conditions

$$
\begin{aligned}
\zeta(x, t) &= \eta(x + 2\pi, t), \\
\xi(x, t) &= \xi(x + 2\pi, t) = 0, \\
\zeta_t(x, 0) &= 0, \\
\zeta(x, 0) &= 0, \\
\xi(0, t) &= 0, \\
x &\in D,
\end{aligned}
\tag{84}
$$

$$
\text{where } \delta > 0, \ \beta > 0.
$$

In accordance with the same method as in Section 3, for any $T > 0$ and almost all $\omega \in \Omega$, we have

$$
\begin{aligned}
\zeta &\in C\left(0, T; H_0^1(D)\right), \\
\xi &\in C\left(0, T; H^{2\alpha}(D) \cap H_0^\alpha(D)\right).
\end{aligned}
\tag{85}
$$

Therefore, we can get the following estimate:

$$
\mathbb{E} \left(\|\eta_x\|^2 + \left\| (-\Delta)^\alpha \xi \right\|^2 \right) \leq C, \tag{86}
$$

where a positive constant C is independent of T. Moreover, for any $T > 0$,

$$
\mathbb{E} \sup_{0 \leq t \leq T} \left(\|\eta_x\|^2 + \left\| (-\Delta)^\alpha \xi \right\|^2 \right) \leq C_T \tag{87}
$$

holds for a positive constant C_T dependent on T. Set $B^{k,M} = (U^{k,M}, N^{k,M})$ as the solution of the following equations:

$$iU_t^{k,M} - (-\Delta)^\alpha U^{k,M} - \chi_M\left(\left\|B^{k,M}\right\|_{V_1}\right) P^k\left(n^{k,M} u^{k,M}\right)$$
$$+ i\delta U^{k,M} = f^k, \tag{88}$$

$$N_t^{k,M} + \chi_M\left(\left\|B^{k,M}\right\|_{V_1}\right) P^k\left(\left|u^{k,M}\right|^2\right)_x + \beta N^{k,M} = g^k, \tag{89}$$

$$N^{k,M}(x,0) = P^k n_0, \tag{90}$$

$$U^{k,M}(x,0) = P^k u_0.$$

Let $n^{k,M} = N^{k,M} + P^k\zeta$, $u^{k,M} = U^{k,M} + P^k\xi$, and $\chi_M \in C_0^\infty(\mathbb{R})$ satisfy $\chi_M(r) = 1$ for $|r| \leq M$ and $\chi_M(r) = 0$ for $|r| \geq 2M$. It is easy to see that (88)-(89) are random differential equations with Lipschitz nonlinearity in finite dimension. Afterwards, for almost all $\omega \in \Omega$, we have a unique solution $(U^{k,M}, N^{k,M})$ for (88)-(89). We define the stopping time as follows:

$$\tau_M = \inf\left\{t > 0 : \|B\|_{V_1}^{k,M} \geq M\right\}, \tag{91}$$

if the set $\{\|B\|_M^{k,M} \geq M\}$ is nonempty, or else $\tau_M = \infty$. Because τ_M is increasing in M, let $\tau_\infty = \lim_{M\to\infty} \tau_M$ almost surely. For $t < \tau_M$, we obtain

$$\left(U^{k,M}, N^{k,M}\right) + \left(P^k\xi, P^k\zeta\right), \tag{92}$$

satisfying (82). On the basis of the estimates given in Section 3.2 and (86)-(87), for any $t \geq 0$, we obtain

$$\mathbb{E}\left\|\left(U^{k,M}, N^{k,M}\right)\right\|_{V_1}^2 \leq C\left(f, g, q_1, q_2, n_0, u_0\right), \tag{93}$$

where the positive constant $C(f, g, q_1, q_2, n_0, u_0)$, which is dependent on T and M. And for $T > 0$, we obtain

$$\mathbb{E} \sup_{0 \leq t \leq T \wedge \tau_M} \left\|\left(U^{k,M}, N^{k,M}\right)\right\|_{V_1}^2$$
$$\leq C_T\left(f, g, q_1, q_2, n_0, u_0\right), \tag{94}$$

with the positive constant $C(f, g, q_1, q_2, n_0, u_0)$, which is dependent on T but independent of M. Let $T \wedge \tau_M = \min\{T, \tau_M\}$. On the other side, we get

$$\mathbb{E}\left\|U^{k,M}\left(T \wedge \tau_M\right), N^{k,M}\left(T \wedge \tau_M\right)\right\|_{V_1}^2$$
$$\geq \mathbb{E}\left[I\left(\tau_M \leq T\right)\right.$$
$$\left. \cdot \left|\left(U^{k,M}\left(T \wedge \tau_M\right), N^{k,M}\left(T \wedge \tau_M\right)\right)\right|_{V_1}^2\right] \geq M^2 \mathbb{P}\left(\tau\right.$$
$$\left. \leq T\right), \tag{95}$$

where $I(\tau_M \leq T) = 1$ for $\tau_M \leq T$ and $I(\tau_M \leq T) = 0$ for $\tau_M > T$. Then, by (93), we get

$$\mathbb{P}\left(\tau_M \leq T\right) \leq \frac{1}{M^2} C\left(f, g, q_1, q_2, n_0, u_0\right). \tag{96}$$

In the light of the above estimate and the Borel-Cantelli lemma, for any $T > 0$, we obtain

$$\mathbb{P}\left(\tau_\infty > T\right) = 1. \tag{97}$$

So we know that

$$\left(U^k, N^k\right) = \lim_{M\to\infty}\left(U^{k,M}, N^{k,M}\right) \tag{98}$$

satisfies the following random differential equations:

$$iU_t^k - (-\Delta)^\alpha U^k - P^k\left(n^k, u^k\right) + i\delta U^k = f^k, \tag{99}$$

$$N_t^k + P^k\left(\left|u^{k,M}\right|^2\right)_x + \beta N^k = g^k, \tag{100}$$

with initial conditions

$$U^k(0) = P^k u_0,$$
$$N^k(0) = P^k n_0. \tag{101}$$

Then (U^k, N^k) satisfies the estimates (93) and (94), and for any $t \geq 0$, we find that $(u^k, n^k) = (U^k, N^k) + (P^k\xi, P^k\zeta)$ is the unique global solution of (82).

And then, we will investigate (99)-(100) for fixed ω. Firstly, by (94), for any $T > 0$, we can know that

$$\mathbb{P}\left(\bigcup_{L=1}^\infty \bigcap_{l=1}^\infty \cap_{k=l}^\infty \left\{\sup_{0 \leq t \leq T}\left\|\left(U^k, N^k\right)\right\|_{V_1}^2 \geq L\right\}\right) = 0. \tag{102}$$

We let

$$\widetilde{\Omega} = \bigcup_{L=1}^\infty \bigcap_{l=1}^\infty \cap_{k=l}^\infty \left\{\sup_{0 \leq t \leq T}\left\|\left(U^k, N^k\right)\right\|_{V_1}^2 \leq L\right\}. \tag{103}$$

Then $\mathbb{P}(\Omega \setminus \widetilde{\Omega}) = 0$. Thus, for any fixed $\omega \in \widetilde{\Omega}$, there exists $r(\omega)$ with $0 < r(\omega) < \infty$ such that

$$\sup_{0 \leq t \leq T}\left\|\left(U^k, N^k\right)\right\|_{V_1}^2 \leq r(\omega). \tag{104}$$

Then we can extract a subsequence of (U^k, N^k), which are still denoted by (U^k, N^k), such that

$$U^k \overset{*}{\rightharpoonup} U \quad \text{in } L^\infty\left(0, T; H^{2\alpha}(D) \cap H_0^\alpha(D)\right),$$
$$N^k \overset{*}{\rightharpoonup} N \quad \text{in } L^\infty\left(0, T; H_0^1(D)\right). \tag{105}$$

These convergences are sufficient to pass the limit $k \to \infty$ in linear terms; however, in fact, for nonlinear terms, we need a strong convergence of U^k. From (100) and the estimate (104), it is easy to get $U_t^k \in L^\infty(0, T; L^2(D))$. Further, we can extract a subsequence of U^k which is still denoted by U^k such that

$$U^k \longrightarrow U \quad \text{strongly in } L^\infty\left(0, T; H_0^1(D)\right). \tag{106}$$

For the nonlinear term, we can pass the limit $k \to \infty$ by a standard procedure. Thus, we prove that $(U, N) \in L^\infty(0, T; V_1)$ is a weak solution of

$$iU_t - (-\Delta)^\alpha U - nu + i\delta U = f,$$
$$N_t + \left(|u|^2\right)_x + \beta N = g \tag{107}$$

with initial conditions

$$U(0) = u_0,$$

$$N(0) = n_0, \tag{108}$$

$$x \in D.$$

Then $(u, n) = (U, N) + (\xi, \zeta)$ is a solution of (1)-(2) and satisfies the estimates given in Section 3.2. The continuity of the solution can be proved in the following. For any $\omega \in \widetilde{\Omega}$, $(|u|^2)_x \in L^\infty(0, T; H_1^0(D))$, it is obvious that $N_t = -(|u|^2)_x - \beta N + g \in H_1^1(D)$. Then, we obtain that there is $N \in C(0, T; H_0^1(D))$ for almost all $\omega \in \Omega$ by Lemma 3 in [23]. Noticing $U_t \in L^\infty(0, T; L^2(D))$ almost surely and applying similar methods, we can get $U \in C(0, T; H^{2\alpha}(D) \cap H_0^\alpha(D))$ according to [21]. So we get $(u, n) \in C(0, T; V_1)$ almost surely by definition of U and N. Thus, the solution (u, n) is continuous from $[0, T]$ to V_1 almost surely.

Because the noise is additive, we can use the same approach as [7]. Therefore, the solution (u, n) is unique almost surely in $L^\infty(0, T; V_1)$. □

Theorem 8. *If* $(u_0, n_0) \in V_2$, $q_2 \in H^2(D)$, $q_1 \in H^{3\alpha}(D)$, *and* $f, g \in H^2(D)$, *then there exists a unique solution* $(u, n) \in (\mathbb{R}^+; V_2)$ *almost surely satisfy (1)-(2). Moreover,* (u, n) *is continuous from* $\mathbb{R}^+$ *to* V_2.

As a matter of fact, from Theorems 7 and 8, a continuous random dynamical system can be defined in V_1 and V_2, respectively. Then we can, respectively, construct a random attractor endowed with the weak topology for the continuous random dynamical system in V_1 and V_2.

The proof of Theorem 8 is similar to that of Theorem 7. Here we omit the detail of the proof.

5. Proofs of Theorems 10 and 11

In this section, the asymptotic behavior of solution for the corresponding problem is studied. We will construct a random attractor for stochastic fractional long-short wave equations in v_1 equipped with the weak topology. In [13–16], we can obtain some basic concepts related to random attractors for random dynamical systems.

Motivated by [13, 14], we can find the following existence result for a random attractor for a continuous RDS. It is a sufficient condition for the existence of random attractors.

Theorem 9 (see [13, 14]). *Assume* Φ *is a RDS on a Polish space* (E, d) *and there exists a random compact set* $K(\omega)$ *absorbing every bounded deterministic set* $D \subset E$. *Then we define a global random attractor for RDS* Φ *by*

$$\mathscr{A}(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}. \tag{109}$$

Next step, according to Theorem 9 and the priori estimates in Section 3, we research the random attractors for the stochastic long-short wave equations in V_1 and V_2.

Theorem 10. *If* $(u_0, n_0) \in V_1$, $q_1 \in H^1(D)$, $q_2 \in H^{2\alpha}(D)$, *and* $f, g \in H^1(D)$, *then (1)-(2) have a global random weak attractor* $\mathscr{A}(\omega)$ *which is a random tempered compact set in* V_1 *endowed with the weak topology.*

Proof. On the basis of the former analysis, we can consider the properties of solution a (U, N) of the system and it (107) has a unique solution $(U, N) \in C(0, T; V_1)$ for almost all $\omega \in \Omega$. Noticing that system (107) has coefficients driven by θ_t, a random dynamical system can be defined by (U, N) on V_1. Thus, $(u, n) = (U + \xi, N + \zeta)$ also defines a continuous random dynamical system on V_1, which is denoted by $\Phi(t, \omega)$, and $\Phi(t, \omega)$ is weakly continuous almost surely on V_1. $B(0, r)$ denotes the ball center at 0 with radius r in V_1. Using estimates made in accordance with Section 3, there is a random variable $R(\omega)$ such that, for any $r > 0$, $(u, n) \in B(0, r)$. Therefore, there exists a random time $t_r(\omega) > 0$, such that, for all $t > t_r(\omega)$ and almost all $\omega \in \Omega$,

$$\left\| \Phi(t, \theta_{-t}\omega)(u_0, n_0) \right\|_{V_1} \leq R(\omega). \tag{110}$$

We define the random attractor $\mathscr{A}(\omega)$ as follows:

$$\mathscr{A}(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}^{V_1^\omega}, \tag{111}$$

where the closure is taken with respect to the weak topology of V_1. Next we prove that $\mathscr{A}(\omega)$ is tempered. According to the estimates obtained in Section 3, we obtain

$$\mathbb{E} \sup_{0 \leq t \leq 1} R^2(\theta_t\omega) < \infty. \tag{112}$$

Then, from Birkhoff's ergodic Theorem [24],

$$\lim_{s \to \pm\infty} \frac{\sup_{t \in [0,1]} R^2(\theta_{t+s}\omega)}{s} = 0 \tag{113}$$

on a θ-invariant subset of Ω with full probability measure; that is, $R(\omega)$ is tempered. So we get that $\mathscr{A}(\omega)$ is tempered. The proof is completed. □

Theorem 11. *If* $(u_0, n_0) \in V_2$, $q_1 \in H^2(D)$, $q_2 \in H^{3\alpha}(D)$, *and* $f, g \in H^2(D)$, *then (1)-(2) have a global random weak attractor* $\mathscr{A}(\omega)$ *which is a random tempered compact set in* V_2 *endowed with the weak topology.*

Based on Theorem 10, we can prove Theorem 11 using the same methods and ideas of proof. More precisely, there exists a random attractor for the stochastic fractional long-short wave equations in V_2.

Theorem 12. *If* $(u_0, n_0) \in V_2$, $q_1 \in H^2(D)$, $q_2 \in H^{4\alpha}(D)$, *then (1)-(2) have one stationary measure on* V_1 *and* V_2.

Proof. If $(u_0, n_0) \in V_2$, (1)-(2) has a unique solution (u, n) with $(u(0), n(0)) = (u_0, n_0)$ by the results given in Sections 3 and 4, which, for any $t > 0$, satisfies

$$\mathbb{E}\left(\left\| (-\Delta)^\alpha n \right\|^2 + \left\| (-\Delta)^\alpha u_x \right\|^2 \right) \leq C, \tag{114}$$

for a positive constant $C > 0$ which is independent of $t > 0$.

Let μ_t be the distribution of (n_t, n, E) for $t \geq 0$. According to the classical Bogolyubov-Krylov argument [22], we define

$$\overline{\mu}_t = \frac{1}{t} \int_0^t \mu_s ds \qquad (115)$$

as

$$\overline{\mu}_t (\Gamma) = \frac{1}{t} \int_0^t \mu_s (\Gamma) \, ds \qquad (116)$$

for any Borel set Γ of V_1: namely, $\Gamma \in \mathscr{B}(V_1)$. From (114), we obtain

$$\int_{V_1} \|(u, n)\|_{V_2}^2 \, \overline{\mu}_t \, (dv) = \frac{1}{t} \int_0^t \mathbb{E} \, \|(u(s), n(s))\|_{V_2}^2 \, ds \qquad (117)$$

$$\leq C.$$

According to Chebyshev's inequality and the fact that V_2 has a compact embedding into V_1, $\{\overline{\mu}_t\}_{t \geq 0}$ is tight in V_1. Then there exists a sequence $\{\overline{\mu}_{tk}\}$ with $t_k \to \infty$ as $k \to \infty$ and a probability measure μ on V_1 such that $\{\overline{\mu}_{tk}\} \to \mu$ weakly as $k \to \infty$. Thus, using the standard argument as in [24], μ is a stationary measure for stochastic fractional long-short wave equations on V_1. In addition, from (114), μ is in fact supported on V_2; that is to say, μ is a stationary measure for stochastic fractional long-short wave equations on V_2. The proof is completed. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

References

[1] R. H. J. Grimshaw, "Modulation of an internal gravity-wave packet, and the resonance with the mean motion," *Studies in Applied Mathematics*, vol. 56, no. 3, pp. 241–266, 1977.

[2] D. R. Nicholson and M. V. Goldman, "Damped nonlinear Schrödinger equation," *The Physics of Fluids*, vol. 19, no. 10, pp. 1621–1625, 1976.

[3] B. Guo, "The global solution for one class of the system of LS nonlinear wave interaction," *Journal of Mathematical Research and Exposition*, vol. 7, no. 1, pp. 69–76, 1987.

[4] B. L. Guo, "The periodic initial value problems and initial value problems for class of generalized long-short type equations," *J. Engineering Math*, vol. 8, pp. 47–53, 1991.

[5] X. Y. Du and B. L. Guo, "The global attracttor for LS type equation in R1," *Acta Mathematicae Applicatae Sinica*, vol. 28, pp. 723–734, 2005.

[6] B. Guo and L. Chen, "Orbital stability of solitary waves of the long wave-short wave resonance equations," *Mathematical Methods in the Applied Sciences*, vol. 21, no. 10, pp. 883–894, 1998.

[7] B. Guo and B. Wang, "Attractors for the long-short wave equations," *Journal of Partial Differential Equations*, vol. 11, no. 4, pp. 361–383, 1998.

[8] Y. Li, "Long time behavior for the weakly damped driven long-wave-short-wave resonance equations," *Journal of Differential Equations*, vol. 223, no. 2, pp. 261–289, 2006.

[9] R.-F. Zhang, "Existence of global attractor for LS type equations," *Journal of Mathematical Research and Exposition*, vol. 26, no. 4, pp. 708–714, 2006.

[10] E. Weinan, X. Li, and E. Vanden-Eijnden, "Some recent progress in multiscale modeling," in *Multiscale Modelling and Simulation*, P. Koumoutsakos and S. Attiner, Eds., vol. 39 of *Lecture Notes in Computational Science and Engineering*, pp. 3–21, Springer, Berlin, Germany, 2004.

[11] P. Imkeller and A. H. Monahan, "Conceptual stochastic climate models," *Stochastics and Dynamics*, vol. 2, pp. 311–326, 2002.

[12] Y. Wang, L. Liu, and Y. Wu, "Positive solutions for a nonlocal fractional differential equation," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 11, pp. 3599–3605, 2011.

[13] H. Crauel and F. Flandoli, "Attractors for random dynamical systems," *Probability Theory and Related Fields*, vol. 100, no. 3, pp. 365–393, 1994.

[14] H. Crauel, A. Debussche, and F. Flandoli, "Random attractors," *Journal of Dynamics and Differential Equations*, vol. 9, no. 2, pp. 307–341, 1997.

[15] B. Wang, "Random attractors for the stochastic Benjamin-BONa-Mahony equation on unbounded domains," *Journal of Differential Equations*, vol. 246, no. 6, pp. 2506–2537, 2009.

[16] B. Wang, "Random attractors for the stochastic FitzHugh-Nagumo system on unbounded domains," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2811–2828, 2009.

[17] D. Li, Y. Guo, and B. Guo, "Random attractor and stationary measure for stochastic long-short wave equations," *Communications in Mathematical Sciences*, vol. 13, no. 2, pp. 539–555, 2015.

[18] H. Lu and J. Xin, "The existence of a global attractor for the (2+1)-dimensional long wave-short wave resonance interaction equation," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 73, no. 12, pp. 3767–3778, 2010.

[19] B. Zhu, L. Liu, and Y. Wu, "Existence and uniqueness of global mild solutions for a class of nonlinear fractional reaction-diffusion equations with delay," *Computers & Mathematics with Applications*, 2016.

[20] J. U. Kim, "On the stochastic porous medium equation," *Journal of Differential Equations*, vol. 220, no. 1, pp. 163–194, 2006.

[21] J.-L. Lions and E. Magenes, *Problems aux Limites Non Homogenes et Applications*, Dunod, Paris, France, 1968.

[22] G. da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, vol. 44, Cambridge University Press, Cambridge, UK, 1992.

[23] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, vol. 68 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 2nd edition, 1997.

[24] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional System*, vol. 229, Cambridge University Press, Cambridge, UK, 1996.