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Research Article

Global Positive Periodic Solutions for Periodic Two-Species Competitive Systems with Multiple Delays and Impulses

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A set of easily verifiable sufficient conditions are derived to guarantee the existence and the global stability of positive periodic solutions for two-species competitive systems with multiple delays and impulses, by applying some new analysis techniques. This improves and extends a series of the well-known sufficiency theorems in the literature about the problems mentioned previously.

1. Introduction

Throughout this paper, we make the following notation and assumptions:

let $\omega > 0$ be a constant and

$C_\omega = \{x \mid x \in C(R,R),\ x(t+\omega) = x(t)\}$, with the norm being defined by $|x|_0 = \max_{t\in[0,\omega]}|x(t)|$;

$C^1_\omega = \{x \mid x \in C^1(R,R),\ x(t+\omega) = x(t)\}$, with the norm being defined by $\|x\| = \max_{t\in[0,\omega]}\{|x|_0, |x'|_0\}$;

$PC = \{x \mid x : R \to R^+, \lim_{s\to t} x(s) = x(t)$, if $t \neq t_k, \lim_{t\to t_k^-} x(t) = x(t_k), \lim_{t\to t_k^+} x(t)$ exists, $k \in Z^+\}$;

$PC^1 = \{x \mid x : R \to R^+, x' \in PC\}$;

$PC_\omega = \{x \mid x \in PC, x(t+\omega) = x(t)\}$, with the norm being defined by $|x|_0 = \max_{t\in[0,\omega]}|x(t)|$;

$PC^1_\omega = \{x \mid x \in PC^1, x(t+\omega) = x(t)\}$, with the norm being defined by $\|x\| = \max_{t\in[0,\omega]}\{|x|_0, |x'|_0\}$;

then those spaces are all Banach spaces. We also denote that

$$\overline{f} = \frac{1}{\omega}\int_0^\omega f(t)\,dt, \qquad f^L = \min_{t\in[0,\omega]} f(t), \qquad f^M = \max_{t\in[0,\omega]} f(t), \tag{1}$$

for any $f \in PC_\omega$.

In this paper, we investigate the existence, uniqueness, and global stability of the positive periodic solution for two corresponding periodic Lotka-Volterra competitive systems involving multiple delays and impulses:

$$x_1'(t) = x_1(t)\Big[r_1(t) - a_1(t)x_1(t) + \sum_{i=1}^{n} b_{1i}(t)x_1(t-\tau_i(t)) - \sum_{j=1}^{m} c_{1j}(t)x_2(t-\delta_j(t))\Big],$$

$$\begin{aligned}
x_2'(t) &= x_2(t)\Bigg[r_2(t) - a_2(t)x_2(t) \\
&\quad + \sum_{j=1}^{m} b_{2j}(t)x_2\left(t-\eta_j(t)\right) \\
&\quad - \sum_{i=1}^{n} c_{2i}(t)x_1\left(t-\sigma_i(t)\right)\Bigg], \quad t \neq t_k, \\
\Delta x_l(t) &= x_l(t^+) - x_l(t) = \theta_{lk}x_l(t), \\
&\quad l = 1,2, \quad k = 1,2,\ldots, \quad t = t_k,
\end{aligned} \tag{2}$$

$$\begin{aligned}
x_1'(t) &= x_1(t)\Bigg[r_1(t) - a_1(t)x_1(t) \\
&\quad - \sum_{i=1}^{n} b_{1i}(t)x_1\left(t-\tau_i(t)\right) \\
&\quad - \sum_{j=1}^{m} c_{1j}(t)x_2\left(t-\delta_j(t)\right)\Bigg], \\
x_2'(t) &= x_2(t)\Bigg[r_2(t) - a_2(t)x_2(t) \\
&\quad - \sum_{j=1}^{m} b_{2j}(t)x_2\left(t-\eta_j(t)\right) \\
&\quad - \sum_{i=1}^{n} c_{2i}(t)x_1\left(t-\sigma_i(t)\right)\Bigg], \quad t \neq t_k, \\
\Delta x_l(t) &= x_l(t^+) - x_l(t) = \theta_{lk}x_l(t), \\
&\quad l = 1,2, \quad k = 1,2,\ldots, \quad t = t_k,
\end{aligned} \tag{3}$$

with initial conditions

$$\begin{gathered}
x_l(\xi) = \phi_l(\xi), \qquad x_l'(\xi) = \phi_l'(\xi), \\
\xi \in [-\tau, 0], \qquad \phi_l(0) > 0, \\
\phi_l \in C\left([-\tau,0), R^+\right) \bigcap C^1\left([-\tau,0), R^+\right), \quad l = 1,2,
\end{gathered} \tag{4}$$

where $a_1(t)$, $a_2(t)$, $b_{1i}(t)$, $b_{2j}(t)$, $c_{1j}(t)$, and $c_{2i}(t)$ are all in PC_ω. Also $\tau_i(t)$, $\delta_j(t)$, $\eta_j(t)$, and $\sigma_i(t)$ are all in PC_ω^1 with $\tau_i(t) > 0$, $\delta_j(t) > 0$, $\eta_j(t) > 0$, $\sigma_i(t) > 0$, $t \in [0,\omega]$, $\tau = \max\{\tau_i(t), \delta_j(t), \eta_j(t), \sigma_i(t)\}$, $\tau_i'(t) < 1$, $\delta_j'(t) < 1$, $\eta_j'(t) < 1$, $\sigma_i'(t) < 1$ $(i = 1,2,\ldots,n,\ j = 1,2,\ldots,m)$. Furthermore, the intrinsic growth rates $r_1(t), r_2(t) \in PC_\omega$ are with $\int_0^\omega r_l(t)dt > 0$, $(l = 1,2)$. For the ecological justification of (2) and (3) and similar types refer to [1–10].

In [1], Freedman and Wu proposed the following periodic single-species population growth models with periodic delay:

$$y'(t) = y(t)\left[r(t) - a(t)y(t) + b(t)y(t-\tau(t))\right]. \tag{5}$$

They had assumed that the net birth $r(t)$, the self-inhibition rate $a(t)$, and the delay $\tau(t)$ are continuously differentiable ω-periodic functions, and $r(t) > 0$, $a(t) > 0$, $b(t) \geq 0$, and $\tau(t) \geq 0$ for $t \in R$. The positive feedback term $b(t)y(t-\tau(t))$ in the average growth rate of species has a positive time delay (the sign of the time delay term is positive), which is a delay due to gestation (see [1, 2]). They had established sufficient conditions which guarantee that system (5) has a positive periodic solution which is globally asymptotically stable.

In [3], Fan and Wang investigated the following periodic single-species population growth models with periodic delay:

$$y'(t) = y(t)\left[r(t) - a(t)y(t) - b(t)y(t-\tau(t))\right]. \tag{6}$$

They had assumed that the net birth $r(t)$, the self-inhibition rate $a(t)$, and the delay $\tau(t)$ are continuously differentiable ω-periodic functions, and $r(t) > 0$, $a(t) > 0$, $b(t) \geq 0$, and $\tau(t) \geq 0$ for $t \in R$. The negative feedback term $-b(t)y(t-\tau(t))$ in the average growth rate of species has a negative time delay (the sign of the time delay term is negative), which can be regarded as the deleterious effect of time delay on a species growth rate (see [4–6]). They had derived sufficient conditions for the existence and global attractivity of positive periodic solutions of system (6). But the discussion of global attractivity is only confined to the special case when the periodic delay is constant.

Alvarez and Lazer [7] and Ahmad [8] have studied the following two-species competitive system without delay:

$$\begin{aligned}
y_1'(t) &= y_1(t)\left[r_1(t) - a_1(t)y_1(t) - c_1(t)y_2(t)\right], \\
y_2'(t) &= y_2(t)\left[r_2(t) - a_2(t)y_2(t) - c_2(t)y_1(t)\right].
\end{aligned} \tag{7}$$

They had derived sufficient conditions for the existence and global attractivity of positive periodic solutions of system (7) by using differential inequalities and topological degree, respectively. In fact, in many practical situations the time delay occurs so often. A more realistic model should include some of the past states of the system. Therefore, in [10], Liu et al. considered two corresponding periodic Lotka-Volterra competitive systems involving multiple delays:

$$\begin{aligned}
y_1'(t) &= y_1(t)\Bigg[r_1(t) - a_1(t)y_1(t) + \sum_{i=1}^{n} b_{1i}(t)y_1\left(t-\tau_i(t)\right) \\
&\quad - \sum_{j=1}^{m} c_{1j}(t)y_2\left(t-\rho_j(t)\right)\Bigg], \\
y_2'(t) &= y_2(t)\Bigg[r_2(t) - a_2(t)y_2(t) + \sum_{j=1}^{m} b_{2j}(t)y_2\left(t-\eta_j(t)\right) \\
&\quad - \sum_{i=1}^{n} c_{2i}(t)y_1\left(t-\sigma_i(t)\right)\Bigg],
\end{aligned} \tag{8}$$

$$y_1'(t) = y_1(t)\left[r_1(t) - a_1(t)y_1(t) - \sum_{i=1}^{n} b_{1i}(t)y_1(t-\tau_i(t)) - \sum_{j=1}^{m} c_{1j}(t)y_2(t-\rho_j(t))\right],$$

$$y_2'(t) = y_2(t)\left[r_2(t) - a_2(t)y_2(t) - \sum_{j=1}^{m} b_{2j}(t)y_2(t-\eta_j(t)) - \sum_{i=1}^{n} c_{2i}(t)y_1(t-\sigma_i(t))\right], \tag{9}$$

where $b_{1i}(t)$, $b_{2j}(t) \in C(R,[0,+\infty))$, $a_1(t)$, $a_2(t)$, $c_{1j}(t)$, $c_{2i}(t) \in C(R,[0,+\infty))$, $\tau_i(t)$, $\rho_j(t)$, $\eta_j(t)$, and $\sigma_i(t) \in C^1(R,[0,+\infty))$ $(i = 1,2,\ldots,n; j = 1,2,\ldots,m)$ are ω-periodic functions. Here, the intrinsic growth rates $r_k(t) \in C(R,R)$ are ω-periodic functions with $\int_0^\omega r_k(t)dt > 0$ $(k = 1,2)$. They had derived the same criteria for the existence and globally asymptotic stability of positive periodic solutions of the above two competitive systems by using Gaines and Mawhin's continuation theorem of coincidence degree theory and by means of a suitable Lyapunov functional.

However, the ecological system is often deeply perturbed by human exploitation activities such as planting, harvesting, and so on, which makes it unsuitable to be considered continually. For having a more accurate description of such a system, we need to consider the impulsive differential equations. The theory of impulsive differential equations not only is richer than the corresponding theory of differential equations without impulses, but also represents a more natural framework for mathematical modeling of many real world phenomena [11–13]. Recently, some impulsive equations have been recently introduced in population dynamics in relation to population ecology [14–26] and chemotherapeutic treatment [27, 28]. However, to the best of the authors' knowledge, to this day, few scholars have done works on the existence, uniqueness, and global stability of positive periodic solution of (2) and (4). One could easily see that systems (5)–(9) are all special cases of systems (2) and (3). Therefore, we propose and study the systems (2) and (3) in this paper.

For the sake of generality and convenience, we always make the following fundamental assumptions.

(H_1) $a_1(t)$, $a_2(t)$, $b_{1i}(t)$, $b_{2j}(t)$, $c_{1j}(t)$, $c_{2i}(t)$, $r_1(t)$, and $r_2(t)$ are all in PC_ω; $\tau_i(t)$, $\delta_j(t)$, $\eta_j(t)$, and $\sigma_i(t)$ are all in PC_ω^1 with $\tau_i(t) > 0$, $\delta_j(t) > 0$, $\eta_j(t) > 0$, $\sigma_i(t) > 0$, $t \in [0,\omega]$, $\tau = \max\{\tau_i(t), \delta_j(t), \eta_j(t), \sigma_i(t)\}$, $\tau_i'(t) < 1$, $\delta_j'(t) < 1$, $\eta_j'(t) < 1$, and $\sigma_i'(t) < 1$ $(i = 1,2,\ldots,n; j = 1,2,\ldots,m)$.

(H_2) $[t_k]_{k\in N}$ satisfies $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = +\infty$, θ_{lk} $(i = 1,2)$ are constants, and there exists a positive integer $q > 0$ such that $t_{k+q} = t_k + \omega$, $\theta_{l(k+q)} = \theta_{lk}$. Without loss of generality, we can assume that $t_k \neq 0$ and $[0,\omega] \cap \{t_k\} = t_1, t_2, \ldots, t_m$, and then $q = m$.

(H_3) $\{\theta_{lk}\}$ is a real sequence such that $\theta_{lk} + 1 > 0$, $\prod_{0<t_k<t}(1+\theta_{lk})$, $l = 1,2$ is an ω-periodic function.

Definition 1. A function $x_l : R \to (0,+\infty)$, $l = 1,2$ is said to be a positive solution of (2) and (3), if the following conditions are satisfied:

(a) $x_l(t)$ is absolutely continuous on each (t_k, t_{k+1});

(b) for each $k \in Z_+$, $x_l(t_k^+)$ and $x_l(t_k^-)$ exist, and $x_l(t_k^-) = x_l(t_k)$;

(c) $x_l(t)$ satisfies the first equation of (2) and (3) for almost everywhere (for short a.e.) in $[0,\infty] \setminus \{t_k\}$ and satisfies $x_l(t_k^+) = (1+\theta_{lk})x_l(t_k)$ for $t = t_k$, $k \in Z_+ = \{1,2,\ldots\}$.

Under the above hypotheses (H_1)–(H_3), we consider the following nonimpulsive delay differential equation:

$$y_1'(t) = y_1(t)\left[r_1(t) - A_1(t)y_1(t) + \sum_{i=1}^{n} B_{1i}(t)y_1(t-\tau_i(t)) - \sum_{j=1}^{m} C_{1j}(t)y_2(t-\delta_j(t))\right],$$

$$y_2'(t) = x_2(t)\left[r_2(t) - A_2(t)y_2(t) + \sum_{j=1}^{m} B_{2j}(t)y_2(t-\eta_j(t)) - \sum_{i=1}^{n} C_{2i}(t)y_1(t-\sigma_i(t))\right], \tag{10}$$

$$y_1'(t) = y_1(t)\left[r_1(t) - A_1(t)y_1(t) - \sum_{i=1}^{n} B_{1i}(t)y_1(t-\tau_i(t)) - \sum_{j=1}^{m} C_{1j}(t)y_2(t-\delta_j(t))\right],$$

$$y_2'(t) = x_2(t)\left[r_2(t) - A_2(t)y_2(t) - \sum_{j=1}^{m} B_{2j}(t)y_2(t-\eta_j(t)) - \sum_{i=1}^{n} C_{2i}(t)y_1(t-\sigma_i(t))\right], \tag{11}$$

with the initial conditions

$$y_l(t) = \varphi_l(\xi), \qquad \xi \in [-\tau, 0],$$
$$\tau = \max\{\tau_i(t), \delta_j(t), \eta_j(t), \sigma_i(t)\}, \tag{12}$$
$$\varphi_l(0) > 0, \qquad \varphi_l \in C([-\tau, 0], R_+),$$

where

$$\begin{aligned} A_l(t) &= a_l(t) \prod_{0<t_k<t} (1+\theta_{lk}), \\ B_{1i}(t) &= b_{1i}(t) \prod_{0<t_k<t-\tau_i(t)} (1+\theta_{1k}), \\ B_{2j}(t) &= b_{2j}(t) \prod_{0<t_k<t-\eta_j(t)} (1+\theta_{2k}), \\ C_{1j}(t) &= c_{1j}(t) \prod_{0<t_k<t-\rho_j(t)} (1+\theta_{2k}), \\ C_{2i}(t) &= c_{2i}(t) \prod_{0<t_k<t-\sigma_i(t)} (1+\theta_{1k}), \\ l = 1,2; \quad & i = 1,2,\ldots,n; \quad j = 1,2,\ldots,m. \end{aligned} \tag{13}$$

The following lemmas will be used in the proofs of our results. The proof of Lemma 2 is similar to that of Theorem 1 in [25].

Lemma 2. *Suppose that* (H_1)*–*(H_3) *hold; then*

(1) *if* $y(t) = (y_1(t), y_2(t))^T$ *is a solution of* (10)–(12) *on* $[-\tau, +\infty)$, *then* $x_l(t) = \prod_{0<t_k<t}(1+\theta_{lk})y_l(t)$ $(l = 1,2)$ *is a solution of* (2)–(4) *on* $[-\tau, +\infty)$;

(2) *if* $x(t) = (x_1(t), x_2(t))^T$ *is a solution of* (2)–(4) *on* $[-\tau, +\infty)$, *then* $y_l(t) = \prod_{0<t_k<t}(1+\theta_{lk})^{-1}x_l(t)$ $(l = 1,2)$ *is a solution of* (10)–(12) *on* $[-\tau, +\infty)$.

Proof. (1) It is easy to see that $x_l(t) = \prod_{0<t_k<t}(1+\theta_{lk})y_l(t)$ $(l = 1,2)$ is absolutely continuous on every interval $(t_k, t_{k+1}]$, $t \neq t_k$, $k = 1,2,\ldots$, and

$$\begin{aligned} x_1'(t) &- x_1(t)\Bigg[r_1(t) - a_1(t)x_1(t) + \sum_{i=1}^{n} b_{1i}(t)x_1(t-\tau_i(t)) - \sum_{j=1}^{m} c_{1j}(t)x_2(t-\delta_j(t))\Bigg] \\ &= \prod_{0<t_k<t}(1+\theta_{1k})y_1'(t) - \prod_{0<t_k<t}(1+\theta_{1k})y_1(t) \\ &\quad\times\Bigg[r_1(t) - a_1(t)\prod_{0<t_k<t}(1+\theta_{1k})y_1(t) + \sum_{i=1}^{n} b_{1i}(t)\prod_{0<t_k<t-\tau_i(t)}(1+\theta_{1k})y_1(t-\tau_i(t)) \\ &\qquad - \sum_{j=1}^{m} c_{1j}(t)\prod_{0<t_k<t-\rho_j(t)}(1+\theta_{2k}) \times y_2(t-\delta_j(t))\Bigg] \\ &= \prod_{0<t_k<t}(1+\theta_{1k})\big(y_1'(t) - y_1(t)\big) \\ &\quad\times\Bigg[r_1(t) - A_1(t)y_1(t) - \sum_{i=1}^{n} B_{1i}(t)y_1(t-\tau_i(t)) - \sum_{j=1}^{m} C_{1j}(t)y_2(t-\rho_j(t))\Bigg] = 0. \end{aligned} \tag{14}$$

On the other hand, for any $t = t_k$, $k = 1,2,\ldots$,

$$\begin{aligned} x_1(t_k^+) &= \lim_{t\to t_k^+}\prod_{0<t_j<t}(1+\theta_{1k})y_1(t) = \prod_{0<t_j\le t_k}(1+\theta_{1k})y_1(t_k), \\ x_1(t_k) &= \prod_{0<t_j<t_k}(1+\theta_{1k})y_1(t_k); \end{aligned} \tag{15}$$

thus

$$\Delta x_1(t_k^+) = (1+\theta_{1k})y_1(t_k), \tag{16}$$

which implies that $x_1(t)$ is a solution of (2); similarly, we can prove that $x_2(t)$ is also a solution of (3). Therefore, $x_l(t)$, $l = 1,2$ are solutions of (2)–(4) on $[-\tau, +\infty)$. Similarly, if $y(t) = (y_1(t), y_2(t))^T$ is a solution of (10)–(12) on $[-\tau, +\infty)$, we can prove that $x_l(t)$ $(l = 1,2)$ are solutions of (2)–(4) on $[-\tau, +\infty)$.

(2) Since $x_l(t) = \prod_{0<t_k<t}(1+\theta_{lk})y_l(t)$ $(l = 1,2)$ is absolutely continuous on every interval $(t_k, t_{k+1}]$, $t \neq t_k$, $k = 1,2,\ldots$, and in view of (15), it follows that for any $k = 1,2,\ldots$,

$$\begin{aligned} y_1(t_k^+) &= \prod_{0<t_j\le t_k}(1+\theta_{1k})^{-1}x_1(t_k^+) \\ &= \prod_{0<t_j<t_k}(1+\theta_{1k})^{-1}x_1(t_k) = y_1(t_k), \end{aligned}$$

$$\begin{aligned} y_1(t_k^-) &= \prod_{0<t_j<t_k} (1+\theta_{1k})^{-1} x_1(t_k^-) \\ &= \prod_{0<t_j\le t_k^-} (1+\theta_{1k})^{-1} x_1(t_k^-) = y_1(t_k), \end{aligned} \tag{17}$$

which implies that $y_1(t)$ is continuous on $[-\tau,+\infty)$. It is easy to prove that $y_1(t)$ is absolutely continuous on $[-\tau,+\infty)$. Similarly, we can prove that $y_2(t)$ is absolutely continuous on $[-\tau,+\infty)$. Similar to the proof of (1), we can check that $y_l(t)=\prod_{0<t_k<t}(1+\theta_{lk})^{-1}x_l(t)$ $(l=1,2)$ are solutions of (10)–(12) on $[-\tau,+\infty)$. If $x(t)=(x_1(t),x_2(t))^T$ is a solution of (2)–(4) on $[-\tau,+\infty)$ by the same method, we can prove that $y_l(t)=\prod_{0<t_k<t}(1+\theta_{lk})^{-1}x_l(t)$ $(l=1,2)$ are solutions of (10)–(12) on $[-\tau,+\infty)$. The proof of Lemma 2 is completed. □

From Lemma 2, if we want to discuss the existence and global asymptotic stability of positive periodic solutions of systems (2)–(4), we only discuss the existence of the existence and global asymptotic stability of positive periodic solutions of systems (10)–(12).

The organization of this paper is as follows. In Section 2, we introduce several useful definitions and lemmas. In Section 3, first, we study the existence of at least one periodic solution of systems (2)–(4) by using continuation theorem proposed by Gaines and Mawhin (see [9]). Second, we investigate the global asymptotic stability of positive periodic solutions of the above systems by using the method of Lyapunov functional. As applications in Section 4, we study some particular cases of systems (2)–(4) which have been investigated extensively in the references mentioned previously.

2. Preliminaries

In this section, we will introduce some concepts and some important lemmas which are useful for the next section.

Let X, Z be two real Banach spaces, let $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, and let $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim\operatorname{Ker}L = \operatorname{condim}\operatorname{Im}L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q)$, it follows that $L|_{\operatorname{Dom} L\cap \operatorname{Ker} P} : (I-P)X \to \operatorname{Im} L$ is invertible; we denote the inverse of that map by K_p. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exist isomorphisms $J : \operatorname{Im} Q \to \operatorname{Ker} L$. Let PC_ω denote the space of ω-periodic functions $\Psi : J \to R$ which are continuous for $t \neq t_k$, are continuous from the left for $t \in R$, and have discontinuities of the first kind at point $t = t_k$. We also denote that $PC_\omega^1 = \{\Psi \in PC_\omega : \Psi' \in PC_\omega\}$.

Definition 3 (see [11]). The set $F \in PC_\omega$ is said to be quasiequicontinuous in $[0,\omega]$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in F$, $k \in N^+$, $t_1, t_2 \in (t_{k-1}, t_k) \cap [0,\omega]$, and $|t_1 - t_2| < \delta$, then $|x(t_1) - x(t_2)| < \epsilon$.

Definition 4. Let $x^*(t) = (x_1^*(t), x_2^*(t))^T$ be a strictly positive periodic solution of (2)–(4). One says that $x^*(t)$ is globally attractive if any other solution $x(t) = (x_1(t), x_2(t))^T$ of (2)–(4) has the property $\lim_{t\to+\infty}|x_i^*(t) - x_i(t)| = 0$, $i = 1, 2$.

Lemma 5. *The region $R_+^2 = \{(x_1,x_2) : x_1(0) > 0, x_2(0) > 0\}$ is the positive invariable region of the systems* (2)–(4).

Proof. By the definition of $x_l(t)$ $(l = 1, 2)$ we have $x_l(0) > 0$. In view of having

$$\begin{aligned} x_1(t) = x_1(0)\exp\Bigg\{\int_0^t \Bigg[& r_1(\xi) - a_1(\xi)x_1(\xi) \\ & + \sum_{i=1}^n b_{1i}(\xi)x_1(\xi-\tau_i(\xi)) \\ & - \sum_{j=1}^m c_{1j}(\xi)x_2(\xi-\delta_j(\xi))\Bigg]d\xi\Bigg\}, \\ & \qquad t\in[0,t_1], \\ x_1(t) = x_1(t_k)\exp\Bigg\{\int_0^t \Bigg[& r_1(\xi) - a_1(\xi)x_1(\xi) \\ & + \sum_{i=1}^n b_{1i}(\xi)x_1(\xi-\tau_i(\xi)) \\ & - \sum_{j=1}^m c_{1j}(\xi)x_2(\xi-\delta_j(\xi))\Bigg]d\xi\Bigg\}, \\ & \qquad t\in(t_k,t_{k+1}], \end{aligned}$$

$$x_1(t_k^+) = (1+\theta_{1k})x_1(t_k) > 0, \quad k \in N;$$

$$\begin{aligned} x_2(t) = x_2(0)\exp\Bigg\{\int_0^t \Bigg[& r_2(\xi) - a_2(\xi)x_2(\xi) \\ & + \sum_{j=1}^m b_{2j}(\xi)x_2(\xi-\eta_j(\xi)) \\ & - \sum_{i=1}^n c_{2i}(\xi)x_1(\xi-\sigma_i(\xi))\Bigg]d\xi\Bigg\}, \\ & \qquad t\in[0,t_1], \\ x_2(t) = x_2(t_k)\exp\Bigg\{\int_0^t \Bigg[& r_2(\xi) - a_2(\xi)x_2(\xi) \\ & + \sum_{j=1}^m b_{2j}(\xi)x_2(\xi-\eta_j(\xi)) \\ & - \sum_{i=1}^n c_{2i}(\xi)x_1(\xi-\sigma_i(\xi))\Bigg]d\xi\Bigg\}, \\ & \qquad t\in(t_k,t_{k+1}], \end{aligned}$$

$$x_2(t_k^+) = (1+\theta_{2k})\, x_2(t_k) > 0, \quad k \in N, \tag{18}$$

$$\begin{aligned}
x_1(t) &= x_1(0)\exp\left\{\int_0^t \left[r_1(\xi) - a_1(\xi)x_1(\xi) - \sum_{i=1}^n b_{1i}(\xi)x_1(\xi-\tau_i(\xi)) - \sum_{j=1}^m c_{1j}(\xi)x_2(\xi-\delta_j(\xi))\right]d\xi\right\}, \quad t \in [0,t_1],\\
x_1(t) &= x_1(t_k)\exp\left\{\int_0^t \left[r_1(\xi) - a_1(\xi)x_1(\xi) - \sum_{i=1}^n b_{1i}(\xi)x_1(\xi-\tau_i(\xi)) - \sum_{j=1}^m c_{1j}(\xi)x_2(\xi-\delta_j(\xi))\right]d\xi\right\}, \quad t \in (t_k,t_{k+1}],\\
x_1(t_k^+) &= (1+\theta_{1k})\,x_1(t_k) > 0, \quad k \in N;\\
x_2(t) &= x_2(0)\exp\left\{\int_0^t \left[r_2(\xi) - a_2(\xi)x_2(\xi) - \sum_{j=1}^m b_{2j}(\xi)x_2(\xi-\eta_j(\xi)) - \sum_{i=1}^n c_{2i}(\xi)x_1(\xi-\sigma_i(\xi))\right]d\xi\right\}, \quad t \in [0,t_1],\\
x_2(t) &= x_2(t_k)\exp\left\{\int_0^t \left[r_2(\xi) - a_2(\xi)x_2(\xi) - \sum_{j=1}^m b_{2j}(\xi)x_2(\xi-\eta_j(\xi)) - \sum_{i=1}^n c_{2i}(\xi)x_1(\xi-\sigma_i(\xi))\right]d\xi\right\}, \quad t \in (t_k,t_{k+1}],\\
x_2(t_k^+) &= (1+\theta_{2k})\,x_2(t_k) > 0, \quad k \in N.
\end{aligned} \tag{19}$$

Then the solution of (2)–(4) is positive. The proof of Lemma 5 is completed. □

Lemma 6 (see [19, 29]). *Suppose that $\sigma \in C_\omega^1$ and $\sigma'(t) < 1$, $t \in [0,\omega]$. Then the function $t - \sigma(t)$ has a unique inverse $\mu(t)$ satisfying $\mu \in C(R,R)$ with $\mu(a+\omega) = \mu(a) + \omega$ $\forall a \in R$, and if $g \in PC_\omega$, $\tau'(t) < 1$, $t \in [0,\omega]$, then $g(\mu(t)) \in PC_\omega$.*

Proof. Since $\sigma'(t) < 1$, $t \in [0,\omega]$, and $t - \sigma(t)$ is continuous on R, it follows that $t - \sigma(t)$ has a unique inverse function $\mu(t) \in C(R,R)$ on R. Hence, it suffices to show that $\mu(a + \omega) = \mu(a) + \omega$, $\forall a \in R$. For any $a \in R$, by the condition $\sigma'(t) < 1$, one can find that $t - \sigma(t) = a$ exists as a unique solution t_0 and $t - \sigma(t) = a + \omega$ exists as a unique solution t_1; that is, $t_0 - \sigma(t_0) = a$ and $t_1 - \sigma(t_1) = a + \omega$; that is, $\mu(a) = t_0 = \sigma(t_0) + a$ and $\mu(a+\omega) = t_1$.

As

$$\begin{aligned}
a + \omega + \sigma(t_0) - \sigma(a + \omega + \sigma(t_0)) &= a + \omega + \sigma(t_0) - \sigma(a + \sigma(t_0))\\
&= a + \omega + \sigma(t_0) - \sigma(t_0) = a + \omega,
\end{aligned} \tag{20}$$

it follows that $t_1 = a + \omega + \sigma(t_0)$. Since $\mu(a+\omega) = t_1$, we have $\mu(a+\omega) = t_1 = a+\omega+\sigma(t_0)$ and $\mu(a+\omega) = t_1 = \mu(a)+\omega$. We can easily obtain that if $g \in PC_\omega$, $\tau'(t) < 1$, $t \in [0,\omega]$, then $g(\mu(t+\omega)) = g(\mu(t)+\omega) = g(\mu(t))$, $t \in R$, where $\mu(t)$ is the unique inverse function of $t - \tau(t)$, which together with $\mu \in C(R,R)$ implies that $g(\mu(t)) \in PC_\omega$. The proof of Lemma 6 is completed. □

Lemma 7 (see [9]). *Let X and Z be two Banach spaces, and let $L : \operatorname{Dom} L \subset X \to Z$ be a Fredholm operator with index zero. $\Omega \subset X$ is an open bounded set, and let $N : \overline{\Omega} \to Z$ be L-compact on $\overline{\Omega}$. Suppose that*

(a) *$Lx \neq \lambda Nx$ for each $\lambda \in (0,1)$ and $x \in \partial\Omega \cap \operatorname{Dom} L$;*

(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \ker L$;*

(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then, equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

Lemma 8 (see [11]). *The set $F \subset PC_\omega$ is relatively compact if only if*

(1) *F is bounded, that is, $\|x\| \leq M$, for each x, and some $M > 0$;*

(2) *F is quasiequicontinuous in $[0,\omega]$.*

Lemma 9 (see [30]). *Assume that $f(t)$, $g(t)$ are continuous nonnegative functions defined on the interval $[\alpha,\beta]$; then there exists $\xi \in [\alpha,\beta]$ such that $\int_\alpha^\beta f(t)g(t)dt = f(\xi)\int_\alpha^\beta g(t)dt$.*

Lemma 10 (see [20, 31]). *Suppose that $\phi(t)$ is a differently continuous ω-periodic function on R with ($\omega > 0$); then, for any $t^* \in R$, the following inequality holds:*

$$\max_{t\in[t^*,t^*+\omega]} \Phi(t) \leq |\Phi(t^*)| + \frac{1}{2}\left[\int_0^\omega |\Phi'(t)|\,dt\right]. \tag{21}$$

Lemma 11 (see Barbalat's Lemma [32]). *Let $f(t)$ be a nonnegative function defined on $[0,+\infty)$ such that $f(t)$ is integrable and uniformly continuous on $[0,+\infty)$; then $\lim_{t\to+\infty} f(t)=0$.*

In the following section, we only discuss the existence and global asymptotic stability of positive periodic solutions of systems (10)–(12).

3. Existence and Global Asymptotic Stability

Since $\tau_i'(t)<1$, $\delta_j'(t)<1$, $\eta_j'(t)<1$, $\sigma_i'(t)<1$, $t\in[0,\omega]$, by Lemma 6, we see that all $t-\tau_i(t)$ have their inverse functions. Throughout the following part, we set $\alpha_i(t)$, $\beta_i(t)$, $\mu_j(t)$, and $\nu_j(t)$ to represent the inverse function of $t-\tau_i(t)$, $t-\sigma_i(t)$, $t-\delta_j(t)$, and $t-\eta_j(t)$, respectively. Obviously, $\alpha_i(t)$, $\beta_i(t)$, $\mu_j(t)$, $\nu_j(t)\in PC_\omega^1$. We also denote that

$$\begin{aligned}
F_1(t) &= A_1(t)-\sum_{i=1}^{n}\frac{B_{1i}(\alpha_i(t))}{1-\tau_i'(\alpha_i(t))},\\
F_2(t) &= A_2(t)-\sum_{j=1}^{m}\frac{B_{2j}(\nu_j(t))}{1-\eta_j'(\nu_j(t))},\\
F_1^*(t) &= A_1(t)+\sum_{i=1}^{n}\frac{B_{1i}(\alpha_i(t))}{1-\tau_i'(\alpha_i(t))},\\
F_2^*(t) &= A_2(t)+\sum_{j=1}^{m}\frac{B_{2j}(\nu_j(t))}{1-\eta_j'(\nu_j(t))},\\
G_1(t) &= \sum_{j=1}^{m}\frac{C_{1j}(\mu_j(t))}{1-\delta_j'(\mu_j(t))},\\
G_2(t) &= \sum_{i=1}^{n}\frac{C_{2i}(\beta_i(t))}{1-\sigma_i'(\beta_i(t))}.
\end{aligned} \tag{22}$$

Theorem 12. *In addition to (H_1)–(H_3), assume that one of the following conditions hold:*

(H_4) $\overline{r_1}F_1^{*M}<\overline{r_2}G_2^L$, $\overline{r_2}F_2^{*M}<\overline{r_1}G_1^L$;

(H_5) $\overline{r_1}F_1^{*L}>\overline{r_2}G_2^M$, $\overline{r_2}F_2^{*L}>\overline{r_1}G_1^M$.

Then systems (3) and (4) have at least one positive ω-periodic solution, where $F_1^*(t)$, $F_2^*(t)$, $G_1(t)$, and $G_2(t)$ are defined in (22).

Proof. Since the solutions of systems (11) and (12) remain positive for $t\ge 0$, we carry out the change of variable $u_i(t)=\ln y_i(t)$ $(i=1,2)$; then (11) can be transformed to

$$\begin{aligned}
u_1'(t) &= r_1(t)-A_1(t)e^{u_1(t)}-\sum_{i=1}^{n}B_{1i}(t)e^{u_1(t-\tau_i(t))}\\
&\quad-\sum_{j=1}^{m}C_{1j}(t)e^{u_2(t-\delta_j(t))},\\
u_2'(t) &= r_2(t)-A_2(t)e^{u_2(t)}-\sum_{j=1}^{m}B_{2j}(t)e^{u_2(t-\eta_j(t))}\\
&\quad-\sum_{i=1}^{n}C_{2i}(t)e^{u_1(t-\sigma_i(t))}.
\end{aligned} \tag{23}$$

It is easy to see that if system (23) has one ω-periodic solution $(u_1^*(t),u_2^*(t))^T$, then $(y_1^*(t),y_2^*(t))^T=(e^{u_1^*(t)},e^{u_2^*(t)})^T$ is a positive ω-periodic solution of system (10); that is to say, $(x_1^*(t),x_2^*(t))^T=(\prod_{0<t_k<t}(1+\theta_{1k})e^{u_1^*(t)},\prod_{0<t_k<t}(1+\theta_{2k})e^{u_2^*(t)})^T$ is a positive ω-periodic solution of system (2). Therefore, it suffices to prove that system (23) has a ω-periodic solution. In order to use Lemma 7 to (23), we take

$$\begin{aligned}
X=Z=\Big\{u(t)=(u_1(t),u_2(t))^T\mid u_i(t)\\
\in C(R,R^2):u_i(t+\omega)=u_i(t),l=1,2\Big\}
\end{aligned} \tag{24}$$

and define

$$\begin{aligned}
\|u\| &= \sum_{i=1}^{2}\sup_{t\in[0,\omega]}|u_l(t)|,\\
u(t) &= (u_1(t),u_2(t))^T\in X\ (\text{or } Z).
\end{aligned} \tag{25}$$

Then X and Z are Banach spaces when they are endowed with the norm $\|\cdot\|$. Let $N:X\to Z$ with

$$Nu=\begin{bmatrix} r_1(t)-A_1(t)e^{u_1(t)}-\sum_{i=1}^{n}B_{1i}(t)e^{u_1(t-\tau_i(t))}-\sum_{j=1}^{m}C_{1j}(t)e^{u_2(t-\delta_j(t))}\\ r_2(t)-A_2(t)e^{u_2(t)}-\sum_{j=1}^{m}B_{2j}(t)e^{u_2(t-\eta_j(t))}-\sum_{i=1}^{n}C_{2i}(t)e^{u_1(t-\sigma_i(t))}\end{bmatrix}=\begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}\quad\text{for any } u\in X, \tag{26}$$

and define

$$\begin{aligned}
Lu=u';\quad Pu&=\frac{1}{\omega}\int_0^\omega u(t)\,dt,\quad u\in X;\\
Qz&=\frac{1}{\omega}\int_0^\omega z(t)\,dt,\quad z\in Z.
\end{aligned} \tag{27}$$

It is not difficult to show that

$$\begin{aligned}
\operatorname{Ker}L &= \{u\in X\mid u=h\in R^2\},\\
\operatorname{Im}L &= \left\{z\in Z\mid \int_0^\omega z(s)\,ds=0\right\},
\end{aligned} \tag{28}$$

and $\dim\operatorname{Ker}L = 2 = \operatorname{codim}\operatorname{Im}L$. So, $\operatorname{Im} L$ is closed in Z, and L is a Fredholm mapping of index zero. It is trivial to show that P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q). \tag{29}$$

Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$K_P z = \int_0^t z(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)\,ds\,dt. \tag{30}$$

Thus, for $u \in X$

$$QNu = \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega \left[r_1(t) - A_1(t)e^{u_1(t)} - \sum_{i=1}^n B_{1i}(t)e^{u_1(t-\tau_i(t))} - \sum_{j=1}^m C_{1j}(t)e^{u_2(t-\delta_j(t))} \right] dt \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega \left[r_2(t) - A_2(t)e^{u_2(t)} - \sum_{j=1}^m B_{2j}(t)e^{u_2(t-\eta_j(t))} - \sum_{i=1}^n C_{2i}(t)e^{u_1(t-\sigma_i(t))} \right] dt \end{pmatrix}, \tag{31}$$

$$K_P(I-Q)Nu = \begin{pmatrix} \displaystyle\int_0^t f_1(\xi)\,d\xi \\ \displaystyle\int_0^t f_2(\xi)\,d\xi \end{pmatrix} - \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega\int_0^t f_1(\xi)\,d\xi\,dt \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega\int_0^t f_2(\xi)\,d\xi\,dt \end{pmatrix} - \begin{pmatrix} \left(\dfrac{t}{\omega} - \dfrac{1}{2}\right)\displaystyle\int_0^\omega f_1(\xi)\,d\xi \\ \left(\dfrac{t}{\omega} - \dfrac{1}{2}\right)\displaystyle\int_0^\omega f_2(\xi)\,d\xi \end{pmatrix}. \tag{32}$$

Clearly, QN and $K_P(I - Q)N$ are continuous. By applying Ascoli-Arzela theorem, one can easily show that $QN(\overline{\Omega})$, $K_P(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is obviously bounded. Thus, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$. Now, we reach the position to search for an appropriate open bounded set $\Omega \subset X$ for the application of Lemma 7. Corresponding to the operating equation $Lu = \lambda Nu$, $\lambda \in (0, 1)$, we have

$$\begin{aligned} u_1'(t) &= \lambda \left[r_1(t) - A_1(t)e^{u_1(t)} - \sum_{i=1}^n B_{1i}(t)e^{u_1(t-\tau_i(t))} - \sum_{j=1}^m C_{1j}(t)e^{u_2(t-\delta_j(t))} \right], \\ u_2'(t) &= \lambda \left[r_2(t) - A_2(t)e^{u_2(t)} - \sum_{j=1}^m B_{2j}(t)e^{u_2(t-\eta_j(t))} - \sum_{i=1}^n C_{2i}(t)e^{u_1(t-\sigma_i(t))} \right]. \end{aligned} \tag{33}$$

Since $u(t) = (u_1(t), u_2(t))^T$ is a ω-periodic function, we need only to prove the result in the interval $[0, \omega]$. Integrating (33) over the interval $[0, \omega]$ leads to the following:

$$\begin{aligned} &\int_0^\omega \left[r_1(t) - A_1(t)e^{u_1(t)} - \sum_{i=1}^n B_{1i}(t)e^{u_1(t-\tau_i(t))} - \sum_{j=1}^m C_{1j}(t)e^{u_2(t-\delta_j(t))} \right] dt = 0, \\ &\int_0^\omega \left[r_2(t) - A_2(t)e^{u_2(t)} - \sum_{j=1}^m B_{2j}(t)e^{u_2(t-\eta_j(t))} - \sum_{i=1}^n C_{2i}(t)e^{u_1(t-\sigma_i(t))} \right] dt = 0. \end{aligned} \tag{34}$$

Hence, we have

$$\begin{aligned} &\int_0^\omega \left[A_1(t)e^{u_1(t)} + \sum_{i=1}^n B_{1i}(t)e^{u_1(t-\tau_i(t))} + \sum_{j=1}^m C_{1j}(t)e^{u_2(t-\delta_j(t))} \right] dt = \overline{r_1}\omega, \\ &\int_0^\omega \left[A_2(t)e^{u_2(t)} + \sum_{j=1}^m B_{2j}(t)e^{u_2(t-\eta_j(t))} + \sum_{i=1}^n C_{2i}(t)e^{u_1(t-\sigma_i(t))} \right] dt = \overline{r_2}\omega. \end{aligned} \tag{35}$$

Note that $u(t) = (u_1(t), u_2(t)) \in X$, and then there exists ζ_l, $\xi_l \in [0, \omega]$ $(l = 1, 2)$ such that

$$u_l(\zeta_l) = \inf_{t\in[0,\omega]} u_l(t), \qquad u_l(\xi_l) = \sup_{t\in[0,\omega]} u_l(t), \qquad l = 1, 2. \tag{36}$$

Since $\tau_i'(t) < 1$, we can let $s = t - \tau_i(t)$, that is, $t = \alpha_i(s)$ $(i = 1, 2, \ldots, n)$; then

$$\int_0^\omega B_{1i}(t)e^{u_1(t-\tau_i(t))}dt = \int_{-\tau_i(0)}^{\omega-\tau_i(\omega)} \frac{B_{1i}(\alpha_i(s))}{1 - \tau_i'(\alpha_i(s))} e^{u_1(s)} ds. \tag{37}$$

According to Lemma 7, we know that $((B_{1i}(\alpha_i(s)))/(1-\tau_i'(\alpha_i(s))))e^{u_1(s)} \in C_\omega$. Thus,

$$\int_{-\tau_i(0)}^{\omega-\tau_i(\omega)} \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} e^{u_1(s)} ds = \int_0^\omega \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} e^{u_1(s)} ds. \tag{38}$$

By (37) and (38), we have

$$\int_0^\omega B_{1i}(t) e^{u_1(t-\tau_i(t))} dt = \int_0^\omega \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} e^{u_1(s)} ds. \tag{39}$$

Similarly, we obtain

$$\begin{aligned}
&\int_0^\omega C_{1j}(t) e^{u_2(t-\delta_j(t))} dt = \int_0^\omega \frac{C_{1j}(\mu_j(s))}{1-\delta_j'(\mu_j(s))} e^{u_2(s)} ds,\\
&\int_0^\omega B_{2j}(t) e^{u_1 2(t-\eta_j(t))} dt = \int_0^\omega \frac{B_{2j}(\nu_j(s))}{1-\eta_j'(\nu_j(s))} e^{u_2(s)} ds,\\
&\int_0^\omega C_{2i}(t) e^{u_1(t-\sigma_i(t))} dt = \int_0^\omega \frac{C_{2i}(\beta_i(s))}{1-\sigma_i'(\beta_i(s))} e^{u_1(s)} ds.
\end{aligned} \tag{40}$$

It follows from (35), (39), and (40) that we get

$$\begin{aligned}
&\int_0^\omega \left[\left(A_1(s) + \sum_{i=1}^n \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} \right) e^{u_1(s)} + \sum_{j=1}^m \frac{C_{1j}(\mu_j(s))}{1-\delta_j'(\mu_j(s))} e^{u_2(s)} \right] ds = \overline{r_1}\omega,\\
&\int_0^\omega \left[\left(A_2(s) + \sum_{j=1}^m \frac{B_{2j}(\nu_j(s))}{1-\eta_j'(\nu_j(s))} \right) e^{u_2(s)} + \sum_{i=1}^n \frac{C_{2i}(\beta_i(s))}{1-\sigma_i'(\beta_i(s))} e^{u_1(s)} \right] ds = \overline{r_2}\omega.
\end{aligned} \tag{41}$$

Thus, from (41) we get

$$\begin{aligned}
&\int_0^\omega F_1^*(s) e^{u_1(s)} ds + \int_0^\omega G_1(s) e^{u_2(s)} ds = \overline{r_1}\omega,\\
&\int_0^\omega F_2^*(s) e^{u_2(s)} ds + \int_0^\omega G_2(s) e^{u_1(s)} ds = \overline{r_2}\omega,
\end{aligned} \tag{42}$$

where $F_1^*(s)$, $F_2^*(s)$, $G_1(s)$, and $G_2(s)$ are defined by (22). On the other hand, by Lemma 7, we can see that $\alpha_i(\omega) = \alpha_i(0)+\omega$, so we can derive

$$\begin{aligned}
\int_0^\omega \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} ds &= \int_{\alpha_i(0)}^{\alpha_i(\omega)} \frac{B_{1i}(t)\left(1-\tau_i'(\alpha_i(t))\right)}{1-\tau_i'(\alpha_i(t))} dt\\
&= \int_0^\omega B_{1i}(t) dt = \overline{B_{1i}}\omega.
\end{aligned} \tag{43}$$

Thus, from (43) we get

$$\begin{aligned}
\overline{F_1^*}\omega &= \int_0^\omega F_1^*(s) ds = \int_0^\omega \left[A_1(s) + \sum_{i=1}^n \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} \right] ds\\
&= \left(\overline{A_1} - \sum_{i=1}^n \overline{B_{1i}} \right)\omega,\\
\overline{F_2^*}\omega &= \int_0^\omega F_2^*(s) ds = \int_0^\omega \left[A_2(s) + \sum_{i=1}^n \frac{B_{2j}(\nu_j(s))}{1-\eta_j'(\nu_j(s))} \right] ds\\
&= \left(\overline{A_2} + \sum_{i=1}^n \overline{B_{2j}} \right)\omega,\\
\overline{G_1}\omega &= \int_0^\omega G_1(s) ds\\
&= \int_0^\omega \sum_{j=1}^m \frac{C_{1j}(\mu_j(s))}{1-\delta_j'(\mu_j(s))} ds = \sum_{j=1}^m \overline{C_{1j}}\omega,\\
\overline{G_2}\omega &= \int_0^\omega G_2(s) ds\\
&= \int_0^\omega \sum_{i=1}^n \frac{C_{2i}(\beta_i(s))}{1-\sigma_i'(\beta_i(s))} ds = \sum_{i=1}^n \overline{C_{2i}}\omega.
\end{aligned} \tag{44}$$

On one hand, by (42), we have

$$\begin{aligned}
G_1^L \int_0^\omega e^{u_2(s)} ds \le \int_0^\omega G_1(s) e^{u_2(s)} ds \le \overline{r_1}\omega,\\
G_2^L \int_0^\omega e^{u_1(s)} ds \le \int_0^\omega G_2(s) e^{u_1(s)} ds \le \overline{r_2}\omega,
\end{aligned} \tag{45}$$

which implies that

$$\begin{aligned}
\int_0^\omega e^{u_2(s)} ds \le \frac{\overline{r_1}\omega}{G_1^L},\\
\int_0^\omega e^{u_1(s)} ds \le \frac{\overline{r_2}\omega}{G_2^L}.
\end{aligned} \tag{46}$$

On the other hand, by (42), the integral mean value theorem that there are λ_1, λ_2, ρ_1, and $\rho_2 \in [0,\omega]$ such that

$$\begin{aligned}
F_1^*(\lambda_1) \int_0^\omega e^{u_1(s)} ds + G_1(\rho_1) \int_0^\omega e^{u_2(s)} ds = \overline{r_1}\omega,\\
F_2^*(\lambda_2) \int_0^\omega e^{u_2(s)} ds + G_2(\rho_2) \int_0^\omega e^{u_1(s)} ds = \overline{r_2}\omega.
\end{aligned} \tag{47}$$

By (H_4), we have $G_1^L G_2^L > F_1^{*M} F_2^{*M}$, which, together with (47), lead to

$$\int_0^\omega e^{u_1(s)} ds = \frac{\overline{r_2}\omega G_1(\rho_1) - \overline{r_1}\omega F_2^*(\lambda_2)}{G_1(\rho_1) G_2(\rho_2) - F_1^*(\lambda_1) F_2^*(\lambda_2)} \geq \frac{\overline{r_2}\omega G_1^L - \overline{r_1}\omega F_2^{*M}}{G_1^M G_2^M - F_1^{*L} F_2^{*L}} := \Gamma_1 \omega,$$
$$\int_0^\omega e^{u_2(s)} ds = \frac{\overline{r_1}\omega G_2(\rho_2) - \overline{r_2}\omega F_1^*(\lambda_1)}{G_1(\rho_1) G_2(\rho_2) - F_1^*(\lambda_1) F_2^*(\lambda_2)} \geq \frac{\overline{r_1}\omega G_2^L - \overline{r_2}\omega F_1^{*M}}{G_1^M G_2^M - F_1^{*L} F_2^{*L}} := \Gamma_2 \omega. \tag{48}$$

Again, by (H_4), one can deduce that the following inequalities:

$$\frac{\overline{r_2}\omega}{G_2^L} \geq \frac{\overline{r_2}\omega G_1^L - \overline{r_1}\omega F_2^{*M}}{G_1^M G_2^M - F_1^{*L} F_2^{*L}} := \Gamma_1 \omega > 0,$$
$$\frac{\overline{r_1}\omega}{G_1^L} \geq \frac{\overline{r_1}\omega G_2^L - \overline{r_2}\omega F_1^{*M}}{G_1^M G_2^M - F_1^{*L} F_2^{*L}} := \Gamma_2 \omega > 0. \tag{49}$$

It follows from (46), (48), and (49) that

$$\Gamma_2 \omega \leq \int_0^\omega e^{u_2(s)} ds \leq \frac{\overline{r_1}\omega}{G_1^L}, \qquad \Gamma_1 \omega \leq \int_0^\omega e^{u_1(s)} ds \leq \frac{\overline{r_2}\omega}{G_2^L}, \tag{50}$$

which together with (36) yield

$$\Gamma_2 \leq e^{u_2(\zeta_2)}, \qquad e^{u_2(\xi_2)} \leq \frac{\overline{r_1}}{G_1^L}, \qquad \Gamma_1 \leq e^{u_1(\zeta_1)}, \qquad e^{u_1(\xi_1)} \leq \frac{\overline{r_2}}{G_2^L}, \tag{51}$$

which implies that

$$\ln \Gamma_2 \leq u_2(\xi_2), \qquad u_2(\zeta_2) \leq \ln \frac{\overline{r_1}}{G_1^L}, \qquad \ln \Gamma_1 \leq u_1(\xi_1), \qquad u_1(\zeta_1) \leq \ln \frac{\overline{r_2}}{G_2^L}. \tag{52}$$

From the first equation of (32), we get

$$\begin{aligned}
\int_0^\omega \left| u_1'(t) \right| dt
&= \lambda \int_0^\omega \Bigg| r_1(t) - A_1(t) e^{u_1(t)} - \sum_{i=1}^n B_{1i}(t) e^{u_1(t-\tau_i(t))} - \sum_{j=1}^m C_{1j}(t) e^{u_2(t-\delta_j(t))} \Bigg| dt \\
&\leq \int_0^\omega |r_1(t)| dt + \int_0^\omega \left[A_1(t) e^{u_1(t)} + \sum_{i=1}^n B_{1i}(t) e^{u_1(t-\tau_i(t))} + \sum_{j=1}^m C_{1j}(t) e^{u_2(t-\delta_j(t))} \right] dt \\
&= \int_0^\omega |r_1(t)| dt + \int_0^\omega \left[A_1(s) e^{u_1(s)} + \sum_{i=1}^n \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} e^{u_1(s)} + \sum_{j=1}^m \frac{C_{1j}(\mu_j(s))}{1-\delta_j'(\mu_j(s))} e^{u_2(s)} \right] ds \\
&= \int_0^\omega |r_1(t)| dt + \int_0^\omega \left[\left(A_1(s) + \sum_{i=1}^n \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} \right) e^{u_1(s)} + \sum_{j=1}^m \frac{C_{1j}(\mu_j(s))}{1-\delta_j'(\mu_j(s))} e^{u_2(s)} \right] ds \\
&= \int_0^\omega |r_1(t)| dt + \int_0^\omega \left[F_1^*(s) e^{u_1(s)} + G_1(s) e^{u_2(s)} \right] ds \\
&\leq \overline{R_1}\omega + F_1^{*M} \int_0^\omega e^{u_1(s)} ds + G_1^M \int_0^\omega e^{u_2(s)} ds,
\end{aligned} \tag{53}$$

where $\overline{R_1} = (1/\omega)\int_0^\omega |r_1(t)| dt$, $F_1^*(s)$, $G_1(s)$ are defined by (22). By (46) and (53), we obtain

$$\int_0^\omega \left| u_1'(t) \right| dt \leq \overline{R_1}\omega + F_1^{*M} \frac{\overline{r_2}\omega}{G_2^L} + G_1^M \frac{\overline{r_1}\omega}{G_1^L} := \Delta_1. \tag{54}$$

Similarly, by the second equation of (32), we get

$$\int_0^\omega \left| u_2'(t) \right| dt \leq \overline{R_2}\omega + F_2^{*M} \frac{\overline{r_1}\omega}{G_1^L} + G_2^M \frac{\overline{r_2}\omega}{G_2^L} := \Delta_2, \tag{55}$$

where $\overline{R_2} = (1/\omega)\int_0^\omega |r_2(t)|dt$, $F_2^*(s)$, $G_2(s)$ are defined by (22). From (52), (54), and (55) and Lemma 10, it follows that for $t \in [0,\omega]$ that

$$\begin{aligned} u_1(t) &\le u_1(\zeta_1) + \frac{1}{2}\int_0^\omega \left|u_1'(t)\right| dt \le \ln \frac{\overline{r_2}}{G_2^L} + \frac{1}{2}\Delta_1, \\ u_2(t) &\le u_2(\zeta_1) + \frac{1}{2}\int_0^\omega \left|u_2'(t)\right| dt \le \ln \frac{\overline{r_1}}{G_1^L} + \frac{1}{2}\Delta_2, \\ u_1(t) &\ge u_1(\xi_1) - \frac{1}{2}\int_0^\omega \left|u_1'(t)\right| dt \ge \ln \Gamma_1 - \frac{1}{2}\Delta_1, \\ u_2(t) &\ge u_2(\xi_1) - \frac{1}{2}\int_0^\omega \left|u_2'(t)\right| dt \ge \ln \Gamma_2 - \frac{1}{2}\Delta_2. \end{aligned} \tag{56}$$

Let

$$\begin{aligned} R_1 &= \max\left\{\left|\ln \frac{\overline{r_2}}{G_2^L} + \frac{1}{2}\Delta_1\right|, \left|\ln \Gamma_1 - \frac{1}{2}\Delta_1\right|\right\}, \\ R_2 &= \max\left\{\left|\ln \frac{\overline{r_1}}{G_1^L} + \frac{1}{2}\Delta_2\right|, \left|\ln \Gamma_2 - \frac{1}{2}\Delta_2\right|\right\}. \end{aligned} \tag{57}$$

It follows from (56)–(57) that

$$\begin{aligned} \sup_{t\in[0,\omega]} \left|u_1(t)\right| &\le R_1, \\ \sup_{t\in[0,\omega]} \left|u_2(t)\right| &\le R_2. \end{aligned} \tag{58}$$

Clearly, Γ_l, Δ_l, R_l $(l = 1,2)$ are independent of λ, respectively. Note that $\int_0^\omega F_l(t)dt \le F_l^M\omega$, $\int_0^\omega G_l(t)dt \le G_l^L\omega$ $(l = 1,2)$. From (44), we have

$$\begin{aligned} &\overline{A_1} + \sum_{i=1}^n \overline{B_{1i}} = \overline{F_1^*} \le F_1^{*M}, \qquad G_1^L \le \overline{G_1} = \sum_{j=1}^m \overline{C_{1j}}; \\ &\overline{A_2} + \sum_{j=1}^m \overline{B_{2j}} = \overline{F_2^*} \le F_2^{*M}, \qquad G_2^L \le \overline{G_2} = \sum_{i=1}^n \overline{C_{2i}}, \end{aligned} \tag{59}$$

which deduces that

$$\begin{aligned} \overline{r_1}\left(\overline{A_1} + \sum_{i=1}^n \overline{B_{1i}}\right) &= \overline{r_1}F_1^* \le \overline{r_1}F_1^{*M} < \overline{r_2}G_2^L \\ &\le \overline{r_2}\overline{G_2} = \overline{r_2}\sum_{i=1}^n \overline{C_{2i}}; \\ \overline{r_2}\left(\overline{A_2} + \sum_{j=1}^m \overline{B_{2j}}\right) &= \overline{r_2}\overline{F_2^*} \le \overline{r_2}F_2^{*M} < \overline{r_1}G_1^L \\ &\le \overline{r_1}\overline{G_1} = \overline{r_1}\sum_{j=1}^m \overline{C_{1j}}, \end{aligned} \tag{60}$$

which implies that

$$\begin{aligned} \overline{r_1}\left(\overline{A_1} + \sum_{i=1}^n \overline{B_{1i}}\right) &\le \overline{r_2}\sum_{i=1}^n \overline{C_{2i}}; \\ \overline{r_2}\left(\overline{A_2} + \sum_{j=1}^m \overline{B_{2j}}\right) &\le \overline{r_1}\sum_{j=1}^m \overline{C_{1j}}. \end{aligned} \tag{61}$$

Hence

$$\left(\overline{A_1} + \sum_{i=1}^n \overline{B_{1i}}\right)\left(\overline{A_2} + \sum_{j=1}^m \overline{B_{2j}}\right) \le \sum_{j=1}^m \overline{C_{1j}}\sum_{i=1}^n \overline{C_{2i}}. \tag{62}$$

From (61) and (62), it is easy to show that the system of algebraic equations

$$\begin{aligned} \overline{r_1} - \left(\overline{A_1} + \sum_{i=1}^n \overline{B_{1i}}\right)e^{u_1} - \sum_{j=1}^m \overline{C_{1j}}e^{u_2} &= 0, \\ \overline{r_2} - \left(\overline{A_2} + \sum_{j=1}^m \overline{B_{2j}}\right)e^{u_2} - \sum_{i=1}^n \overline{C_{2i}}e^{u_1} &= 0 \end{aligned} \tag{63}$$

has a unique solution $(u_1^*, u_2^*) \in R^2$. In view of (58), we can take sufficiently large R such that $R > R_1 + R_2$, $R > |u_1^*| + |u_2^*|$ and define $\Omega = \{u(t) = (u_1(t), u_2(t))^T \in X : \|u\| < R\}$, and it is clear that Ω satisfies condition (a) of Lemma 7. Letting $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, then u is a constant vector in R^2 with $\|u\| = R$. Then

$$QNu = \begin{pmatrix} \overline{r_1} - \left(\overline{A_1} + \sum_{i=1}^n \overline{B_{1i}}\right)e^{u_1} - \sum_{j=1}^m \overline{C_{1j}}e^{u_2} \\ \overline{r_2} - \left(\overline{A_2} + \sum_{j=1}^m \overline{B_{2j}}\right)e^{u_2} - \sum_{i=1}^n \overline{C_{2i}}e^{u_1} \end{pmatrix} \ne 0; \tag{64}$$

that is, condition (b) of Lemma 7 holds. In order to verify condition (c) in the Lemma 7, by (62) and the formula for Brouwer degree, a straightforward calculation shows that

$$\begin{aligned} &\deg\{JQNu, \operatorname{Ker} L \cap \partial\Omega, 0\} \\ &\quad = \operatorname{sign}\left\{\left(\sum_{j=1}^m \overline{C_{1j}}\sum_{i=1}^n \overline{C_{2i}} - \left(\overline{A_1} + \sum_{i=1}^n \overline{B_{1i}}\right)\right.\right. \\ &\qquad \left.\left.\times\left(\overline{A_2} + \sum_{j=1}^m \overline{B_{2j}}\right)\right)e^{(u_1^* + u_2^*)}\right\} \ne 0. \end{aligned} \tag{65}$$

By now we have proved that all the requirements in Lemma 7 hold. Hence system (32) has at least one ω-periodic solution, say $(u_1^*, u_2^*)^T$. Setting $y_1^*(t) = e^{u_1^*(t)}$, $y_2^*(t) = e^{u_2^*(t)}$, then $(y_1^*(t), y_2^*(t))^T$ has at least one positive ω-periodic solution of systems (11) and (12). Furthermore, setting $x_1^*(t) = \prod_{0<t_k<t}(1 + \theta_{1k})y_1^*(t)$, $x_2^*(t) = \prod_{0<t_k<t}(1 + \theta_{2k})y_2^*(t)$, then $(x_1^*(t), x_2^*(t))^T$ has at least one positive ω-periodic solution

of systems (3) and (4). If (H_5) holds, similarly, we can prove that systems (2) and (4) have at least one positive ω-periodic solution. The proof of Theorem 12 is complete. □

We now proceed to the discussion on the uniqueness and global stability of the ω-periodic solution $x^*(t)$ in Theorem 12. It is immediate that if $x^*(t)$ is globally asymptotically stable, then $x^*(t)$ is unique in fact.

Theorem 13. *In addition to (H_1)–(H_3), assume further that*

$$(H_6)\ F_1^{*L}F_2^{*L} > G_1^M G_2^M.$$

Then systems (3) *and* (4) *have a unique positive ω-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t))^T$ which is globally asymptotically stable.*

Proof. Letting $x^*(t) = (x_1^*(t), x_2^*(t))^T$ be a positive ω-periodic solution of (3) and (4), then $y^*(t) = (y_1^*(t), y_2^*(t))^T$, $(y_l^*(t) = \prod_{0<t_k<t}(1+\theta_{lk})^{-1}x_l^*(t))$ $(l = 1, 2)$ is the positive ω-periodic solution of system (11) and (12), and let $y_l(t) = (y_1(t), y_2(t))^T$ be any positive solution of system (11) with the initial conditions (12). It follows from Theorem 12 that there exist positive constants T, r_l, R_l, such that, for all $t \ge T$,

$$r_l \le y_l^*(t) \le R_l, \quad l = 1, 2. \tag{66}$$

By the assumptions of Theorem 12, we can obtain $F_1^{*L}F_2^{*L} > G_1^M G_2^M$, and then there exist constants $\alpha_1 > 0$, $\alpha_2 > 0$; we can choose a positive constant ε such that

$$F_1^{*L}\alpha_1 - G_2^M\alpha_2 = \varepsilon, \qquad F_2^{*L}\alpha_2 - G_1^M\alpha_1 = \varepsilon. \tag{67}$$

In the following, we always assume that α_1 and α_2 satisfy (67). We define

$$\begin{aligned} V_1(t) = {} & \alpha_1 \left|\ln y_1(t) - \ln y_1^*(t)\right| \\ & + \alpha_2 \left|\ln y_2(t) - \ln y_2^*(t)\right|. \end{aligned} \tag{68}$$

Calculating the upper right derivative of $V_1(t)$ along solutions of (11), it follows that

$$\begin{aligned} & D^+V_1(t) \\ & = \sum_{l=1}^{2}\alpha_l\left(\frac{\dot y_l(t)}{y_l(t)} - \frac{\dot y_l^*(t)}{y_l^*(t)}\right)\operatorname{sgn}(y_l(t) - y_l^*(t)) \\ & \le \operatorname{sgn}(y_1(t) - y_1^*(t))\,\alpha_l \\ & \quad \times \Bigg\{ - A_1(t)(y_1(t) - y_1^*(t)) \\ & \qquad - \sum_{i=1}^{n} B_{1i}(t)(y_1(t-\tau_i(t)) - y_1^*(t-\tau_i(t))) \\ & \qquad - \sum_{j=1}^{m} C_{1j}(t)\left(y_2\left(t-\delta_j(t)\right) - y_2^*\left(t-\delta_j(t)\right)\right)\Bigg\} \\ & \quad + \operatorname{sgn}(y_2(t) - y_2^*(t))\,\alpha_2 \\ & \quad \times \Bigg\{ - A_2(t)(y_2(t) - y_2^*(t)) \\ & \qquad - \sum_{j=1}^{m} B_{2j}(t)\left(y_2\left(t-\eta_j(t)\right) - y_2^*\left(t-\eta_j(t)\right)\right) \\ & \qquad - \sum_{i=1}^{n} C_{2i}(t)(y_1(t-\sigma_i(t)) - y_1^*(t-\sigma_i(t)))\Bigg\} \\ & \le -\alpha_l A_1(t)\left|y_1(t) - y_1^*(t)\right| \\ & \quad + \sum_{i=1}^{n}\alpha_l B_{1i}(t)\left|y_1(t-\tau_i(t)) - y_1^*(t-\tau_i(t))\right| \\ & \quad + \sum_{j=1}^{m}\alpha_l C_{1j}(t)\left|y_2\left(t-\delta_j(t)\right) - y_2^*\left(t-\delta_j(t)\right)\right| \\ & \quad - \alpha_2 A_2(t)\left|y_2(t) - y_2^*(t)\right| + \sum_{j=1}^{m}\alpha_2 B_{2j}(t) \\ & \quad \times \left|y_2\left(t-\eta_j(t)\right) - y_2^*\left(t-\eta_j(t)\right)\right| \\ & \quad + \sum_{i=1}^{n}\alpha_2 C_{2i}(t)\left|y_1(t-\sigma_i(t)) - y_1^*(t-\sigma_i(t))\right|. \end{aligned} \tag{69}$$

We also define

$$\begin{aligned} V_2(t) = {} & \sum_{i=1}^{n}\alpha_1\int_{t-\tau_i(t)}^{t}\frac{B_{1i}(\alpha_i(\xi))}{1-\tau_i'(\alpha_i(\xi))}\left|y_1(\xi) - y_1^*(\xi)\right| d\xi \\ & + \sum_{j=1}^{m}\alpha_1\int_{t-\delta_j(t)}^{t}\frac{C_{1j}\left(\mu_j(\xi)\right)}{1-\delta_j'\left(\mu_j(\xi)\right)}\left|y_2(\xi) - y_2^*(\xi)\right| d\xi \\ & + \sum_{j=1}^{m}\alpha_2\int_{t-\rho_j(t)}^{t}\frac{B_{2j}\left(\nu_j(\xi)\right)}{1-\rho_j'\left(\nu_j(\xi)\right)}\left|y_2(\xi) - y_2^*(\xi)\right| d\xi \\ & + \sum_{i=1}^{n}\alpha_2\int_{t-\sigma_i(t)}^{t}\frac{C_{2i}(\beta_i(\xi))}{1-\sigma_i'(\beta_i(\xi))}\left|y_1(\xi) - y_1^*(\xi)\right| d\xi. \end{aligned} \tag{70}$$

Calculating the upper right derivative of $V_2(t)$ along solutions of (11), it follows that

$$\begin{aligned}D^+V_2(t) = &\sum_{i=1}^{n}\alpha_1\frac{B_{1i}(\alpha_i(t))}{1-\tau_i'(\alpha_i(t))}\left|y_1(t)-y_1^*(t)\right|\\&-\sum_{i=1}^{n}\alpha_1\frac{B_{1i}(t)}{1-\tau_i'(t)}\left(1-\tau_i'(t)\right)\\&\times\left|y_1(t-\tau_i(t))-y_1^*(t-\tau_i(t))\right|\\&+\sum_{j=1}^{m}\alpha_1\frac{C_{1j}\left(\mu_j(t)\right)}{1-\delta_j'\left(\mu_j(t)\right)}\left|y_2(t)-y_2^*(t)\right|\\&-\sum_{j=1}^{m}\alpha_1\frac{C_{1j}(t)}{1-\delta_j'(t)}\left(1-\delta_j'(t)\right)\\&\times\left|y_2\left(t-\delta_j(t)\right)-y_2^*\left(t-\delta_j(t)\right)\right|\\&+\sum_{j=1}^{m}\alpha_2\frac{B_{2j}\left(\nu_j(t)\right)}{1-\eta_j'\left(\nu_j(t)\right)}\left|y_2(t)-y_2^*(t)\right|\\&-\sum_{j=1}^{m}\alpha_1\frac{B_{2j}(t)}{1-\eta_j'(t)}\left(1-\eta_j'(t)\right)\\&\times\left|y_2\left(t-\eta_j(t)\right)-y_2^*\left(t-\eta_j(t)\right)\right|\\&+\sum_{i=1}^{n}\alpha_2\frac{C_{2i}(\beta_i(t))}{1-\sigma_i'(\beta_i(t))}\\&\times\left|y_1(t)-y_1^*(t)\right|-\sum_{i=1}^{n}\alpha_2\frac{C_{2i}(t)}{1-\sigma_i'(t)}\\&\times\left(1-\sigma_i'(t)\right)\left|y_1(t-\sigma_i(t))-y_1^*(t-\sigma_i(t))\right|.\end{aligned}\tag{71}$$

We define a Lyapunov functional $V(t)$ as follows:

$$V(t)=V_1(t)+V_2(t).\tag{72}$$

Calculating the upper right derivative of $V(t)$ along solutions of (11), it follows that

$$\begin{aligned}D^+V(t) = &-\alpha_l A_1(t)\left|y_1(t)-y_1^*(t)\right|\\&+\sum_{i=1}^{n}\alpha_1\frac{B_{1i}(\alpha_i(t))}{1-\tau_i'(\alpha_i(t))}\left|y_1(t)-y_1^*(t)\right|\\&+\sum_{j=1}^{m}\alpha_1\frac{C_{1j}\left(\mu_j(t)\right)}{1-\delta_j'\left(\mu_j(t)\right)}\left|y_2(t)-y_2^*(t)\right|\\&-\alpha_2A_2(t)\left|y_2(t)-y_2^*(t)\right|\\&+\sum_{j=1}^{m}\alpha_2\frac{B_{2j}\left(\nu_j(t)\right)}{1-\eta_j'\left(\nu_j(t)\right)}\left|y_2(t)-y_2^*(t)\right|\\&+\sum_{i=1}^{n}\alpha_2\frac{C_{2i}(\beta_i(t))}{1-\sigma_i'(\beta_i(t))}\left|y_1(t)-y_1^*(t)\right|\\=&-\left[\alpha_lA_1(t)-\sum_{i=1}^{n}\alpha_1\frac{B_{1i}(\alpha_i(t))}{1-\tau_i'(\alpha_i(t))}\right.\\&\left.-\sum_{i=1}^{n}\alpha_2\frac{C_{2i}(\beta_i(t))}{1-\sigma_i'(\beta_i(t))}\right]\left|y_1(t)-y_1^*(t)\right|\\&-\left[\alpha_2A_2(t)-\sum_{j=1}^{m}\alpha_1\frac{C_{1j}\left(\mu_j(t)\right)}{1-\delta_j'\left(\mu_j(t)\right)}\right.\\&\left.-\sum_{j=1}^{m}\alpha_2\frac{B_{2j}\left(\nu_j(t)\right)}{1-\eta_j'\left(\nu_j(t)\right)}\right]\left|y_2(t)-y_2^*(t)\right|\\\le&\left(\alpha_lF_1^L-\alpha_2G_2^M\right)\left|y_1(t)-y_1^*(t)\right|\\&-\left(\alpha_2F_2^L-\alpha_1G_1^M\right)\left|y_2(t)-y_2^*(t)\right|\\=&-\varepsilon\left(\left|y_1(t)-y_1^*(t)\right|+\left|y_2(t)-y_2^*(t)\right|\right).\end{aligned}\tag{73}$$

So by (73), we have

$$\begin{aligned}&\varepsilon\int_0^t\left(\left|y_1(\xi)-y_1^*(\xi)\right|+\left|y_2(\xi)-y_2^*(\xi)\right|\right)d\xi\\&\quad+V(t)\le V(0)<+\infty,\quad t\ge0,\end{aligned}\tag{74}$$

where

$$\begin{aligned}V(0)=&\,\alpha_1\left|\ln y_1(0)-\ln y_1^*(0)\right|+\alpha_2\left|\ln y_2(0)-\ln y_2^*(0)\right|\\&+\sum_{i=1}^{n}\alpha_1\int_{-\tau_i(0)}^{0}\frac{B_{1i}(\alpha_i(\xi))}{1-\tau_i'(\alpha_i(\xi))}\left|y_1(\xi)-y_1^*(\xi)\right|d\xi\\&+\sum_{j=1}^{m}\alpha_1\int_{-\delta_j(0)}^{0}\frac{C_{1j}\left(\mu_j(\xi)\right)}{1-\delta_j'\left(\mu_j(\xi)\right)}\left|y_2(\xi)-y_2^*(\xi)\right|d\xi\\&+\sum_{j=1}^{m}\alpha_2\int_{-\rho_j(0)}^{0}\frac{B_{2j}\left(\nu_j(\xi)\right)}{1-\rho_j'\left(\nu_j(\xi)\right)}\\&\times\left|y_2(\xi)-y_2^*(\xi)\right|d\xi\\&+\sum_{i=1}^{n}\alpha_2\int_{-\sigma_i(0)}^{0}\frac{C_{2i}(\beta_i(\xi))}{1-\sigma_i'(\beta_i(\xi))}\left|y_1(\xi)-y_1^*(\xi)\right|d\xi,\end{aligned}\tag{75}$$

which implies that

$$\int_0^t \left(\left|y_1(\xi) - y_1^*(\xi)\right| + \left|y_2(\xi) - y_2^*(\xi)\right|\right) d\xi \le \frac{V(0)}{\varepsilon}. \quad (76)$$

By (76), it is obvious that $|y_1(t) - y_1^*(t)| + |y_2(t) - y_2^*(t)|$ is bounded.

On the other hand, we know that

$$\alpha_1 \left|\ln y_1(t) - \ln y_1^*(t)\right| + \alpha_2 \left|\ln y_2(t) - \ln y_2^*(t)\right| \le V(t) \le V(0) < +\infty, \quad t \ge 0, \quad (77)$$

which implies that

$$\left|\ln y_1(t) - \ln y_1^*(t)\right| \le \frac{V(0)}{\alpha_1}, \qquad \left|\ln y_2(t) - \ln y_2^*(t)\right| \le \frac{V(0)}{\alpha_2}, \quad (78)$$

which, together with (66), yield

$$r_1 e^{-V(0)/\alpha_1} \le y_1(t) \le R_1 e^{V(0)/\alpha_1} < +\infty, \qquad r_2 e^{-V(0)/\alpha_2} \le y_2(t) \le R_2 e^{V(0)/\alpha_2} < +\infty. \quad (79)$$

From (66) and (79), it follows that $y_l(t)$ $(l = 1, 2)$ is bounded for $t \ge 0$. Hence, $y_1(t) - y_1^*(t)$, $y_2(t) - y_2^*(t)$, and their derivatives remain bounded on $[0, +\infty)$. So $|y_1(t) - y_1^*(t)|$, $|y_2(t) - y_2^*(t)|$ are uniformly continuous on $[0, +\infty)$. By Lemma 11, we have

$$\lim_{t\to+\infty} \left|y_l(s) - y_l^*(s)\right| = \lim_{t\to+\infty} \left[\prod_{0<t_k<t} (1+\theta_{lk})^{-1} \left|x_l^*(s) - x_l(s)\right|\right] = 0, \quad l = 1, 2. \quad (80)$$

Therefore

$$\lim_{t\to+\infty} \left|x_l(s) - x_l^*(s)\right| = 0, \quad l = 1, 2. \quad (81)$$

By Theorems 7.4 and 8.2 in [30], we know that the periodic positive solution $x^*(t) = (x_1^*(t), x_2^*(t))^T$ is uniformly asymptotically stable. The proof of Theorem 13 is completed. □

Theorem 14. *In addition to* (H_1)*–*(H_3)*, assume that one of the following conditions holds:*

(H_7) $\overline{r_1} F_1^M < \overline{r_2} G_2^L$, $\overline{r_2} F_2^M < \overline{r_1} G_1^L$;

(H_8) $\overline{r_1} F_1^L > \overline{r_2} G_2^M$, $\overline{r_2} F_2^L > \overline{r_1} G_1^M$.

Then systems (2) *and* (4) *have at least one positive* ω*-periodic solution, where* $F_1(t)$*,* $F_2(t)$*,* $G_1(t)$*, and* $G_2(t)$ *are defined in* (22).

Proof. Since the solutions of systems (10) and (12) remain positive for $t \ge 0$, we carry out the change of variable $u_i(t) = \ln y_i(t)$ $(i = 1, 2)$, and then (10) can be transformed to

$$\begin{aligned} u_1'(t) &= r_1(t) - A_1(t) e^{u_1(t)} + \sum_{i=1}^{n} B_{1i}(t) e^{u_1(t-\tau_i(t))} - \sum_{j=1}^{m} C_{1j}(t) e^{u_2(t-\delta_j(t))}, \\ u_2'(t) &= r_2(t) - A_2(t) e^{u_2(t)} + \sum_{j=1}^{m} B_{2j}(t) e^{u_2(t-\eta_j(t))} - \sum_{i=1}^{n} C_{2i}(t) e^{u_1(t-\sigma_i(t))}. \end{aligned} \quad (82)$$

It is easy to see that if system (82) has one ω-periodic solution $(u_1^*(t), u_2^*(t))^T$, then $(y_1^*(t), y_2^*(t))^T = (e^{u_1^*(t)}, e^{u_2^*(t)})^T$ is a positive ω-periodic solution of systems (10) and (12); that is to say, $(x_1^*(t), x_2^*(t))^T = (\prod_{0<t_k<t}(1+\theta_{1k}) e^{u_1^*(t)}, \prod_{0<t_k<t}(1+\theta_{2k}) e^{u_2^*(t)})^T$ is a positive ω-periodic solution of systems (2) and (4). Therefore, it suffices to prove that system (82) has a ω-periodic solution. In order to use Lemma 6 for (81), we take

$$X = Z = \left\{u(t) = (u_1(t), u_2(t))^T \mid u_i(t) \in C(R, R^2) : u_i(t+\omega) = u_i(t), l = 1, 2\right\} \quad (83)$$

and define

$$\|u\| = \sum_{i=1}^{2} \sup_{t\in[0,\omega]} \left|u_l(t)\right|, \qquad u(t) = (u_1(t), u_2(t))^T \in X \text{ (or } Z). \quad (84)$$

Then X and Z are Banach spaces when they are endowed with the norm $\|\cdot\|$. Let $N : X \to Z$ with

$$Nu = \begin{bmatrix} r_1(t) - A_1(t) e^{u_1(t)} + \sum_{i=1}^{n} B_{1i}(t) e^{u_1(t-\tau_i(t))} - \sum_{j=1}^{m} C_{1j}(t) e^{u_2(t-\delta_j(t))} \\ r_2(t) - A_2(t) e^{u_2(t)} + \sum_{j=1}^{m} B_{2j}(t) e^{u_2(t-\eta_j(t))} - \sum_{i=1}^{n} C_{2i}(t) e^{u_1(t-\sigma_i(t))} \end{bmatrix} = \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix} \quad \text{for any } u \in X, \quad (85)$$

and define

$$Lu = u'; \quad Pu = \frac{1}{\omega}\int_0^\omega u(t)\,dt, \quad u \in X;$$
$$Qz = \frac{1}{\omega}\int_0^\omega z(t)\,dt, \quad z \in Z. \tag{86}$$

It is not difficult to show that

$$\operatorname{Ker} L = \left\{u \in X \mid u = h \in R^2\right\},$$
$$\operatorname{Im} L = \left\{z \in Z \mid \int_0^\omega z(s)\,ds = 0\right\}, \tag{87}$$

and $\dim\operatorname{Ker}L = 2 = \operatorname{codim}\operatorname{Im}L$. So, $\operatorname{Im} L$ is closed in Z, and L is a Fredholm mapping of index zero. It is trivial to show that P, Q are continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ exists and is given by

$$K_P z = \int_0^t z(s)\,ds - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)\,ds\,dt. \tag{88}$$

Thus, for $u \in X$

$$QNu = \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega \left[r_1(t) - A_1(t)e^{u_1(t)} + \sum_{i=1}^n B_{1i}(t)e^{u_1(t-\tau_i(t))} - \sum_{j=1}^m C_{1j}(t)e^{u_2(t-\delta_j(t))}\right]dt \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega \left[r_2(t) - A_2(t)e^{u_2(t)} + \sum_{j=1}^m B_{2j}(t)e^{u_2(t-\eta_j(t))} - \sum_{i=1}^n C_{2i}(t)e^{u_1(t-\sigma_i(t))}\right]dt \end{pmatrix},$$
$$K_P(I-Q)Nu = \begin{pmatrix} \displaystyle\int_0^t g_1(\xi)\,d\xi \\ \displaystyle\int_0^t g_2(\xi)\,d\xi \end{pmatrix} - \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega\int_0^t g_1(\xi)\,d\xi\,dt \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega\int_0^t g_2(\xi)\,d\xi\,dt \end{pmatrix} - \begin{pmatrix} \left(\dfrac{t}{\omega} - \dfrac{1}{2}\right)\displaystyle\int_0^\omega g_1(\xi)\,d\xi \\ \left(\dfrac{t}{\omega} - \dfrac{1}{2}\right)\displaystyle\int_0^\omega g_2(\xi)\,d\xi \end{pmatrix}. \tag{89}$$

Clearly, QN and $K_P(I - Q)N$ are continuous. By applying Ascoli-Arzela theorem, one can easily show that $QN(\overline{\Omega})$, $K_P(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is obviously bounded. Thus, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset X$. Now, we reach the position to search for an appropriate open bounded set $\Omega \subset X$ for the application of Lemma 6. Corresponding to the operating equation $Lu = \lambda Nu$, $\lambda \in (0, 1)$, we have

$$u_1'(t) = \lambda\left[r_1(t) - A_1(t)e^{u_1(t)} + \sum_{i=1}^n B_{1i}(t)e^{u_1(t-\tau_i(t))} - \sum_{j=1}^m C_{1j}(t)e^{u_2(t-\delta_j(t))}\right],$$
$$u_2'(t) = \lambda\left[r_2(t) - A_2(t)e^{u_2(t)} + \sum_{j=1}^m B_{2j}(t)e^{u_2(t-\eta_j(t))} - \sum_{i=1}^n C_{2i}(t)e^{u_1(t-\sigma_i(t))}\right]. \tag{90}$$

Since $u(t) = (u_1(t), u_2(t))^T$ is a ω-periodic function, we need only to prove the result in the interval $[0, \omega]$. Integrating (90) over the interval $[0, \omega]$ leads to the following:

$$\int_0^\omega \left[r_1(t) - A_1(t)e^{u_1(t)} + \sum_{i=1}^n B_{1i}(t)e^{u_1(t-\tau_i(t))} - \sum_{j=1}^m C_{1j}(t)e^{u_2(t-\delta_j(t))}\right]dt = 0,$$
$$\int_0^\omega \left[r_2(t) - A_2(t)e^{u_2(t)} + \sum_{j=1}^m B_{2j}(t)e^{u_2(t-\eta_j(t))} - \sum_{i=1}^n C_{2i}(t)e^{u_1(t-\sigma_i(t))}\right]dt = 0. \tag{91}$$

Hence, we have

$$\int_0^\omega \left[A_1(t)e^{u_1(t)} - \sum_{i=1}^n B_{1i}(t)e^{u_1(t-\tau_i(t))} + \sum_{j=1}^m C_{1j}(t)e^{u_2(t-\delta_j(t))}\right]dt = \overline{r_1}\omega,$$
$$\int_0^\omega \left[A_2(t)e^{u_2(t)} - \sum_{j=1}^m B_{2j}(t)e^{u_2(t-\eta_j(t))} + \sum_{i=1}^n C_{2i}(t)e^{u_1(t-\sigma_i(t))}\right]dt = \overline{r_2}\omega. \tag{92}$$

Noting that $u(t) = (u_1(t), u_2(t)) \in X$, then there exists ζ_l, $\xi_l \in [0,\omega]$ $(l = 1,2)$ such that

$$u_l(\zeta_l) = \inf_{t\in[0,\omega]} u_l(t), \quad u_l(\xi_l) = \sup_{t\in[0,\omega]} u_l(t), \quad l = 1,2. \tag{93}$$

Since $\tau_i'(t) < 1$, we can let $s = t - \tau_i(t)$, that is, $t = \alpha_i(s)$, $i = 1,2,\ldots,n$, and then

$$\int_0^\omega B_{1i}(t)\, e^{u_1(t-\tau_i(t))} dt = \int_{-\tau_i(0)}^{\omega-\tau_i(\omega)} \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} e^{u_1(s)} ds. \tag{94}$$

According to Lemma 6, we know that $((B_{1i}(\alpha_i(s)))/(1 - \tau_i'(\alpha_i(s))))e^{u_1(s)} \in C_\omega$. Thus,

$$\int_{-\tau_i(0)}^{\omega-\tau_i(\omega)} \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} e^{u_1(s)} ds = \int_0^\omega \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} e^{u_1(s)} ds. \tag{95}$$

By (37) and (38), we have

$$\int_0^\omega B_{1i}(t)\, e^{u_1(t-\tau_i(t))} dt = \int_0^\omega \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} e^{u_1(s)} ds. \tag{96}$$

Similarly, we obtain

$$\begin{aligned} \int_0^\omega C_{1j}(t)\, e^{u_2(t-\delta_j(t))} dt &= \int_0^\omega \frac{C_{1j}(\mu_j(s))}{1-\delta_j'(\mu_j(s))} e^{u_2(s)} ds, \\ \int_0^\omega B_{2j}(t)\, e^{u_1 2(t-\eta_j(t))} dt &= \int_0^\omega \frac{B_{2j}(\nu_j(s))}{1-\eta_j'(\nu_j(s))} e^{u_2(s)} ds, \\ \int_0^\omega C_{2i}(t)\, e^{u_1(t-\sigma_i(t))} dt &= \int_0^\omega \frac{C_{2i}(\beta_i(s))}{1-\sigma_i'(\beta_i(s))} e^{u_1(s)} ds. \end{aligned} \tag{97}$$

It follows from (92), (96), and (97) that we get

$$\begin{aligned} &\int_0^\omega \left[\left(A_1(s) - \sum_{i=1}^n \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))}\right) e^{u_1(s)} + \sum_{j=1}^m \frac{C_{1j}(\mu_j(s))}{1-\delta_j'(\mu_j(s))} e^{u_2(s)}\right] ds = \overline{r_1}\omega, \\ &\int_0^\omega \left[\left(A_2(s) - \sum_{j=1}^m \frac{B_{2j}(\nu_j(s))}{1-\eta_j'(\nu_j(s))}\right) e^{u_2(s)} + \sum_{i=1}^n \frac{C_{2i}(\beta_i(s))}{1-\sigma_i'(\beta_i(s))} e^{u_1(s)}\right] ds = \overline{r_2}\omega. \end{aligned} \tag{98}$$

Thus from (98) we get

$$\begin{aligned} \int_0^\omega F_1(s)\, e^{u_1(s)} ds + \int_0^\omega G_1(s)\, e^{u_2(s)} ds &= \overline{r_1}\omega, \\ \int_0^\omega F_2(s)\, e^{u_2(s)} ds + \int_0^\omega G_2(s)\, e^{u_1(s)} ds &= \overline{r_2}\omega. \end{aligned} \tag{99}$$

where $F_1(s)$, $F_2(s)$, $G_1(s)$, and $G_2(s)$ are defined by (22). On the other hand, by Lemma 6, we can see that $\alpha_i(\omega) = \alpha_i(0)+\omega$, so we can derive

$$\begin{aligned} \int_0^\omega \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} ds &= \int_{\alpha_i(0)}^{\alpha_i(\omega)} \frac{B_{1i}(t)\,(1-\tau_i'(\alpha_i(t)))}{1-\tau_i'(\alpha_i(t))} dt \\ &= \int_0^\omega B_{1i}(t)\, dt = \overline{B_{1i}}\omega. \end{aligned} \tag{100}$$

Thus, from (99) and (100), we get

$$\begin{aligned} \overline{F_1}\omega &= \int_0^\omega F_1(s)\, ds = \int_0^\omega \left[A_1(s) - \sum_{i=1}^n \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))}\right] ds \\ &= \left(\overline{A_1} - \sum_{i=1}^n \overline{B_{1i}}\right)\omega, \\ \overline{F_2}\omega &= \int_0^\omega F_2(s)\, ds = \int_0^\omega \left[A_2(s) - \sum_{i=1}^n \frac{B_{2j}(\nu_j(s))}{1-\eta_j'(\nu_j(s))}\right] ds \\ &= \left(\overline{A_2} - \sum_{i=1}^n \overline{B_{2j}}\right)\omega, \\ \overline{G_1}\omega &= \int_0^\omega G_1(s)\, ds = \int_0^\omega \sum_{j=1}^m \frac{C_{1j}(\mu_j(s))}{1-\delta_j'(\mu_j(s))} ds \\ &= \sum_{j=1}^m \overline{C_{1j}}\omega, \\ \overline{G_2}\omega &= \int_0^\omega G_2(s)\, ds = \int_0^\omega \sum_{i=1}^n \frac{C_{2i}(\beta_i(s))}{1-\sigma_i'(\beta_i(s))} ds \\ &= \sum_{i=1}^n \overline{C_{2i}}\omega. \end{aligned} \tag{101}$$

By (99), on one hand, we have

$$\begin{aligned} G_1^L \int_0^\omega e^{u_2(s)} ds &\le \int_0^\omega G_1(s)\, e^{u_2(s)} ds \le \overline{r_1}\omega, \\ G_2^L \int_0^\omega e^{u_1(s)} ds &\le \int_0^\omega G_2(s)\, e^{u_1(s)} ds \le \overline{r_2}\omega, \end{aligned} \tag{102}$$

which implies that

$$\begin{aligned} \int_0^\omega e^{u_2(s)} ds &\le \frac{\overline{r_1}\omega}{G_1^L}, \\ \int_0^\omega e^{u_1(s)} ds &\le \frac{\overline{r_2}\omega}{G_2^L}. \end{aligned} \tag{103}$$

On the other hand, by (99) the integral mean value theorem that there is λ_1, λ_2, ρ_1, and $\rho_2 \in [0,\omega]$ such that

$$\begin{aligned} F_1(\lambda_1) \int_0^\omega e^{u_1(s)} ds + G_1(\rho_1) \int_0^\omega e^{u_2(s)} ds &= \overline{r_1}\omega, \\ F_2(\lambda_2) \int_0^\omega e^{u_2(s)} ds + G_2(\rho_2) \int_0^\omega e^{u_1(s)} ds &= \overline{r_2}\omega. \end{aligned} \tag{104}$$

By (H_7), we have $G_1^L G_2^L > F_1^M F_2^M$, which together with (104), lead to the following:

$$\int_0^\omega e^{u_2(s)} ds = \frac{\overline{r_1}\omega G_2(\rho_2) - \overline{r_2}\omega F_1(\lambda_1)}{G_1(\rho_1) G_2(\rho_2) - F_1(\lambda_1) F_2(\lambda_2)} \geq \frac{\overline{r_1}\omega G_2^L - \overline{r_2}\omega F_1^M}{G_1^M G_2^M - F_1^L F_2^L} := \Gamma_4 \omega,$$
$$\int_0^\omega e^{u_1(s)} ds = \frac{\overline{r_2}\omega G_1(\rho_1) - \overline{r_1}\omega F_2(\lambda_2)}{G_1(\rho_1) G_2(\rho_2) - F_1(\lambda_1) F_2(\lambda_2)} \geq \frac{\overline{r_2}\omega G_1^L - \overline{r_1}\omega F_2^M}{G_1^M G_2^M - F_1^L F_2^L} := \Gamma_3 \omega. \quad (105)$$

Again, by (H_7), one can deduce that the following inequalities:

$$\frac{\overline{r_1}\omega}{G_1^L} \geq \frac{\overline{r_1}\omega G_2^L - \overline{r_2}\omega F_1^M}{G_1^M G_2^M - F_1^L F_2^L} := \Gamma_4 \omega > 0,$$
$$\frac{\overline{r_2}\omega}{G_2^L} \geq \frac{\overline{r_2}\omega G_1^L - \overline{r_1}\omega F_2^M}{G_1^M G_2^M - F_1^L F_2^L} := \Gamma_3 \omega > 0. \quad (106)$$

It follows from (103), (105), and (106) that

$$\Gamma_4 \omega \leq \int_0^\omega e^{u_2(s)} ds \leq \frac{\overline{r_1}\omega}{G_1^L},$$
$$\Gamma_3 \omega \leq \int_0^\omega e^{u_1(s)} ds \leq \frac{\overline{r_2}\omega}{G_2^L}, \quad (107)$$

which, together with (92) yield

$$\Gamma_4 \leq e^{u_2(\zeta_2)}, \qquad e^{u_2(\xi_2)} \leq \frac{\overline{r_1}}{G_1^L},$$
$$\Gamma_3 \leq e^{u_1(\zeta_1)}, \qquad e^{u_1(\xi_1)} \leq \frac{\overline{r_2}}{G_2^L}, \quad (108)$$

which implies that

$$\ln \Gamma_4 \leq u_2(\xi_2), \qquad u_2(\zeta_2) \leq \ln \frac{\overline{r_1}}{G_1^L},$$
$$\ln \Gamma_3 \leq u_1(\xi_1), \qquad u_1(\zeta_1) \leq \ln \frac{\overline{r_2}}{G_2^L}. \quad (109)$$

From the first equation of (90), we get

$$\begin{aligned}
&\int_0^\omega \left|u_1'(t)\right| dt \\
&= \lambda \int_0^\omega \left| r_1(t) - A_1(t) e^{u_1(t)} + \sum_{i=1}^n B_{1i}(t) e^{u_1(t-\tau_i(t))} - \sum_{j=1}^m C_{1j}(t) e^{u_2(t-\delta_j(t))} \right| dt \\
&\leq \int_0^\omega \left|r_1(t)\right| dt + \int_0^\omega \left[A_1(t) e^{u_1(t)} + \sum_{i=1}^n B_{1i}(t) e^{u_1(t-\tau_i(t))} + \sum_{j=1}^m C_{1j}(t) e^{u_2(t-\delta_j(t))} \right] dt \\
&= \int_0^\omega \left|r_1(t)\right| dt + \int_0^\omega \left[A_1(s) e^{u_1(s)} + \sum_{i=1}^n \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} e^{u_1(s)} + \sum_{j=1}^m \frac{C_{1j}(\mu_j(s))}{1-\delta_j'(\mu_j(s))} e^{u_2(s)} \right] ds \\
&= \int_0^\omega \left|r_1(t)\right| dt + \int_0^\omega \left[\left(A_1(s) + \sum_{i=1}^n \frac{B_{1i}(\alpha_i(s))}{1-\tau_i'(\alpha_i(s))} \right) e^{u_1(s)} + \sum_{j=1}^m \frac{C_{1j}(\mu_j(s))}{1-\delta_j'(\mu_j(s))} e^{u_2(s)} \right] ds \\
&= \int_0^\omega \left|r_1(t)\right| dt + \int_0^\omega \left[F_1^*(s) e^{u_1(s)} + G_1(s) e^{u_2(s)} \right] ds \\
&\leq \overline{R_1}\omega + F_1^{*M} \int_0^\omega e^{u_1(s)} ds + G_1^M \int_0^\omega e^{u_2(s)} ds,
\end{aligned} \quad (110)$$

where $\overline{R_1} = (1/\omega)\int_0^\omega |r_1(t)| dt$, $F_1^*(s)$, $G_1(s)$ are defined by (22). By (103) and (110), we obtain

$$\int_0^\omega \left|u_1'(t)\right| dt \leq \overline{R_1}\omega + F_1^{*M} \frac{\overline{r_2}\omega}{G_2^L} + G_1^M \frac{\overline{r_1}\omega}{G_1^L} := \Delta_3. \quad (111)$$

Similarly, by the second equation of (90), we get

$$\int_0^\omega \left|u_2'(t)\right| dt \leq \overline{R_2}\omega + F_2^{*M} \frac{\overline{r_1}\omega}{G_1^L} + G_2^M \frac{\overline{r_2}\omega}{G_2^L} := \Delta_4, \quad (112)$$

where $\overline{R_2} = (1/\omega)\int_0^\omega |r_2(t)|dt$, $F_2^*(s)$, $G_2(s)$ are defined by (22). From (109), (111), and (112) and Lemma 10, it follows that for $t \in [0,\omega]$

$$u_1(t) \le u_1(\zeta_1) + \frac{1}{2}\int_0^\omega \left|u_1'(t)\right| dt \le \ln\frac{\overline{r_2}}{G_2^L} + \frac{1}{2}\Delta_3,$$
$$u_2(t) \le u_2(\zeta_1) + \frac{1}{2}\int_0^\omega \left|u_2'(t)\right| dt \le \ln\frac{\overline{r_1}}{G_1^L} + \frac{1}{2}\Delta_4, \tag{113}$$

$$u_1(t) \ge u_1(\xi_1) - \frac{1}{2}\int_0^\omega \left|u_1'(t)\right| dt \ge \ln\Gamma_1 - \frac{1}{2}\Delta_3,$$
$$u_2(t) \ge u_2(\xi_1) - \frac{1}{2}\int_0^\omega \left|u_2'(t)\right| dt \ge \ln\Gamma_2 - \frac{1}{2}\Delta_4. \tag{114}$$

Let

$$R_3 = \max\left\{\left|\ln\frac{\overline{r_2}}{G_2^L} + \frac{1}{2}\Delta_3\right|, \left|\ln\Gamma_3 - \frac{1}{2}\Delta_3\right|\right\},$$
$$R_4 = \max\left\{\left|\ln\frac{\overline{r_1}}{G_1^L} + \frac{1}{2}\Delta_4\right|, \left|\ln\Gamma_4 - \frac{1}{2}\Delta_4\right|\right\}. \tag{115}$$

It follows from (113)–(115) that

$$\sup_{t\in[0,\omega]}\left|u_1(t)\right| \le R_3,$$
$$\sup_{t\in[0,\omega]}\left|u_2(t)\right| \le R_4. \tag{116}$$

Clearly, Γ_l, Δ_l, R_l $(l = 3, 4)$ are independent of λ, respectively. Note that $\int_0^\omega F_l(t)dt \le F_l^M\omega$, $\int_0^\omega G_l(t)dt \le G_l^L\omega$, $l = 1, 2$. From (44), we have

$$\overline{A_1} - \sum_{i=1}^n \overline{B_{1i}} = \overline{F_1} \le F_1^M, \qquad G_1^L \le \overline{G_1} = \sum_{j=1}^m \overline{C_{1j}};$$
$$\overline{A_2} - \sum_{j=1}^m \overline{B_{2j}} = \overline{F_2} \le F_2^M, \qquad G_2^L \le \overline{G_2} = \sum_{i=1}^n \overline{C_{2i}}, \tag{117}$$

which deduces that

$$\overline{r_1}\left(\overline{A_1} - \sum_{i=1}^n \overline{B_{1i}}\right) = \overline{r_1}\,\overline{F_1} \le \overline{r_1}F_1^M < \overline{r_2}G_2^L \le \overline{r_2}\,\overline{G_2} = \overline{r_2}\sum_{i=1}^n \overline{C_{2i}};$$
$$\overline{r_2}\left(\overline{A_2} - \sum_{j=1}^m \overline{B_{2j}}\right) = \overline{r_2}\,\overline{F_2} \le \overline{r_2}F_2^M < \overline{r_1}G_1^L \le \overline{r_1}\,\overline{G_1} = \overline{r_1}\sum_{j=1}^m \overline{C_{1j}}, \tag{118}$$

which implies that

$$\overline{r_1}\left(\overline{A_1} - \sum_{i=1}^n \overline{B_{1i}}\right) \le \overline{r_2}\sum_{i=1}^n \overline{C_{2i}};$$
$$\overline{r_2}\left(\overline{A_2} - \sum_{j=1}^m \overline{B_{2j}}\right) \le \overline{r_1}\sum_{j=1}^m \overline{C_{1j}}. \tag{119}$$

Hence

$$\left(\overline{A_1} - \sum_{i=1}^n \overline{B_{1i}}\right)\left(\overline{A_2} - \sum_{j=1}^m \overline{B_{2j}}\right) \le \sum_{j=1}^m \overline{C_{1j}}\sum_{i=1}^n \overline{C_{2i}}. \tag{120}$$

From (119) and (120), it is easy to show that the system of algebraic equations

$$\overline{r_1} - \left(\overline{A_1} - \sum_{i=1}^n \overline{B_{1i}}\right)e^{u_1} - \sum_{j=1}^m \overline{C_{1j}}e^{u_2} = 0,$$
$$\overline{r_2} - \left(\overline{A_2} - \sum_{j=1}^m \overline{B_{2j}}\right)e^{u_2} - \sum_{i=1}^n \overline{C_{2i}}e^{u_1} = 0 \tag{121}$$

has a unique solution $(u_1^*, u_2^*) \in R^2$. In view of (116), we can take sufficiently large R such that $R > R_3 + R_4$, $R > |u_1^*| + |u_2^*|$ and define $\Omega = \{u(t) = (u_1(t), u_2(t))^T \in X : \|u\| < R\}$, and it is clear that Ω satisfies condition (a) of Lemma 7. Letting $u \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, then u is a constant vector in R^2 with $\|u\| = R$. Then

$$QNu = \begin{pmatrix} \overline{r_1} - \left(\overline{A_1} - \sum_{i=1}^n \overline{B_{1i}}\right)e^{u_1} - \sum_{j=1}^m \overline{C_{1j}}e^{u_2} \\ \overline{r_2} - \left(\overline{A_2} - \sum_{j=1}^m \overline{B_{2j}}\right)e^{u_2} - \sum_{i=1}^n \overline{C_{2i}}e^{u_1} \end{pmatrix} \neq 0. \tag{122}$$

That is, condition (b) of Lemma 7 holds. In order to verify condition (c) in the Lemma 7, by (120) and the formula for Brouwer degree, a straightforward calculation shows that

$$\deg\{JQNu, \operatorname{Ker} L \cap \partial\Omega, 0\} = \operatorname{sign}\left\{\left(\sum_{j=1}^m \overline{C_{1j}}\sum_{i=1}^n \overline{C_{2i}} - \left(\overline{A_1} - \sum_{i=1}^n \overline{B_{1i}}\right) \times \left(\overline{A_2} - \sum_{j=1}^m \overline{B_{2j}}\right)\right)e^{(u_1^*+u_2^*)}\right\} \neq 0. \tag{123}$$

By now we have proved that all requirements in Lemma 7 hold. Hence system (82) has at least one ω-periodic solution, say $(u_1^*, u_2^*)^T$. Setting $y_1^*(t) = e^{u_1^*(t)}$, $y_2^*(t) = e^{u_2^*(t)}$, then $(y_1^*(t), y_2^*(t))^T$ has at least one positive ω-periodic solution of systems (10) and (12). Furthermore, setting $x_1^*(t) = \prod_{0<t_k<t}(1 + \theta_{1k})y_1^*(t)$, $x_2^*(t) = \prod_{0<t_k<t}(1 + \theta_{2k})y_2^*(t)$, then

$(x_1^*(t), x_2^*(t))^T$ has at least one positive ω-periodic solution of systems (2) and (4). If (H_8) holds, similarly we can prove that systems (2) and (4) have at least one positive ω-periodic solution. The proof of Theorem 14 is complete. □

We now proceed to the discussion on the uniqueness and global stability of the ω-periodic solution $x^*(t)$ in Theorem 14. It is immediate that if $x^*(t)$ is globally asymptotically stable, then $x^*(t)$ is unique in fact.

Theorem 15. *In addition to (H_1)–(H_3), assume further that*

(H_9) $F_1^L F_2^L > G_1^M G_2^M$.

Then systems (2) and (4) have a unique positive ω-periodic solution $x^*(t) = (x_1^*(t), x_2^*(t))^T$ which is globally asymptotically stable.

Proof. Letting $x^*(t) = (x_1^*(t), x_2^*(t))^T$ be a positive ω-periodic solution of (2) and (4), then $y^*(t) = (y_1^*(t), y_2^*(t))^T$(where $y_l^*(t) = \prod_{0<t_k<t}(1+\theta_{lk})^{-1}x_l^*(t)$, $l=1,2$) is the positive ω-periodic solution of systems (10) and (12), and let $y_l(t) = (y_1(t), y_2(t))^T$ be any positive solution of system (10) with the initial conditions (12). It follows from Theorem 14 that there exist positive constants T, r_l, R_l, such that for all $t \geq T$

$$r_l \leq y_l^*(t) \leq R_l, \quad l = 1, 2. \tag{124}$$

By the assumptions of Theorem 14, we can obtain $F_1^L F_2^L > G_1^M G_2^M$; then there exist constants $\alpha_3 > 0$, $\alpha_4 > 0$; we can choose a positive constant ε such that

$$F_1^L \alpha_3 - G_2^M \alpha_4 = \varepsilon, \qquad F_2^L \alpha_4 - G_1^M \alpha_3 = \varepsilon. \tag{125}$$

In the following, we always assume that α_3 and α_4 satisfy (67). We define

$$V_1(t) = \alpha_3 \left|\ln y_1(t) - \ln y_1^*(t)\right| + \alpha_4 \left|\ln y_2(t) - \ln y_2^*(t)\right|. \tag{126}$$

Calculating the upper right derivative of $V_1(t)$ along solutions of (10), it follows that

$$\begin{aligned}
D^+V_1(t) &= \alpha_3 \left(\frac{\dot{y}_1(t)}{y_1(t)} - \frac{\dot{y}_1^*(t)}{y_1^*(t)}\right) \operatorname{sgn}\left(y_1(t) - y_1^*(t)\right) \\
&\quad + \alpha_4 \left(\frac{\dot{y}_2(t)}{y_2(t)} - \frac{\dot{y}_2^*(t)}{y_2^*(t)}\right) \operatorname{sgn}\left(y_2(t) - y_2^*(t)\right) \\
&\leq \operatorname{sgn}\left(y_1(t) - y_1^*(t)\right)\alpha_3 \\
&\quad \times \Bigg\{ - A_1(t)\left(y_1(t) - y_1^*(t)\right) \\
&\quad + \sum_{i=1}^{n} B_{1i}(t)\left(y_1(t-\tau_i(t)) - y_1^*(t-\tau_i(t))\right) \\
&\quad - \sum_{j=1}^{m} C_{1j}(t)\left(y_2\left(t-\delta_j(t)\right) - y_2^*\left(t-\delta_j(t)\right)\right)\Bigg\} \\
&\quad + \operatorname{sgn}\left(y_2(t) - y_2^*(t)\right)\alpha_4 \\
&\quad \times \Bigg\{ - A_2(t)\left(y_2(t) - y_2^*(t)\right) \\
&\quad + \sum_{j=1}^{m} B_{2j}(t)\left(y_2\left(t-\eta_j(t)\right) - y_2^*\left(t-\eta_j(t)\right)\right) \\
&\quad - \sum_{i=1}^{n} C_{2i}(t)\left(y_1(t-\sigma_i(t)) - y_1^*(t-\sigma_i(t))\right)\Bigg\} \\
&\leq -\alpha_3 A_1(t)\left|y_1(t) - y_1^*(t)\right| \\
&\quad + \sum_{i=1}^{n} \alpha_3 B_{1i}(t)\left|y_1(t-\tau_i(t)) - y_1^*(t-\tau_i(t))\right| \\
&\quad + \sum_{j=1}^{m} \alpha_3 C_{1j}(t)\left|y_2\left(t-\delta_j(t)\right) - y_2^*\left(t-\delta_j(t)\right)\right| \\
&\quad - \alpha_4 A_2(t)\left|y_2(t) - y_2^*(t)\right| + \sum_{j=1}^{m} \alpha_4 B_{2j}(t) \\
&\quad \times \left|y_2\left(t-\eta_j(t)\right) - y_2^*\left(t-\eta_j(t)\right)\right| \\
&\quad + \sum_{i=1}^{n} \alpha_4 C_{2i}(t)\left|y_1(t-\sigma_i(t)) - y_1^*(t-\sigma_i(t))\right|.
\end{aligned} \tag{127}$$

We also define

$$\begin{aligned}
V_2(t) &= \sum_{i=1}^{n} \alpha_3 \int_{t-\tau_i(t)}^{t} \frac{B_{1i}(\alpha_i(\xi))}{1-\tau_i'(\alpha_i(\xi))} \left|y_1(\xi) - y_1^*(\xi)\right| d\xi \\
&\quad + \sum_{j=1}^{m} \alpha_3 \int_{t-\delta_j(t)}^{t} \frac{C_{1j}\left(\mu_j(\xi)\right)}{1-\delta_j'\left(\mu_j(\xi)\right)} \left|y_2(\xi) - y_2^*(\xi)\right| d\xi \\
&\quad + \sum_{j=1}^{m} \alpha_4 \int_{t-\rho_j(t)}^{t} \frac{B_{2j}\left(\nu_j(\xi)\right)}{1-\rho_j'\left(\nu_j(\xi)\right)} \left|y_2(\xi) - y_2^*(\xi)\right| d\xi \\
&\quad + \sum_{i=1}^{n} \alpha_4 \int_{t-\sigma_i(t)}^{t} \frac{C_{2i}(\beta_i(\xi))}{1-\sigma_j'(\beta_i(\xi))} \left|y_1(\xi) - y_1^*(\xi)\right| d\xi.
\end{aligned} \tag{128}$$

Calculating the upper right derivative of $V_2(t)$ along solutions of (10), it follows that

$$\begin{aligned}
D^+V_2(t) = {} & \sum_{i=1}^{n} \alpha_3 \frac{B_{1i}(\alpha_i(t))}{1-\tau_j'(\alpha_i(t))} |y_1(t) - y_1^*(t)| \\
& - \sum_{i=1}^{n} \alpha_1 \frac{B_{1i}(t)}{1-\tau_j'(t)} (1-\tau_j'(t)) \\
& \times |y_1(t-\tau_i(t)) - y_1^*(t-\tau_i(t))| \\
& + \sum_{j=1}^{m} \alpha_3 \frac{C_{1j}(\mu_j(t))}{1-\delta_j'(\mu_j(t))} |y_2(t) - y_2^*(t)| \\
& - \sum_{j=1}^{m} \alpha_1 \frac{C_{1j}(t)}{1-\delta_j'(t)} (1-\delta_j'(t)) \\
& \times |y_2(t-\delta_j(t)) - y_2^*(t-\delta_j(t))| \\
& + \sum_{j=1}^{m} \alpha_4 \frac{B_{2j}(\nu_j(t))}{1-\eta_j'(\nu_j(t))} |y_2(t) - y_2^*(t)| \\
& - \sum_{j=1}^{m} \alpha_1 \frac{B_{2j}(t)}{1-\eta_j'(t)} (1-\eta_j'(t)) \\
& \times |y_2(t-\eta_j(t)) - y_2^*(t-\eta_j(t))| \\
& + \sum_{i=1}^{n} \alpha_4 \frac{C_{2i}(\beta_i(t))}{1-\sigma_i'(\beta_i(t))} |y_1(t) - y_1^*(t)| \\
& - \sum_{i=1}^{n} \alpha_2 \frac{C_{2i}(t)}{1-\sigma_i'(t)} (1-\sigma_i'(t)) \\
& \times |y_1(t-\sigma_i(t)) - y_1^*(t-\sigma_i(t))|.
\end{aligned} \tag{129}$$

We define a Lyapunov functional $V(t)$ as follows:

$$V(t) = V_1(t) + V_2(t). \tag{130}$$

Calculating the upper right derivative of $V(t)$ along solutions of (10), it follows that

$$\begin{aligned}
D^+V(t) = {} & -\alpha_3 A_1(t) |y_1(t) - y_1^*(t)| \\
& + \sum_{i=1}^{n} \alpha_3 \frac{B_{1i}(\alpha_i(t))}{1-\tau_i'(\alpha_i(t))} |y_1(t) - y_1^*(t)| \\
& + \sum_{j=1}^{m} \alpha_1 \frac{C_{1j}(\mu_j(t))}{1-\delta_j'(\mu_j(t))} |y_2(t) - y_2^*(t)| \\
& - \alpha_4 A_2(t) |y_2(t) - y_2^*(t)| \\
& + \sum_{j=1}^{m} \alpha_4 \frac{B_{2j}(\nu_j(t))}{1-\eta_j'(\nu_j(t))} |y_2(t) - y_2^*(t)| \\
& + \sum_{i=1}^{n} \alpha_2 \frac{C_{2i}(\beta_i(t))}{1-\sigma_i'(\beta_i(t))} |y_1(t) - y_1^*(t)| \\
= {} & -\left[\alpha_3 A_1(t) - \sum_{i=1}^{n} \alpha_3 \frac{B_{1i}(\alpha_i(t))}{1-\tau_i'(\alpha_i(t))} \right. \\
& \left. - \sum_{i=1}^{n} \alpha_4 \frac{C_{2i}(\beta_i(t))}{1-\sigma_i'(\beta_i(t))} \right] |y_1(t) - y_1^*(t)| \\
& - \left[\alpha_4 A_2(t) - \sum_{j=1}^{m} \alpha_1 \frac{C_{1j}(\mu_j(t))}{1-\delta_j'(\mu_j(t))} \right. \\
& \left. - \sum_{j=1}^{m} \alpha_4 \frac{B_{2j}(\nu_j(t))}{1-\eta_j'(\nu_j(t))} \right] |y_2(t) - y_2^*(t)| \\
\le {} & (\alpha_3 F_1^L - \alpha_4 G_2^M) |y_1(t) - y_1^*(t)| \\
& - (\alpha_4 F_2^L - \alpha_3 G_1^M) |y_2(t) - y_2^*(t)| \\
= {} & -\varepsilon (|y_1(t) - y_1^*(t)| + |y_2(t) - y_2^*(t)|).
\end{aligned} \tag{131}$$

So by (131), we have

$$\begin{aligned}
& \varepsilon \int_0^t (|y_1(\xi) - y_1^*(\xi)| + |y_2(\xi) - y_2^*(\xi)|) \, d\xi \\
& \quad + V(t) \le V(0) < +\infty, \quad t \ge 0,
\end{aligned} \tag{132}$$

where

$$\begin{aligned}
V(0) = {} & \alpha_3 |\ln y_1(0) - \ln y_1^*(0)| + \alpha_4 |\ln y_2(0) - \ln y_2^*(0)| \\
& + \sum_{i=1}^{n} \alpha_3 \int_{-\tau_i(0)}^{0} \frac{B_{1i}(\alpha_i(\xi))}{1-\tau_i'(\alpha_i(\xi))} \\
& \times |y_1(\xi) - y_1^*(\xi)| \, d\xi \\
& + \sum_{j=1}^{m} \alpha_3 \int_{-\delta_j(0)}^{0} \frac{C_{1j}(\mu_j(\xi))}{1-\delta_j'(\mu_j(\xi))} \\
& \times |y_2(\xi) - y_2^*(\xi)| \, d\xi + \sum_{j=1}^{m} \alpha_4 \\
& \times \int_{-\rho_j(0)}^{0} \frac{B_{2j}(\nu_j(\xi))}{1-\rho_j'(\nu_j(\xi))} |y_2(\xi) - y_2^*(\xi)| \, d\xi \\
& + \sum_{i=1}^{n} \alpha_4 \int_{-\sigma_i(0)}^{0} \frac{C_{2i}(\beta_i(\xi))}{1-\sigma_i'(\beta_i(\xi))} \\
& \times |y_1(\xi) - y_1^*(\xi)| \, d\xi,
\end{aligned} \tag{133}$$

which implies that

$$\int_0^t \left(\left|y_1(\xi) - y_1^*(\xi)\right| + \left|y_2(\xi) - y_2^*(\xi)\right|\right) d\xi \le \frac{V(0)}{\varepsilon}. \quad (134)$$

By (134), it is obvious that $|y_1(t) - y_1^*(t)| + |y_2(t) - y_2^*(t)|$ is bounded.

On the other hand, we know that

$$\begin{aligned} &\alpha_3 \left|\ln y_1(t) - \ln y_1^*(t)\right| + \alpha_4 \left|\ln y_2(t) - \ln y_2^*(t)\right| \\ &\quad \le V(t) \le V(0) < +\infty, \quad t \ge 0, \end{aligned} \quad (135)$$

which implies that

$$\begin{aligned} \left|\ln y_1(t) - \ln y_1^*(t)\right| &\le \frac{V(0)}{\alpha_3}, \\ \left|\ln y_2(t) - \ln y_2^*(t)\right| &\le \frac{V(0)}{\alpha_4}, \end{aligned} \quad (136)$$

which, together with (123), yield

$$\begin{aligned} r_1 e^{-V(0)/\alpha_3} &\le y_1(t) \le R_1 e^{V(0)/\alpha_3} < +\infty, \\ r_2 e^{-V(0)/\alpha_4} &\le y_2(t) \le R_2 e^{V(0)/\alpha_4} < +\infty. \end{aligned} \quad (137)$$

From (124) and (137), it follows that $y_l(t)$ $(l = 1, 2)$ are bounded for $t \ge 0$. Hence, $y_1(t) - y_1^*(t)$, $y_2(t) - y_2^*(t)$, and their derivatives remain bounded on $[0, +\infty)$. So $|y_1(t) - y_1^*(t)|$, $|y_2(t) - y_2^*(t)|$ are uniformly continuous on $[0, +\infty)$. By Lemma 11, we have

$$\begin{aligned} &\lim_{t \to +\infty} \left|y_l(s) - y_l^*(s)\right| \\ &\quad = \lim_{t \to +\infty} \left[\prod_{0 < t_k < t} \left(1 + \theta_{ik}\right)^{-1} \left|x_l^*(s) - x_l(s)\right|\right] = 0, \\ &\qquad l = 1, 2. \end{aligned} \quad (138)$$

Therefore

$$\lim_{t \to +\infty} \left|x_l(s) - x_l^*(s)\right| = 0, \quad l = 1, 2. \quad (139)$$

By Theorems 7.4 and 8.2 in [30], we know that the periodic positive solution $x^*(t) = (x_1^*(t), x_2^*(t))^T$ is uniformly asymptotically stable. The proof of Theorem 15 is completed. □

4. Applications

In this section, for some applications of our main results, we will consider some special cases of systems (2) and (3), which have been investigated extensively in [10].

Application 1. consider the following equations:

$$\begin{aligned} y_1'(t) &= y_1(t) \left[r_1(t) - a_1(t) y_1(t) + \sum_{i=1}^n b_{1i}(t) y_1\left(t - \tau_i(t)\right) \right. \\ &\qquad \left. - \sum_{j=1}^m c_{1j}(t) y_2\left(t - \rho_j(t)\right) \right], \\ y_2'(t) &= y_2(t) \left[r_2(t) - a_2(t) y_2(t) + \sum_{j=1}^m b_{2j}(t) y_2\left(t - \eta_j(t)\right) \right. \\ &\qquad \left. - \sum_{i=1}^n c_{2i}(t) y_1\left(t - \sigma_i(t)\right) \right], \\ y_i(0) &> 0, \quad i = 1, 2, \end{aligned} \quad (140)$$

$$\begin{aligned} y_1'(t) &= y_1(t) \left[r_1(t) - a_1(t) y_1(t) - \sum_{i=1}^n b_{1i}(t) y_1\left(t - \tau_i(t)\right) \right. \\ &\qquad \left. - \sum_{j=1}^m c_{1j}(t) y_2\left(t - \rho_j(t)\right) \right], \\ y_2'(t) &= y_2(t) \left[r_2(t) - a_2(t) y_2(t) - \sum_{j=1}^m b_{2j}(t) y_2\left(t - \eta_j(t)\right) \right. \\ &\qquad \left. - \sum_{i=1}^n c_{2i}(t) y_1\left(t - \sigma_i(t)\right) \right], \\ y_i(0) &> 0, \quad i = 1, 2, \end{aligned} \quad (141)$$

which are special cases of systems (2) and (3) without impulse, respectively. By applying Theorems 12–15 to systems (140) and (141), respectively, we obtain the following theorems.

Theorem 16. *In addition to* (H_1)*, assume that the following conditions hold:*

(H_{10}) $\overline{r_1} F_1^L > \overline{r_2} G_2^M$, $\overline{r_2} F_2^L > \overline{r_1} G_1^M$.

Then system (140) *has a unique positive ω-periodic solution* $x^*(t) = (x_1^*(t), x_2^*(t))^T$ *which is globally asymptotically stable, where* $F_1(t)$, $F_2(t)$, $G_1(t)$, *and* $G_2(t)$ *are defined in* (22).

Proof. It is similar to the proof of Theorems 12 and 13, so we omit the details here. □

Theorem 17. *In addition to* (H_1)*, assume further that*

(H_{11}) $\overline{r_1} F_1^{*L} > \overline{r_2} G_2^M$, $\overline{r_2} F_2^{*L} > \overline{r_1} G_1^M$.

Then system (140) *has a unique positive ω-periodic solution* $x^*(t) = (x_1^*(t), x_2^*(t))^T$ *which is globally asymptotically stable, where* $F_1^*(t)$, $F_2^*(t)$, $G_1(t)$, *and* $G_2(t)$ *are defined in* (22).

Proof. It is similar to the proof of Theorems 14 and 15, so we omit the details here.

We consider the following systems:

$$x'(t) = x(t)\left[r(t) - a(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t - \tau_i(t))\right], \quad (142)$$

$$x'(t) = x(t)\left[r(t) - a(t)x(t) - \sum_{i=1}^{n} b_i(t)x(t - \tau_i(t))\right], \quad (143)$$

which are special cases of systems (140) and (141), respectively. From Theorems 17 and 18, we have the following corollary. □

Corollary 18. *In addition to* (H_1), *assume that the following condition holds:*

(H_{12}) $a(t) - \sum_{i=1}^{n}(b_i(\mu_i(t)))/(1 - \tau_i'(\mu_i(t))) > 0.$

Then systems (142) *and* (143) *have a unique positive ω-periodic solution* $x^*(t)$ *which is globally asymptotically stable, where* $\mu_i(t)$ *are the inverses of functions* $t - \tau_i(t)$.

Proof. It is similar to the proof of Theorems 12 and 13, so we omit the details here.

Application 2. Let us consider two delayed two-species competitive systems:

$$\begin{aligned} y_1'(t) &= y_1(t)[r_1(t) - a_1(t)y_1(t) \\ &\quad + b_1(t)y_1(t - \tau(t)) - c_1(t)y_2(t - \delta(t))], \\ y_2'(t) &= y_2(t)[r_2(t) - a_2(t)y_2(t) \\ &\quad + b_2(t)y_2(t - \eta(t)) - c_2(t)y_1(t - \sigma(t))], \\ y_i(0) &> 0, \quad i = 1, 2, \end{aligned} \quad (144)$$

$$\begin{aligned} y_1'(t) &= y_1(t)[r_1(t) - a_1(t)y_1(t) \\ &\quad - b_1(t)y_1(t - \tau(t)) - c_1(t)y_2(t - \delta(t))], \\ y_2'(t) &= y_2(t)[r_2(t) - a_2(t)y_2(t) \\ &\quad - b_2(t)y_2(t - \eta(t)) - c_2(t)y_1(t - \sigma(t))], \\ y_i(0) &> 0, \quad i = 1, 2, \end{aligned} \quad (145)$$

which are special cases of systems (2) and (3) without impulse and $i = j = 1$, respectively. By applying Theorems 12–15 to systems (144) and (145), respectively, we obtain the following theorems. □

Theorem 19. *In addition to* (H_1), *assume that the following conditions hold:*

(H_{12}) $\overline{r_1}C_1^L > \overline{r_2}D_2^M$, $\overline{r_2}C_2^L > \overline{r_1}D_1^M$.

Then system (144) *has a unique positive ω-periodic solution* $x^*(t) = (x_1^*(t), x_2^*(t))^T$ *which is globally asymptotically stable, where* $C_1(t)$, $C_2(t)$, $D_1(t)$, *are* $D_2(t)$ *are defined as follow:*

$$\begin{aligned} C_1(t) &= a_1(t) - \frac{b_1(\alpha(t))}{1 - \tau'(\alpha(t))}, \qquad D_1(t) = \frac{C_1(\mu(t))}{1 - \delta'(\mu(t))}, \\ C_2(t) &= a_2(t) - \frac{b_2(\nu(t))}{1 - \eta'(\nu(t))}, \qquad D_2(t) = \frac{C_2(\beta(t))}{1 - \sigma'(\beta(t))}. \end{aligned} \quad (146)$$

And $\alpha(t)$, $\beta(t)$, $\mu(t)$, *and* $\nu(t)$ *represent the inverse function of* $t - \tau(t)$, $t - \sigma(t)$, $t - \delta(t)$, *and* $t - \eta(t)$, *respectively.*

Proof. It is similar to the proof of Theorems 12 and 13, so we omit the details here. □

Theorem 20. *In addition to* (H_1), *assume further that*

(H_{13}) $\overline{r_1}C_1^{*L} > \overline{r_2}D_2^M$, $\overline{r_2}C_2^{*L} > \overline{r_1}G_1^M$.

Then system (145) *has a unique positive ω-periodic solution* $x^*(t) = (x_1^*(t), x_2^*(t))^T$ *which is globally asymptotically stable, where* $C_1^*(t)$, $C_2^*(t)$, $D_1(t)$, *and* $D_2(t)$ *are defined as follow:*

$$\begin{aligned} C_1^*(t) &= a_1(t) + \frac{b_1(\alpha(t))}{1 - \tau'(\alpha(t))}, \qquad D_1(t) = \frac{C_1(\mu(t))}{1 - \delta'(\mu(t))}, \\ C_2^*(t) &= a_2(t) + \frac{b_2(\nu(t))}{1 - \eta'(\nu(t))}, \qquad D_2(t) = \frac{C_2(\beta(t))}{1 - \sigma'(\beta(t))}. \end{aligned} \quad (147)$$

And $\alpha(t)$, $\beta(t)$, $\mu(t)$, *and* $\nu(t)$ *represent the inverse function of* $t - \tau(t)$, $t - \sigma(t)$, $t - \delta(t)$, *and* $t - \eta(t)$, *respectively.*

Proof. It is similar to the proof of Theorems 14 and 15, so we omit the details here. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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