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Research Article

$\mathscr{H}_-$ Index for Stochastic Linear Discrete-Time Systems

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This paper discusses the $\mathscr{H}_-$ index problem for stochastic linear discrete-time systems. A necessary and sufficient condition of $\mathscr{H}_-$ index is given for such systems in finite horizon. It is proved that when the $\mathscr{H}_-$ index is greater than a given value, the feasibility of $\mathscr{H}_-$ index is equivalent to the solvability of a constrained difference equation. The above result can be applied to the fault detection observer design. Finally, some examples are presented to illustrate the proposed theoretical results.

1. Introduction

Model-based fault detection has attracted increasing attention in recent years because of its importance in reliability, security, and fault tolerance of dynamic systems; see [1–3]. In general, model-based fault detection is related to residual generation, that is, constructing a residual signal and comparing it with a predefined threshold. If the residual exceeds the threshold, an alarm is ringed. However, the residual can change due to the effects of external disturbance and model uncertainty. So fault detection observers must be insensitive to external disturbance and model uncertainty. Some approaches have been given for the design of fault detection observers, such as $\mathscr{H}_\infty$ norm, $\mathscr{H}_2$ norm, and $\mathscr{H}_-$ index, which are to evaluate the effectiveness of a fault detection observer design [2, 3]. The $\mathscr{H}_\infty$ norm characterizes the maximum effect of an input on an output, which plays an important role in robust control and was widely generalized by [4–9]. An upper bound of $\mathscr{H}_\infty$ norm can be described by means of the bounded real lemma. On the contrary, $\mathscr{H}_-$ index is used for measuring the sensitivity of residual to fault, which aims to maximize the minimum effect of fault on the residual output of a fault detection observer; see [3, 10–16] and the reference therein. In [3], $\mathscr{H}_-$ index in zero frequency was defined by using the minimum nonzero singular value. In [13], $\mathscr{H}_-$ index was defined as the minimum singular value over a given frequency range. A necessary and sufficient condition was given by LMIs for the infinite frequency range. The case for finite frequency range was obtained in terms of frequency weighting.

In recent years, the $\mathscr{H}_-$ index in time domain has attracted more attention. In [16], the authors developed a fault residual generator to maximize the fault sensitivity by $\mathscr{H}_-$ index in the finite time domain. Based on $\mathscr{H}_-$ index, the problems considered in [17, 18] were about the optimal fault detection for discrete time-varying linear systems. In [19], a necessary and sufficient condition of the $\mathscr{H}_-$ index for linear continuous time-varying systems in finite horizon was given. Characterization of $\mathscr{H}_-$ index for linear discrete time-varying systems was discussed in [20]. The $\mathscr{H}_-$ index was described by the existence of the solution to a backward difference Riccati equation.

Although there was much work on the $\mathscr{H}_-$ index, little work was concerned with the the $\mathscr{H}_-$ index in stochastic linear systems. In this paper, the characterization of the $\mathscr{H}_-$ index for stochastic linear systems in finite horizon is presented. The definition of the $\mathscr{H}_-$ index is extended to stochastic systems. New necessary and sufficient conditions are given for the $\mathscr{H}_-$ index. The feasibility of the $\mathscr{H}_-$ index is shown to be equivalent to the solvability of a constrained difference equation. As a special case, the stochastic $\mathscr{H}_-$ of square systems is addressed in this paper. Our results can be viewed as the extensions of deterministic systems, which can be applied to the fault detection.

The outline of the paper is organized as follows. Section 2 is devoted to developing some efficient criteria for the linear stochastic $\mathcal{H}_-$ index in finite horizon. Section 3 contains some examples provided to show the efficiency of the proposed results. Finally, we end this paper in Section 4 with a brief conclusion.

Notations. $\mathcal{R}$ is the field of real numbers. $\mathcal{R}^{m\times n}$ is the vector space of all $m\times n$ matrices with entries in $\mathcal{R}$. $\mathcal{S}_n(\mathcal{R})$ is the set of all real symmetric matrices $\mathcal{R}^{n\times n}$. A' denotes the transpose of the complex matrix A. A^{-1} is the inverse of A. Given a positive semidefinite (positive definite) matrix A, we denote it by $A \geq 0$ $(A > 0)$. Let $\mathcal{E}$ denote the mathematical expectation. I_n is $n \times n$ identity matrix. 0_n is $n \times n$ zero matrix. Consider $\mathcal{N}_T = \{0, 1, \ldots, T\}$, $\mathcal{N} = \{0, 1, \ldots\}$. A tall system refers to a system when the number of inputs is less than that of outputs. A wide system is the case of more inputs than outputs. A square system denotes a system when the number of inputs equals the output number.

2. Finite Horizon Stochastic $\mathcal{H}_-$ Index

In this section, we will discuss the finite horizon stochastic $\mathcal{H}_-$ index problem. We give a necessary and sufficient condition for the finite horizon stochastic $\mathcal{H}_-$ index.

Consider the following stochastic system:

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)v(t) \\ &\quad + [A_0(t)x(t) + B_0(t)v(t)]w(t), \\ &\qquad t \in \mathcal{N}_T, \\ z(t) &= C(t)x(t) + D(t)v(t), \quad x(0) = x_0, \end{aligned} \tag{1}$$

where $\{w(t)\}_{t\geq 0}$ is a sequence of one-dimensional independent real random variables defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with $\mathcal{E}[w(t)] = 0$ and $\mathcal{E}[w(s)w(t)] = \delta_{st}$, $s, t \in \mathcal{N}_T$, where δ_{st} is the Kronecker delta. Suppose $v(t)$ and $w(t)$ are independent. $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times l}$, $A_0 \in \mathcal{R}^{n\times n}$, $B_0 \in \mathcal{R}^{n\times l}$, $C \in \mathcal{R}^{q\times n}$, and $D \in \mathcal{R}^{q\times l}$. $x(t) \in \mathcal{R}^n$, $v(t) \in \mathcal{R}^l$, and $z(t) \in \mathcal{R}^q$ are the state, input, and output, respectively.

We define the σ-algebra generated by $w(t)$, $t \in \mathcal{N}_t$, $\mathcal{F}_t = \sigma\{w(s), s \in \mathcal{N}_t\}$, $t \in \mathcal{N}$. $\{\mathcal{F}_t\}_{t\in\mathcal{N}}$ is an increasing sequence of σ-algebra $\mathcal{F}_t \subset \mathcal{F}$ and $w(t)$ is adapted to $\mathcal{F}_t$ for all $t \in \mathcal{N}$. Let $L^2(\Omega, \mathcal{R}^k)$ be the space of $\mathcal{R}^k$-valued random vectors ξ with $\mathcal{E}\|\xi\|^2 < \infty$. $l_w^2(\mathcal{N}_T; \mathcal{R}^k)$ denotes the space of all sequences $y(t) \in L^2(\Omega, \mathcal{R}^k)$ that are $\mathcal{F}_{t-1}$ measurable for all $t \in \mathcal{N}_T$. The l^2-norm of $l_w^2(\mathcal{N}_T; \mathcal{R}^k)$ is defined by $\|y(\cdot)\|_{l_w^2(\mathcal{N}_T;\mathcal{R}^k)} = (\sum_{t\in\mathcal{N}_T} \mathcal{E}[y'(t)y(t)])^{1/2} < \infty$. We suppose that x_0 is deterministic. For any $T \in \mathcal{N}$ and $(x_0, v) \in \mathcal{R}^n \times l_w^2(\mathcal{N}_T; \mathcal{R}^l)$, there exists a unique solution $x(\cdot) = x(\cdot; x_0, v) \in l_w^2(\mathcal{N}_{T+1}; \mathcal{R}^n)$ of (1) with $x(0) = x_0$.

The finite horizon stochastic $\mathcal{H}_-$ index problem of system (1) can be stated as follows.

Definition 1. For stochastic system (1), given $0 \leq T < \infty$, define

$$\begin{aligned} \|\mathcal{G}\|_-^{[0,T]} &= \inf_{v\in l_w^2(\mathcal{N}_T;\mathcal{R}^l), v\neq 0, x_0=0} \frac{\|z(t)\|_{l_w^2(\mathcal{N}_T;\mathcal{R}^q)}}{\|v(t)\|_{l_w^2(\mathcal{N}_T;\mathcal{R}^l)}} \\ &= \inf_{v\in l_w^2(\mathcal{N}_T;\mathcal{R}^l), v\neq 0, x_0=0} \frac{\{\sum_{t\in\mathcal{N}_T} \mathcal{E}[z'(t)z(t)]\}^{1/2}}{\{\sum_{t\in\mathcal{N}_T} \mathcal{E}[v'(t)v(t)]\}^{1/2}}, \end{aligned} \tag{2}$$

which is called $\mathcal{H}_-$ index of (1) in $\mathcal{N}_T$.

Remark 2. Definition 1 describes the smallest sensitivity of stochastic system (1) from input v to output z in time domain. Assume that v is fault signal and z is the residual; then $\|\mathcal{G}\|_-^{[0,T]}$ characterizes the minimal fault sensitivity.

Remark 3. When system (1) is wide, $\|\mathcal{G}\|_-^{[0,T]} = 0$ (see [20]). In this paper, we suppose that system (1) is tall or square.

For any given $\gamma \geq 0$, and $T \in \mathcal{N}$, let

$$\begin{aligned} J_T^\gamma(x_0, v) &= \|z(t)\|^2_{l_w^2(\mathcal{N}_T;\mathcal{R}^q)} - \gamma^2 \|v(t)\|^2_{l_w^2(\mathcal{N}_T;\mathcal{R}^l)} \\ &= \sum_{t=0}^{T} \mathcal{E}\left[z'(t)z(t) - \gamma^2 v'(t)v(t)\right] \\ &= \sum_{t=0}^{T} \mathcal{E}\left[(C(t)x(t) + D(t)v(t))'\right. \\ &\quad \left.\cdot (C(t)x(t) + D(t)v(t)) - \gamma^2 v'(t)v(t)\right], \end{aligned} \tag{3}$$

where $x(\cdot) = x(\cdot; x_0, v)$ is the solution of (1) and $z(\cdot) = z(\cdot; x_0, v)$ is the corresponding output. We will discuss the following optimal control problem:

$$\min J_T^\gamma(x_0, v) \quad \text{for } v \in l_w^2(\mathcal{N}_T; \mathcal{R}^l). \tag{4}$$

Remark 4. Obviously, $\|\mathcal{G}\|_-^{[0,T]} > \gamma$ is equivalent to the following inequality:

$$\begin{aligned} J_T^\gamma(0, v) &= \|z(t)\|^2_{l_w^2(\mathcal{N}_T;\mathcal{R}^q)} - \gamma^2 \|v(t)\|^2_{l_w^2(\mathcal{N}_T;\mathcal{R}^l)} \\ &= \sum_{t=0}^{T} \mathcal{E}\left[z'(t)z(t) - \gamma^2 v'(t)v(t)\right] > 0, \end{aligned} \tag{5}$$

for all $v \in l_w^2(\mathcal{N}_T; \mathcal{R}^l)$, $v \neq 0$, $x_0 = 0$.

Remark 5. When $T = \infty$, (2) and (5) correspond to the infinite horizon $\mathcal{H}_-$ index case.

In the following, we present some useful lemmas, which play important roles throughout the paper.

Lemma 6. *For given $T \in \mathcal{N}$, if $(P(0), P(1), \ldots, P(T+1))$ is an arbitrary family of matrices in $\mathcal{S}_n(\mathcal{R})$, then for any $x_0 \in \mathcal{R}^n$*

$$J_T^\gamma(x_0, v) = x_0' P(0) x_0 - \mathcal{E}\left[x'(T+1) P(T+1) x(T+1)\right] + \sum_{t=0}^{T} \mathcal{E}\left\{\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathcal{M}(t, P(\cdot)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}\right\}, \quad (6)$$

where

$$\begin{aligned}
\mathcal{M}(t, P(\cdot)) &= \begin{bmatrix} \mathcal{L}(P(t+1)) - P(t) & \mathcal{K}(P(t+1)) \\ \mathcal{K}'(P(t+1)) & \mathcal{H}^\gamma(P(t+1)) \end{bmatrix}, \\
\mathcal{L}(P(t+1)) &= A'(t) P(t+1) A(t) + C'(t) C(t) + A_0'(t) P(t+1) A_0(t), \\
\mathcal{K}(P(t+1)) &= A'(t) P(t+1) B(t) + C'(t) D(t) + A_0'(t) P(t+1) B_0(t), \\
\mathcal{H}^\gamma(P(t+1)) &= B'(t) P(t+1) B(t) + D'(t) D(t) - \gamma^2 I + B_0'(t) P(t+1) B_0(t).
\end{aligned} \quad (7)$$

Proof. Since $w(t)$ is independent of $v(t)$, we conclude that $A(t)x(t) + B(t)v(t)$ and $A_0(t)x(t) + B_0(t)v(t)$ are $\mathcal{F}_{t-1}$ measurable and independent of $w(t)$, so

$$\begin{aligned}
&\mathcal{E}\left\{[A(t)x(t) + B(t)v(t)]' P(t+1) \cdot [A_0(t)x(t) + B_0(t)v(t)] w(t)\right\} \\
&= \mathcal{E}\left\{[A_0(t)x(t) + B_0(t)v(t)]' P(t+1) \cdot [A(t)x(t) + B(t)v(t)] w(t)\right\} = 0.
\end{aligned} \quad (8)$$

It follows that

$$\mathcal{E}\left[x'(t+1) P(t+1) x(t+1) - x'(t) P(t) x(t)\right] = \mathcal{E}\left[\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathcal{Q}(t, P(\cdot)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}\right], \quad (9)$$

where

$$\mathcal{Q}(t, P(\cdot)) = \begin{bmatrix} Q_1(t) & Q_2(t) \\ Q_2'(t) & Q_3(t) \end{bmatrix}, \quad (10)$$

where

$$\begin{aligned}
Q_1(t) &= A'(t) P(t+1) A(t) - P(t) + A_0'(t) P(t+1) A_0(t), \\
Q_2(t) &= A'(t) P(t+1) B(t) + A_0'(t) P(t+1) B_0(t), \\
Q_3(t) &= B'(t) P(t+1) B(t) + B_0'(t) P(t+1) B_0(t).
\end{aligned} \quad (11)$$

Take summation from $t = 0$ to T; it yields that

$$\mathcal{E}\left[x'(T+1) P(T+1) x(T+1) - x'(0) P(0) x(0)\right] = \sum_{t=0}^{T} \mathcal{E}\left[\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathcal{Q}(t, P(\cdot)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}\right]. \quad (12)$$

From (3), we get

$$\begin{aligned}
J_T^\gamma(x_0, v) &= \sum_{t=0}^{T} \mathcal{E}\left[z'(t) z(t) - \gamma^2 v'(t) v(t)\right] \\
&= \sum_{t=0}^{T} \mathcal{E}\left[(C(t)x(t) + D(t)v(t))' \cdot (C(t)x(t) + D(t)v(t)) - \gamma^2 v'(t) v(t)\right] \\
&\quad + \sum_{t=0}^{T} \mathcal{E}\left[\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathcal{Q}(t, P(\cdot)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}\right] - \mathcal{E}\left[x'(T+1) P(T+1) x(T+1)\right] + x'(0) P(0) x(0) \\
&= \sum_{t=0}^{T} \mathcal{E}\left\{\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathcal{M}(t, P(\cdot)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}\right\} - \mathcal{E}\left[x'(T+1) P(T+1) x(T+1)\right] + x_0' P(0) x_0,
\end{aligned} \quad (13)$$

which completes the proof. □

Theorem 7. *For (1) and given $\gamma \geq 0$, if the following equation*

$$\begin{aligned}
P(t) &= \mathcal{L}(P(t+1)) - \mathcal{K}(P(t+1)) \mathcal{H}^\gamma(P(t+1))^{-1} \mathcal{K}'(P(t+1)), \\
\mathcal{H}^\gamma(P(t+1)) &> 0, \\
P(T+1) &= 0
\end{aligned} \quad (14)$$

has a solution $P_T(t)$, $\forall t \in \mathcal{N}_T$, then $\|\mathcal{G}\|_{-}^{[0,T]} > \gamma$.

Proof. For any $v \in l_w^2(\mathcal{N}_T; \mathcal{R}^l)$, $v \neq 0$, $x_0 = 0$, by Lemma 6, we have

$$J_T^\gamma(0, v) = \sum_{t=0}^{T} \mathcal{E}\left\{\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathcal{M}(t, P_T(\cdot)) \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}\right\}. \quad (15)$$

By completing squares and considering the first equality in (14), we obtain that

$$\begin{aligned}
J_T^\gamma(0,v) = {} & \sum_{t=0}^{T}\mathscr{E}\left\{x'(t)\left[-P_T(t)+\mathscr{L}(P_T(t+1))\right.\right.\\
& \left.-\mathscr{K}(P_T(T+1))\mathscr{H}^\gamma(P_T(t+1))^{-1}\mathscr{K}'(P_T(T+1))\right]\\
& \left.\cdot x(t)\right\}+\sum_{t=0}^{T}\mathscr{E}\left\{\left[v(t)\right.\right.\\
& \left.+\mathscr{H}^\gamma(P_T(t+1))^{-1}\mathscr{K}'(P_T(t+1))x(t)\right]'\\
& \cdot\mathscr{H}^\gamma(P_T(t+1))\left[v(t)\right.\\
& \left.\left.+\mathscr{H}^\gamma(P_T(t+1))^{-1}\mathscr{K}'(P_T(t+1))x(t)\right]\right\}\\
= {} & \sum_{t=0}^{T}\mathscr{E}\left\{\left[v(t)-v^*(t)\right]'\mathscr{H}^\gamma(P_T(t+1))\left[v(t)\right.\right.\\
& \left.\left.-v^*(t)\right]\right\},
\end{aligned}\tag{16}$$

where $v^*(t)=-\mathscr{H}^\gamma(P_T(t+1))^{-1}\mathscr{K}'(P_T(t+1))x(t)$.

From $\mathscr{H}^\gamma(P_T(t+1))>0$, it is obvious that $J_T^\gamma(0,v)\geq 0$ and $J_T^\gamma(0,v)=0$ if and only if $v(t)=v^*(t)$. Let us substitute $v(t)=v^*(t)$ into system (1). It must be $x(t)=0$, $t\in\mathscr{N}_T$ on the basis of the fact $x_0=0$, which results in $v^*(t)=0$, $t\in\mathscr{N}_T$. Therefore, it is deduced that $J_T^\gamma(0,v)=0$ if and only if $v(t)=v^*(t)=0$, which contradicts the condition $v(t)\neq 0$. Without loss of generality, we assume that $\mathscr{H}^\gamma(P_T(t+1))>\epsilon I_l$, $\epsilon>0$. Then, (16) indicates that $J_T^\gamma(0,v)>\epsilon\|v(t)-v^*(t)\|^2_{l^2_w(\mathscr{N}_T;\mathscr{R}^l)}>0$, which implies that $\|\mathscr{G}\|_-^{[0,T]}>\gamma$. Theorem 7 is proved. □

The necessity of Theorem 7 will be proved by a sequence of lemmas. To this end, we consider the following backward matrix equation:

$$\begin{aligned}
X(t) = {} & \mathscr{L}(X(t+1))+\mathscr{K}(X(t+1))F(t)\\
& +F'(t)\mathscr{H}^\gamma(X(t+1))F(t)\\
& +F'(t)\mathscr{K}'(X(t+1)),\quad t\in\mathscr{N}_T,
\end{aligned}\tag{17}$$

where $F:\mathscr{N}_T\to\mathscr{R}^{l\times n}$ is a given finite sequence of matrices. This equation has a unique solution $X(t)=P_F^\gamma(t)$, $t\in\mathscr{N}_{T+1}$, satisfying $X(T+1)=0$.

By the above, $P_F^\gamma(t)$ is the solution of the following equation:

$$\begin{aligned}
& P_F^\gamma(t)\\
& = \begin{bmatrix} I_n & F'(t)\end{bmatrix}\begin{bmatrix}\mathscr{L}(P_F^\gamma(t+1)) & \mathscr{K}(P_F^\gamma(t+1))\\ \mathscr{K}'(P_F^\gamma(t+1)) & \mathscr{H}^\gamma(P_F^\gamma(t+1))\end{bmatrix}\begin{bmatrix} I_n\\ F(t)\end{bmatrix},\\
& P_F^\gamma(T+1)=0,\\
& \qquad t\in\mathscr{N}_T.
\end{aligned}\tag{18}$$

Lemma 8. *For the given $x_0\in\mathscr{R}^n$, $v\in l^2_w(\mathscr{N}_T;\mathscr{R}^l)$, and $F:\mathscr{N}_T\to\mathscr{R}^{l\times n}$, if the following equation*

$$\begin{aligned}
x_F(t+1) = {} & (A(t)+B(t)F(t))x_F(t)\\
& +(A_0(t)+B_0(t)F(t))x_F(t)w(t)\\
& +B_0(t)v(t)w(t)+B(t)v(t),\\
x_F(0) = {} & x_0,\\
& t\in\mathscr{N}_T
\end{aligned}\tag{19}$$

admits a solution $x_F(t)=x(t;x_0,F(t)x_F(t)+v(t))$, then the cost function $J=J_T^\gamma(x_0,F(t)x_F(t)+v(t))$ is given by

$$\begin{aligned}
J = {} & x_0'P_F^\gamma(0)x_0+\sum_{t=0}^{T}\mathscr{E}\left\{x_F'(t)G'(t)v(t)\right.\\
& \left.+v'(t)G(t)x_F(t)+v'(t)\mathscr{H}^\gamma(P_F^\gamma(t+1))v(t)\right\},
\end{aligned}\tag{20}$$

where $P_F^\gamma(t)$, $t\in\mathscr{N}_{T+1}$ is the solution of (18) and

$$\begin{aligned}
G(t) = {} & \mathscr{K}'(P_F^\gamma(t+1))+\mathscr{H}^\gamma(P_F^\gamma(t+1))F(t),\\
& t\in\mathscr{N}_T.
\end{aligned}\tag{21}$$

Furthermore, for $v=0$,

$$J_T^\gamma(x_0,F(t)x_F(t))=x_0'P_F^\gamma(0)x_0.\tag{22}$$

Proof. By Lemma 6, we can derive that

$$\begin{aligned}
J_T^\gamma(x_0,F(t)x_F(t)+v(t)) = {} & x_0'P_F^\gamma(0)x_0+\sum_{t=0}^{T}\mathscr{E}\left\{\begin{bmatrix}x_F(t)\\ F(t)x_F(t)+v(t)\end{bmatrix}'\mathscr{M}(t,P_F^\gamma(\cdot))\begin{bmatrix}x_F(t)\\ F(t)x_F(t)+v(t)\end{bmatrix}\right\}\\
= {} & x_0'P_F^\gamma(0)x_0\\
& +\sum_{t=0}^{T}\mathscr{E}\left\{x_F'(t)\begin{bmatrix}I_n & F'(t)\end{bmatrix}\begin{bmatrix}\mathscr{L}(P_F^\gamma(t+1))-P_F^\gamma(t) & \mathscr{K}(P_F^\gamma(t+1))\\ \mathscr{K}'(P_F^\gamma(t+1)) & \mathscr{H}^\gamma(P_F^\gamma(t+1))\end{bmatrix}\begin{bmatrix}I_n\\ F(t)\end{bmatrix}x_F(t)\right\}\\
& +\sum_{t=0}^{T}\mathscr{E}\left\{v'(t)G(t)x_F(t)+x_F'(t)G'(t)v(t)+v'(t)\mathscr{H}^\gamma(P_F^\gamma(t+1))v(t)\right\}\\
= {} & x_0'P_F^\gamma(0)x_0+\sum_{t=0}^{T}\mathscr{E}\left\{v'(t)G(t)x_F(t)+x_F'(t)G'(t)v(t)+v'(t)\mathscr{H}^\gamma(P_F^\gamma(t+1))v(t)\right\}.
\end{aligned}\tag{23}$$

For $v=0$, (22) is obvious. □

Next, we will show that matrices $\mathscr{H}^{\gamma}(P_F^{\gamma}(t+1))$, $t \in \mathcal{N}_T$, are invertible.

Lemma 9. *For system (1), assume that, for given $\gamma \geq 0$, $\|\mathscr{G}\|_{-}^{[0,T]} > \gamma$. For given $F : \mathcal{N}_T \to \mathscr{R}^{l\times n}$ and $T \in \mathcal{N}$, if $P_F^{\gamma}(t)$ is the solution of (18), then*

$$\mathscr{H}^{\gamma}\left(P_F^{\gamma}(t+1)\right) \geq \left[\left(\|\mathscr{G}\|_{-}^{[0,T]}\right)^2 - \gamma^2\right] I_l > 0, \quad t \in \mathcal{N}_T. \tag{24}$$

Proof. We first prove $\mathscr{H}^{\gamma}(P_F^{\gamma}(t+1)) \geq 0$ on $\mathcal{N}_T$. Suppose that there exist $t^* \in \mathcal{N}_T$, $\eta > 0$, $u \in \mathscr{R}^l$, and $\|u\| = 1$ such that $u'\mathscr{H}^{\gamma}(P_F^{\gamma}(t^*+1))u \leq -\eta$. Set $F(t) = 0$, $t \in \mathcal{N}_T$, and

$$v(t) = \begin{cases} 0, & t \neq t^*,\ t \in \mathcal{N}_T; \\ u, & t = t^*. \end{cases} \tag{25}$$

According to Lemma 8, we have

$$\begin{aligned} J_T^{\gamma}(0, v) &= \sum_{t=0}^{T} \mathscr{E}\left\{x_F'(t) G'(t) v(t) + v'(t) G(t) x_F(t) \right. \\ &\quad \left. + v'(t) \mathscr{H}^{\gamma}\left(P_F^{\gamma}(t+1)\right) v(t)\right\} = \mathscr{E}\left\{x_F'(t^*) G'(t^*) \right. \\ &\quad \left. \cdot u + u' G(t^*) x_F(t^*) + u' \mathscr{H}^{\gamma}\left(P_F^{\gamma}(t^*+1)\right) u\right\}. \end{aligned} \tag{26}$$

From the definition of $v(t)$, $t \in \mathcal{N}_T$, and (19), it follows that $x_F(t) = 0$ for $t \leq t^*$. Additionally, in view of $\|\mathscr{G}\|_{-}^{[0,T]} > \gamma$, we conclude that

$$0 \leq J_T^{\gamma}(0, v) = u' \mathscr{H}^{\gamma}\left(P_F^{\gamma}(t^*+1)\right) u \leq -\eta, \tag{27}$$

which leads to a contradiction. So, $\mathscr{H}^{\gamma}(P_F^{\gamma}(t+1)) \geq 0$ for any $t \in \mathcal{N}_T$.

Now let $(\|\mathscr{G}\|_{-}^{[0,T]})^2 > \gamma^2 + \rho^2$ for $\rho > 0$ and $\lambda = (\gamma^2 + \rho^2)^{1/2}$. Replacing γ with λ in (18), we obtain the corresponding solution $P_F^{\lambda}(t)$. As in the preceding proof, we have that $\mathscr{H}^{\gamma}(P_F^{\lambda}(t+1)) \geq 0$. For any t_0 and $t \in \mathcal{N}_{T-t_0}$, define $F_{t_0}(t) = F(t+t_0)$. Let $P_{F_{t_0}}^{\lambda}(t)$, $t \in \mathcal{N}_{T-t_0}$, be the solution of (18) with γ and F replaced by λ and F_{t_0}, respectively. Then $P_{F_{t_0}}^{\lambda}(t) = P_F^{\lambda}(t+t_0)$, $t \in \mathcal{N}_{T+1-t_0}$. By (22), it follows that

$$\begin{aligned} x_0' P_F^{\lambda}(t_0) x_0 &= x_0' P_{F_{t_0}}^{\lambda}(0) x_0 = J_{T-t_0}^{\lambda}\left(x_0, F_{t_0} x_{F_{t_0}}\right) \\ &\leq J_{T-t_0}^{\gamma}\left(x_0, F_{t_0} x_{F_{t_0}}\right) = x_0' P_F^{\gamma}(t_0) x_0. \end{aligned} \tag{28}$$

Therefore, $\mathscr{H}^{\lambda}(P_F^{\gamma}(t_0+1)) \geq \mathscr{H}^{\lambda}(P_F^{\lambda}(t_0+1)) \geq 0$, which means that $\mathscr{H}^{\gamma}(P_F^{\gamma}(t+1)) \geq \rho^2 I$ for all $t \in \mathcal{N}_T$ and arbitrary $\rho^2 < (\|\mathscr{G}\|_{-}^{[0,T]})^2 - \gamma^2$. So $\mathscr{H}^{\gamma}(P_F^{\gamma}(t+1)) > [(\|\mathscr{G}\|_{-}^{[0,T]})^2 - \gamma^2] I_l$.

This completes the proof. □

Remark 10. From (24), for $t = T$, $\mathscr{H}^{\gamma}(P_F^{\gamma}(T+1)) = D'(T)D(T) - \gamma^2 I > 0$. If system (1) is time-invariant and satisfies Lemma 9, then

$$D'D - \gamma^2 I > 0. \tag{29}$$

Remark 11. If $I-AB$ or $I-BA$ is invertible, then $A(I-BA)^{-1} = (I-AB)^{-1}A$. By this equality, we see that $C'[I - D(D'D - \gamma^2 I)^{-1}D']C = C'[I - \gamma^{-2}DD']^{-1}C$. If system (1) is square and time-invariant, from (29), we can conclude that

$$\begin{aligned} C'\left[I - D\left(D'D - \gamma^2 I\right)^{-1} D'\right] C \\ = C'\left[I - \gamma^{-2} DD'\right]^{-1} C \leq 0. \end{aligned} \tag{30}$$

However, the above is not true for tall systems.

Now, we discuss the necessity of Theorem 7 and present the following theorem.

Theorem 12. *For system (1), if $\|\mathscr{G}\|_{-}^{[0,T]} > \gamma$ for given $\gamma \geq 0$, then (14) has a unique solution $P_T^{\gamma}(t)$, $t \in \mathcal{N}_{T+1}$, for any $T \geq 0$. Furthermore, $J_T^{\gamma}(x_0, v)$ is minimized with the optimal cost given by*

$$\min_{v \in l_w^2(\mathcal{N}_T; \mathscr{R}^l)} J_T^{\gamma}(x_0, v) = x_0' P_T^{\gamma}(0) x_0 \tag{31}$$

and the optimal control is determined by

$$\begin{aligned} v^*(t) &= F^*(t) x_{F^*}(t), \\ F^*(t) &= -\mathscr{H}^{\gamma}\left(P_T^{\gamma}(t+1)\right)^{-1} \mathscr{K}'\left(P_T^{\gamma}(t+1)\right), \end{aligned} \tag{32}$$

where $x_{F^*}(t)$ satisfies

$$\begin{aligned} x_{F^*}(t+1) &= \left(A(t) + B(t) F^*(t)\right) x_{F^*}(t) \\ &\quad + \left(A_0(t) + B_0(t) F^*(t)\right) x_{F^*}(t) w(t), \\ x_{F^*}(0) &= x_0. \end{aligned} \tag{33}$$

Proof. We first prove that $\|\mathscr{G}\|_{-}^{[0,T]} > \gamma$ means that (14) admits a solution $P_T^{\gamma}(t)$ on $\mathcal{N}_{T+1}$. As $\mathscr{H}^{\gamma}(P_T^{\gamma}(T+1)) = D'(T)D(T) - \gamma^2 I > 0$, it is clear that there exists a solution to (14) at $t = T$; that is,

$$\begin{aligned} P_T^{\gamma}(T) &= \mathscr{L}\left(P_T^{\gamma}(T+1)\right) - \mathscr{K}\left(P_T^{\gamma}(T+1)\right) \\ &\quad \cdot \left(D'(T) D(T) - \gamma^2 I\right) \mathscr{K}'\left(P_T^{\gamma}(T+1)\right). \end{aligned} \tag{34}$$

Suppose (14) does not have a solution on $\mathcal{N}_T$; then there must exist a minimum number $\widetilde{T} \in \mathcal{N}_T$, $0 < \widetilde{T} \leq T$, such that (14) is solvable backward up to $t = \widetilde{T}$. That is to say, $P_T^{\gamma}(\widetilde{T}), P_T^{\gamma}(\widetilde{T}+1), \ldots, P_T^{\gamma}(T+1)$ satisfy (14) but $P_T^{\gamma}(\widetilde{T}-1)$ does not, or $\mathscr{H}^{\gamma}(P_T^{\gamma}(\widetilde{T}))$ is not a positive definite matrix.

Set $\widetilde{F}_T(t) = -\mathscr{H}^{\gamma}(P_T(t+1))^{-1}\mathscr{K}'(P_T^{\gamma}(t+1))$, $t = \widetilde{T}, \widetilde{T}+1, \ldots, T$, and then $\widetilde{F}_T(t)$ is well defined. Let

$$\widetilde{F}(t) = \begin{cases} 0, & t = 0, 1, \ldots, \widetilde{T}-1, \\ \widetilde{F}_T(t), & t = \widetilde{T}, \widetilde{T}+1, \ldots, T. \end{cases} \tag{35}$$

Consider the following equation:

$$P(t) = \mathscr{L}(P(t+1)) + \mathscr{K}(P(t+1))\widetilde{F}(t) + \widetilde{F}'(t)\mathscr{K}'(P(t+1)) + \widetilde{F}'(t)\mathscr{H}^{\gamma}(P(t+1))\widetilde{F}(t), \quad (36)$$

$$\mathscr{H}^{\gamma}(P(t+1)) > 0,$$

$$P(T+1) = 0.$$

Equation (36) admits a solution $\widetilde{P}_T^{\gamma}(t)$, $t \in \mathscr{N}_{T+1}$. Comparing (36) with (14), we arrive at $P_T^{\gamma}(t) = \widetilde{P}_T^{\gamma}(t)$ for $t = \widetilde{T}, \widetilde{T}+1, \ldots, T$. Moreover, along the same line of Lemma 9, we have that $\mathscr{H}^{\gamma}(\widetilde{P}_T^{\gamma}(t+1)) > 0$ on $\mathscr{N}_T$. In particular, $\mathscr{H}^{\gamma}(P_T^{\gamma}(\widetilde{T})) = \mathscr{H}^{\gamma}(\widetilde{P}_T^{\gamma}(\widetilde{T})) > 0$. This is inconsistent with the nonpositiveness of $\mathscr{H}^{\gamma}(P_T^{\gamma}(\widetilde{T}))$. Hence, (14) has a unique solution $P_T^{\gamma}(t)$, $t \in \mathscr{N}_{T+1}$, for any $T \geq 0$.

Next, we suppose that the following equation

$$X(t) = \mathscr{L}(X(t+1)) - \mathscr{K}(X(t+1))\mathscr{H}^{\gamma}(X(t+1))^{-1}\mathscr{K}'(X(t+1)), \quad t \in \mathscr{N}_T; \quad (37)$$

$$X(T+1) = 0$$

admits a solution $P_T^{\gamma}(t)$, $t \in \mathscr{N}_{T+1}$, and then $F^*(t)$ is well defined by (32). If we replace $F(t)$ in (17) by $F^*(t)$, then (18) becomes (37) with $X(t) = P_{F^*}^{\gamma}(t)$, so $P_{F^*}^{\gamma}(t) = P_T^{\gamma}(t)$ for all $t \in \mathscr{N}_{T+1}$. By (21), when $F(t) = F^*(t)$, it yields $G(t) = 0$. By (20), we come to a conclusion that for $v(t) \in l_w^2(\mathscr{N}_T; \mathscr{R}^l)$

$$J_T^{\gamma}(x_0, F^*(t)x_F(t) + v(t)) = x_0' P_T^{\gamma}(0)x_0 + \sum_{t=0}^{T}\mathscr{E}\left\{v'(t)\mathscr{H}^{\gamma}(P_T^{\gamma}(t+1))v(t)\right\}. \quad (38)$$

By (24), we deduce that $v^*(t) = F^*(t)x_{F^*}(t)$ minimizes $J_T^{\gamma}(x_0, v)$ with the optimal value expressed by (31). This proof is complete. □

Lemma 13. *If system (1) is time-invaint, square and $\gamma \geq 0$, then, for any fixed $t \in \mathscr{N}$, $\overline{T}+1 > T+1 > t$,*

$$P_{\overline{T}}^{\gamma}(t) \leq P_T^{\gamma}(t). \quad (39)$$

Proof. Since system (1) is time-invaint, by the time invariance of (37), we have $P_T^{\gamma}(t) = P_{T-t}^{\gamma}(0)$. Without loss of generality, we assume that $t = 0$, $\overline{T} > T$, and $v_T(\cdot)$ is optimal for $x_0 \in \mathscr{R}^n$ on $\mathscr{N}_T$. Let

$$v(t) = \begin{cases} v_T(t), & t \in N_T, \\ -R^{-1}D'Cx(t), & t \in \left\{T+1, T+2, \ldots, \overline{T}\right\} \end{cases} \quad (40)$$

and $R = D'D - \gamma^2 I$. By (31) and Remark 11,

$$\begin{aligned} x_0' P_{\overline{T}}^{\gamma}(0)x_0 &\leq J_{\overline{T}}^{\gamma}(x_0, v) = J_T^{\gamma}(x_0, v_T) + \sum_{t=T+1}^{\overline{T}}\mathscr{E}\left[z'(t) \cdot z(t) - \gamma^2 v'(t)v(t)\right] = J_T^{\gamma}(x_0, v_T) \\ &+ \sum_{t=T+1}^{\overline{T}}\mathscr{E}\left\{x'(t)C'\left[I - DR^{-1}D'\right]Cx(t)\right\} \\ &+ \sum_{t=T+1}^{\overline{T}}\mathscr{E}\left\{\left[v(t) + R^{-1}D'Cx(t)\right]' \cdot R\left[v(t) + R^{-1}D'Cx(t)\right]\right\} = J_T^{\gamma}(x_0, v_T) \\ &+ \sum_{t=T+1}^{\overline{T}}\mathscr{E}\left\{x'(t)C'\left[I - DR^{-1}D'\right]Cx(t)\right\} \\ &\leq J_T^{\gamma}(x_0, v_T) = x_0' P_T^{\gamma}(0)x_0. \end{aligned} \quad (41)$$

This implies (39). □

Based on Theorems 7 and 12, it is easy to get the following main result.

Theorem 14. *For system (1) and a given $\gamma \geq 0$, the following are equivalent.*

(a) $\|\mathscr{G}\|_{-}^{[0,T]} > \gamma$.

(b) *The following equation*

$$P(t) = \mathscr{L}(P(t+1)) - \mathscr{K}(P(t+1))\mathscr{H}^{\gamma}(P(t+1))^{-1}\mathscr{K}(P(t+1))', \quad (42)$$

$$\mathscr{H}^{\gamma}(P(t+1)) > 0,$$

$$P(T+1) = 0$$

admits a unique solution $P_T^{\gamma}(t)$ on $\mathscr{N}_{T+1}$. Moreover, $\min_{v \in l_w^2(\mathscr{N}_T; \mathscr{R}^l)} J_T^{\gamma}(x_0, v) = x_0' P_T^{\gamma}(0)x_0$.

Remark 15. The solution of (42) is not necessarily negative or positive definite.

Theorem 16. *For given $\gamma \geq 0$, if system (1) is time-invariant and square, then the following are equivalent.*

(a) $\|\mathscr{G}\|_{-}^{[0,T]} > \gamma$.

(b) *The following equation*

$$\begin{aligned} P(t) &= A' P(t+1) A + A_0' P(t+1) A_0 + C'C \\ &\quad - \left[A' P(t+1) B + A_0' P(t+1) B_0 + C'D\right] \\ &\quad \cdot \left[B' P(t+1) B + B_0' P(t+1) B_0 + D'D - \gamma^2 I\right]^{-1} \\ &\quad \cdot \left[A' P(t+1) B + A_0' P(t+1) B_0 + C'D\right]', \\ &B' P(t+1) B + B_0' P(t+1) B_0 + D'D - \gamma^2 I > 0, \\ &P(T+1) = 0 \end{aligned} \quad (43)$$

admits a unique solution $P_T^{\gamma}(t) \leq 0$ on $\mathscr{N}_{T+1}$. Moreover, $\min_{v\in l_w^2(\mathscr{N}_T;\mathscr{R}^l)} J_T^{\gamma}(x_0, v) = x_0' P_T^{\gamma}(0) x_0$.

Proof. By Theorem 14, Theorem 16 is established as long as we prove $P_T^{\gamma}(t) \leq 0$.

For any $t_0 \in \mathscr{N}_T, x(t_0) = x \in \mathscr{R}^n$, from Lemma 6 and (43), using completing squares method, it follows that

$$\begin{aligned} J_T^{\gamma}(x, v; t_0) &= \sum_{t=t_0}^{T} \mathscr{E}\left[z'(t) z(t) - \gamma^2 v'(t) v(t)\right] \\ &= x' P_T^{\gamma}(t_0) x + \sum_{t=t_0}^{T} \mathscr{E}\left\{\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' \mathscr{M}\left(t, P_T^{\gamma}(\cdot)\right) \right. \\ &\quad \left. \cdot \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}\right\} = x' P_T^{\gamma}(t_0) x + \sum_{t=t_0}^{T} \mathscr{E}\left\{\left[v(t) - v^*(t)\right]' \right. \\ &\quad \left. \cdot \mathscr{H}^{\gamma}\left(P_T^{\gamma}(t+1)\right)\left[v(t) - v^*(t)\right]\right\}, \end{aligned} \quad (44)$$

where $v^*(t) = -\mathscr{H}^{\gamma}(P_T(t+1))^{-1}\mathscr{K}'(P_T(t+1))x(t)$.

Set $R = D'D - \gamma^2 I$. By completing squares, we have

$$\begin{aligned} J_T^{\gamma}(x, v; t_0) &= \sum_{t=t_0}^{T} \mathscr{E}\left[z'(t) z(t) - \gamma^2 v'(t) v(t)\right] \\ &= \sum_{t=t_0}^{T} \mathscr{E}\left\{x'(t) C' \left[I - DR^{-1}D'\right] Cx(t)\right\} \\ &\quad + \sum_{t=t_0}^{T} \mathscr{E}\left\{\left[v(t) + R^{-1}D'Cx(t)\right]' \right. \\ &\quad \left. \cdot R\left[v(t) + R^{-1}D'Cx(t)\right]\right\} = \sum_{t=t_0}^{T} \mathscr{E}\left\{x'(t) \right. \\ &\quad \left. \cdot C' \left[I - DR^{-1}D'\right] Cx(t)\right\} + \sum_{t=t_0}^{T} \mathscr{E}\left\{\left[v(t) - \tilde{v}(t)\right]' \right. \\ &\quad \left. \cdot R\left[v(t) - \tilde{v}(t)\right]\right\}, \end{aligned} \quad (45)$$

where $\tilde{v}(t) = -R^{-1}D'Cx(t)$.

Based on the above and Remark 11, it is easy to see that

$$\begin{aligned} &\min_{v\in l_w^2(\mathscr{N}_T;\mathscr{R}^l)} J_T^{\gamma}(x, v; t_0) \\ &\quad = J_T^{\gamma}(x, v^*; t_0) = x' P_T^{\gamma}(t_0) x \leq J_T^{\gamma}(x, \tilde{v}; t_0) \\ &\quad = \sum_{t=t_0}^{T} \mathscr{E}\left\{x'(t) C' \left[I - DR^{-1}D'\right] Cx(t)\right\} \leq 0 \end{aligned} \quad (46)$$

for arbitrary $x \in \mathscr{R}^n$. This implies $P_T^{\gamma}(t) \leq 0, t \in \mathscr{N}_{T+1}$. □

Remark 17. For given $\gamma \geq 0$, if system (1) is time-invariant and square, replacing B by $B_\delta = \left[B \ \ 0_{n\times n}\right]$, C by $C_\delta = \left[\begin{smallmatrix} C \\ \delta I_n \end{smallmatrix}\right]$, D by $D_\delta = \left[\begin{smallmatrix} D & 0_{l\times n} \\ 0_{n\times l} & 0_{n\times n} \end{smallmatrix}\right]$, $v(t)$ by $v_\delta(t) = \left[\begin{smallmatrix} v(t) \\ 0_{n\times n} \end{smallmatrix}\right]$, and $z(t)$ by $z_\delta(t)$ in (1), we have the corresponding $\mathscr{H}_-$ index $\|\mathscr{G}_\delta\|_{-}^{[0,T]}$ and the cost

$$\begin{aligned} J_{T,\delta}^{\gamma}(x_0, v) &= \sum_{t=0}^{T} \mathscr{E}\left[z_\delta'(t) z_\delta(t) - \gamma^2 v_\delta'(t) v_\delta(t)\right] \\ &= \sum_{t=0}^{T} \mathscr{E}\left[z'(t) z(t) - \gamma^2 v'(t) v(t) + \delta^2 I\right]. \end{aligned} \quad (47)$$

When $\|\mathscr{G}\|_{-}^{[0,T]} > \gamma$, $\|\mathscr{G}_\delta\|_{-}^{[0,T]} > \gamma$. Applying Theorem 16 to the modified data, we find that the following equation

$$\begin{aligned} &-P(t) + A' P(t+1) A + A_0' P(t+1) A_0 + C'C + \delta^2 I \\ &\quad - \left(A' P(t+1) B + A_0' P(t+1) B_0 + C'D\right) \\ &\quad \cdot \left(B_0' P(t+1) B_0 + B' P(t+1) B + D'D - \gamma^2 I\right)^{-1} \\ &\quad \cdot \left(A' P(t+1) B + A_0' P(t+1) B_0 + C'D\right)' = 0, \\ &B_0' P(t+1) B_0 + B' P(t+1) B + D'D - \gamma^2 I > 0, \\ &P(T+1) = 0 \end{aligned} \quad (48)$$

has a unique solution $P_{\delta,T}^{\gamma}(t) \leq 0$ on $\mathscr{N}_{T+1}$. Moreover, $\min_{v\in l_w^2(\mathscr{N}_T;\mathscr{R}^l)} J_{T,\delta}^{\gamma}(x_0, v) = x_0' P_{\delta,T}^{\gamma}(0) x_0$.

3. Examples

In this section, we present some simple examples to illustrate applications of the results developed in this paper.

Example 1. Consider system (1) with

$$A(t)=\begin{bmatrix}4 & 2^{2-t}\\ 0 & 2\end{bmatrix},\quad B(t)=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix},\quad A_0(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad B_0(t)=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},$$
$$C(t)=\begin{bmatrix}0.1 & 0.5\\ 0.3 & 0.4\\ 3 & 2\end{bmatrix},\quad D(t)=\begin{bmatrix}3 & 4\\ 1 & 5\\ 3 & 2-0.2\times 2^{3-t}\end{bmatrix},\quad \gamma=1.5,\quad T=2. \tag{49}$$

By Theorem 14, we have

$$P_T^{\gamma}(2)=\begin{bmatrix}23.043 & 1.617\\ 1.617 & 6.29\end{bmatrix},\quad P_T^{\gamma}(1)=\begin{bmatrix}247.75 & 140\\ 140 & 72\end{bmatrix},\quad P_T^{\gamma}(0)=\begin{bmatrix}10549 & 9253\\ 9253 & 9084\end{bmatrix}. \tag{50}$$

We can see that $P_T^{\gamma}(t)$ is not necessarily negative definite or positive definite.

Consider the following system:

$$\begin{aligned} x(t+1) &= A(t)x(t)+B(t)f(t) + [A_0(t)x(t)+B_0(t)f(t)]w(t),\\ z(t) &= C(t)x(t)+D(t)f(t),\\ & x(0)=x_0,\ t\in\mathcal{N}_T, \end{aligned} \tag{51}$$

where $x(t)\in\mathcal{R}^n$ is the state, $z(t)\in\mathcal{R}^q$ is the measurement output, and $f(t)\in\mathcal{R}^l$ is the fault input.

The fault detection observer $\mathcal{F}$ has the form

$$\begin{aligned} \hat{x}(t+1) &= A(t)\hat{x}(t)+L[z(t)-\hat{z}(t)] + A_0(t)\hat{x}(t)w(t),\\ \hat{z}(t) &= C(t)\hat{x}(t),\\ r(t) &= V[z(t)-\hat{z}(t)],\\ & x(0)=x_0,\ t\in\mathcal{N}_T, \end{aligned} \tag{52}$$

where $\hat{x}(t)\in\mathcal{R}^n$ is the state estimation, $L\in\mathcal{R}^{n\times q}$ is the gain matrix to be designed, and $V\in\mathcal{R}^{q\times q}$ is a nonsingular weighting matrix.

From the filter $\mathcal{F}$ and system (51), let $e(t)=x(t)-\hat{x}(t)$, and we can express the residual error equation $\mathcal{R}$ as

$$\begin{aligned} e(t+1) &= \overline{A}(t)e(t)+\overline{B}(t)f(t) + \left[\overline{A}_0(t)e(t)+\overline{B}_0(t)f(t)\right]w(t),\quad t\in\mathcal{N}_T\\ r(t) &= \overline{C}(t)x(t)+\overline{D}(t)f(t),\quad x(0)=x_0, \end{aligned} \tag{53}$$

where $\overline{A}(t)=A(t)-LC(t)$, $\overline{B}(t)=B(t)-LD(t)$, $\overline{A}_0(t)=A_0(t)$, $\overline{B}_0(t)=B_0(t)$, $\overline{C}(t)=VC(t)$, and $\overline{D}(t)=VD(t)$. We note that (53) is of the same form as (1). If we take f as input and r as the output, the worst-case fault sensitivity of system (53) is the $\mathcal{H}_-$ index problem discussed in Section 2, and the $\mathcal{H}_-$ index gives a guarantee on the performance of a fault detection observer.

Example 2. Consider system $\sum_1$ formed (53) with coefficients

$$\overline{A}=\begin{bmatrix}4 & 2^{20-t}\\ 0 & 2\end{bmatrix},\quad \overline{B}=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix},\quad \overline{A}_0=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad \overline{B}_0=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix},$$
$$\overline{C}=\begin{bmatrix}0.1 & 4\\ 0.3 & 5\\ 3 & 2-0.1\times 2^{20-t}\end{bmatrix},\quad \overline{D}=\begin{bmatrix}3 & 0.5\\ 1 & 0.4\\ 3 & 2\end{bmatrix},\quad T=20. \tag{54}$$

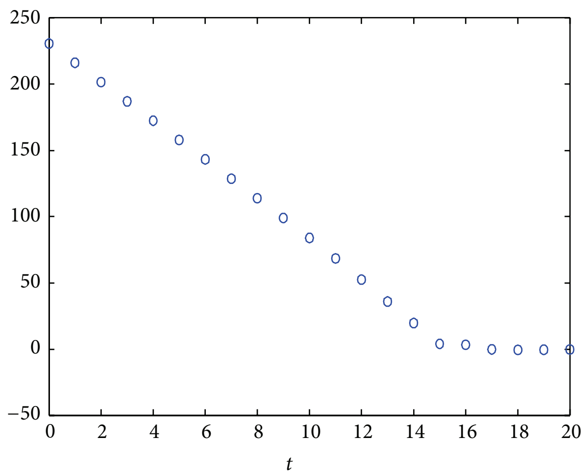

Figure 1: The minimum eigenvalue of $P_{1T}^{\gamma}(t)$.

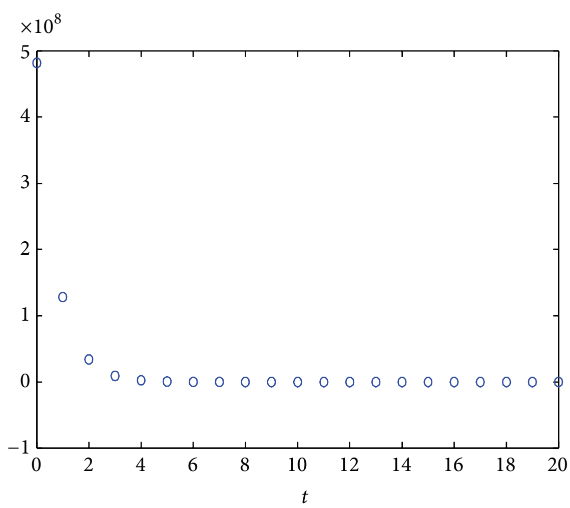

Figure 2: The minimum eigenvalue of $P_{2T}^{\gamma}(t)$.

For $\gamma_1 = \max\gamma = 1.5262$, that is, $\|\mathscr{G}\|_-^{[0,T]} = 1.5262$, by Theorem 14, (42) admits a unique solution $P_{1T}^{\gamma}(t)$. Figure 1 shows the minimum eigenvalue of $P_{1T}^{\gamma}(t)$.

If system $\sum_2$ is the same as system $\sum_1$ except $\overline{A}$ and $\overline{D}$,

$$\overline{A} = \begin{bmatrix} 4 & 3 - 0.2^{20-t} \\ 0 & 2 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 3 & 0.5 \\ 0.1 \times 2^{21-t} & 5 \\ 3 & 1 \end{bmatrix}, \tag{55}$$

with $\gamma_2 = \max\gamma = 1.9617$, that is, $\|\mathscr{G}\|_-^{[0,T]} = 1.9617$, by Theorem 14, (42) admits a unique solution $P_{2T}^{\gamma}(t)$. Figure 2 shows the minimum eigenvalue of $P_{2T}^{\gamma}(t)$.

By comparing the $\mathscr{H}_-$ indexes of system $\sum_1$ and system $\sum_2$, system $\sum_2$ has higher fault detection ability as $\gamma_2 > \gamma_1$.

4. Conclusion

In this paper, we have discussed the stochastic $\mathscr{H}_-$ index of linear discrete-time systems with state and input dependent noise. A necessary and sufficient condition has been presented in finite time horizon. The condition is given by means of the solvability of a constrained difference equation. These results can be used in fault detection. The numerical examples are given to illustrate the proposed methods.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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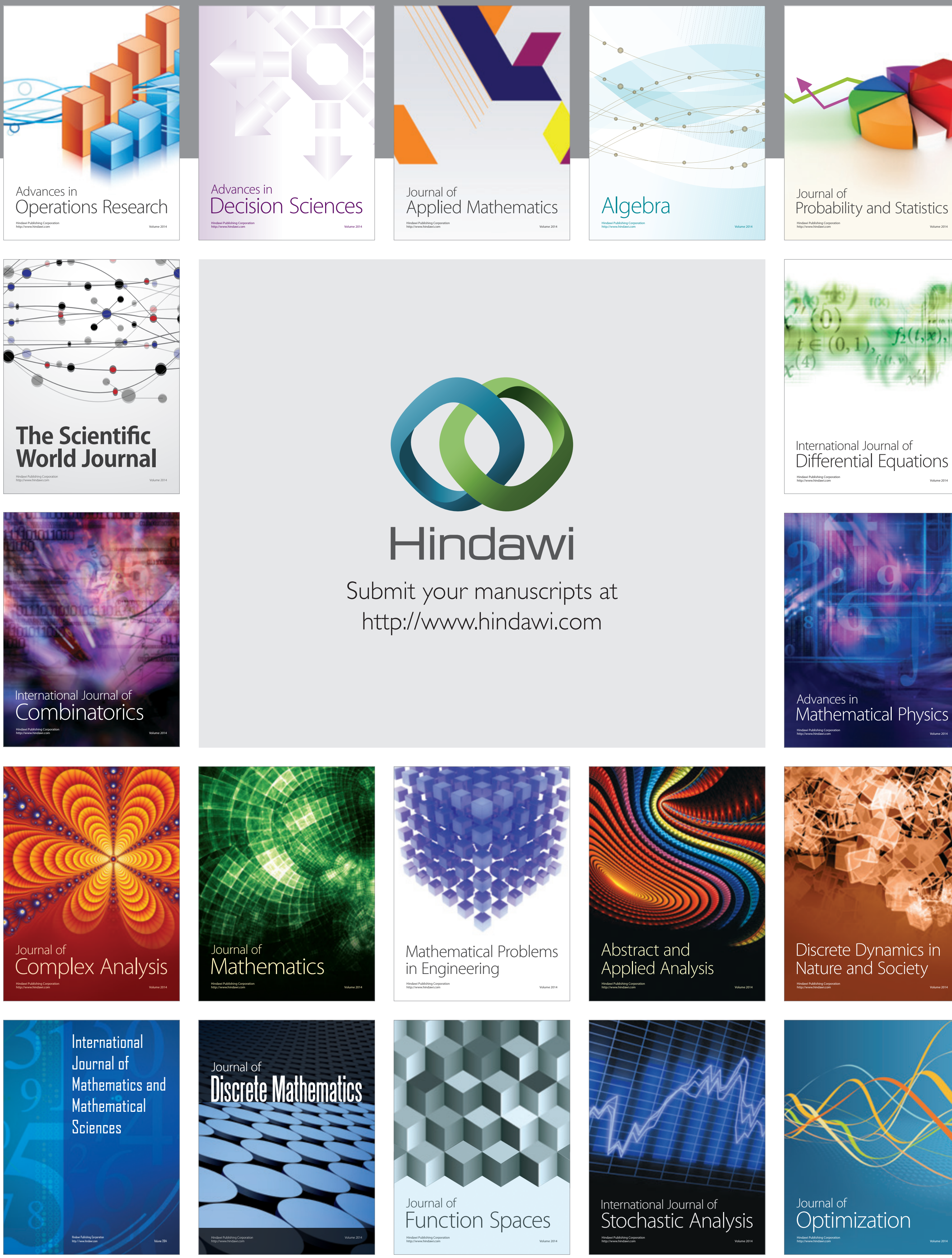
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