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Research Article

Existence of Solutions for Two-Point Boundary Value Problem of Fractional Differential Equations at Resonance

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We establish the existence results for two-point boundary value problem of fractional differential equations at resonance by means of the coincidence degree theory. Furthermore, a result on the uniqueness of solution is obtained. We give an example to demonstrate our results.

1. Introduction

Fractional differential equations have been studied extensively. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications such as physics, chemistry, phenomena arising in engineering, economy, and science; see, for example, [1–5].

Recently, more and more authors have paid their attentions to the boundary value problems of fractional differential equations; see [6–21]. Moreover, there have been many works related to the existence of solutions for boundary value problems at resonance; see [12–21]. It is considerable that there are many papers that have dealt with the solutions of multipoint boundary value problems of fractional differential equations at resonance (see, e.g., [12, 16]).

In [12], Bai and Zhang considered a three-point boundary value problem of fractional differential equations with nonlinear growth given by

$$D_{0^+}^{\alpha} u(t) = f\left(t, u(t), D_{0^+}^{\alpha-1} u(t)\right), \quad 0 < t < 1, \qquad u(0) = 0, \quad u(1) = \sigma u(\eta), \tag{1}$$

where $1 < \alpha \leq 2$, $0 < \eta, \sigma < 1 > 0$, $\sigma\eta^{\alpha-1} = 1$, $D_{0^+}^{\alpha}$ is Riemann-Liouville fractional derivative, and $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are given functions.

In [13], Hu et al. have studied a two-point boundary value problem for fractional differential equation at resonance

$$D_{0^+}^{\alpha} x(t) = f\left(t, x(t), x'(t)\right), \quad 0 \leq t \leq 1, \qquad x(0) = 0, \quad x'(0) = x'(1), \tag{2}$$

where $1 < \alpha \leq 2$, $D_{0^+}^{\alpha}$ is Caputo fractional derivative, and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Carathéodory conditions.

As far as we know, there are few works on the existence of two-point boundary value problems of the fractional differential equations at resonance. Motivated by the works above, we discuss the existence and uniqueness of solutions for the following nonlinear fractional differential equation:

$$\begin{gathered} D_{0^+}^{\alpha} u(t) = f\left(t, u(t), D_{0^+}^{\alpha-1} u(t), D_{0^+}^{\alpha-2} u(t), \ldots, D_{0^+}^{\alpha-(N-1)} u(t)\right), \\ u(0) = D_{0^+}^{\alpha-2} u(0) = \cdots = D_{0^+}^{\alpha-(N-1)} u(0) = 0, \\ D_{0^+}^{\alpha-1} u(0) = D_{0^+}^{\alpha-1} u(1), \end{gathered} \tag{3}$$

where $0 < t < 1$, $N - 1 < \alpha < N$, $D_{0^+}^{\alpha}$ is Riemann-Liouville fractional derivative, and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function.

More precisely, we use the coincidence degree theorem due to Mawhin [22]. The rest of this paper is organized as follows. In Section 2, we give some necessary notations, definitions, and lemmas. In Section 3, we study the existence of solutions of (3) by the coincidence degree theory. Finally, an example is given to illustrate our results in Section 4.

The two-point boundary value problem (3) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{gathered} D_{0^+}^{\alpha} u(t) = 0, \\ u(0) = D_{0^+}^{\alpha-2} u(0) = \cdots = D_{0^+}^{\alpha-(N-1)} u(0) = 0, \\ D_{0^+}^{\alpha-1} u(0) = D_{0^+}^{\alpha-1} u(1), \end{gathered} \tag{4}$$

has $u(t) = c_1 t^{\alpha-1}$ as a nontrivial solution.

2. Preliminaries

In this section, we present the necessary definitions and lemmas from fractional calculus theory. These definitions and properties can be found in the literature. For more details see [1–3].

Definition 1 (see [1]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \tag{5}$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2 (see [1]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \tag{6}$$

where $n - 1 < \alpha \le n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 3 (see [1]). *Let $n-1 < \alpha \le n$, $u \in C(0,1) \bigcap L^1(0,1)$; then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n}, \tag{7}$$

where $C_i \in \mathbb{R}$, $i = 1, 2, \ldots n$.

Lemma 4 (see [1]). *If $\alpha > 0$, $m \in \mathbb{N}$ and $D = d/dx$. If the fractional derivatives $D_{0+}^{\alpha} u(t)$ and $D_{0+}^{\alpha+m} u(t)$ exist, then*

$$D^m D_{0+}^{\alpha} u(t) = D_{0+}^{\alpha+m} u(t). \tag{8}$$

Lemma 5 (see [1]). *The relation*

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(x) = I_{a+}^{\alpha+\beta} f(x) \tag{9}$$

is valid in following cases $\beta > 0$, $\alpha+\beta > 0$, and $f(x) \in L_1(a,b)$.

Now let us recall some notations about the coincidence degree continuation theorem.

Let Y, Z be real Banach spaces, let $L : \operatorname{dom} L \subset Y \to Z$ be a Fredholm map of index zero, and let $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that $\ker L = \operatorname{Im} P$, $\operatorname{Im} L = \ker Q$, and $Y = \ker L \oplus \ker P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ is invertible. We denote the inverse of this map by K_P. If Ω is an open bounded subset of Y, the map N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_{P,Q} N = K_P(I-Q)N : \overline{\Omega} \to Y$ is compact.

Theorem 6. *Let L be a Fredholm operator of index zero and N be L-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied:*

(1) *$Lx \ne \lambda N x$ for each $(x, \lambda) \in [(\operatorname{dom} L \setminus \ker L) \cap \partial\Omega] \times (0,1)$;*

(2) *$Nx \notin \operatorname{Im} L$ for each $x \in \ker L \cap \partial\Omega$;*

(3) *$\deg(JQN|_{\ker L}, \Omega \cap \ker L, 0) \ne 0$, where $Q : Z \to Z$ is a continuous projection as above with $\operatorname{Im} L = \ker Q$ and $J : \operatorname{Im} Q \to \ker L$ is any isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

3. Main Results

In this section, we will prove the existence results for (3).

We use the Banach space $E = C[0,1]$ with the norm $\|u\|_\infty = \max_{0 \le t \le 1} |u(t)|$. For $\alpha > 0$, $N = [\alpha] + 1$, we define a linear space

$$X = \left\{ u \mid u, D_{0+}^{\alpha-i} u \in E,\ i = 1, 2, \ldots, N-1 \right\}. \tag{10}$$

By means of the functional analysis theory, we can prove that X is a Banach space with the norm $\|u\|_X = \|D_{0+}^{\alpha-1} u\|_\infty + \cdots + \|D_{0+}^{\alpha-(N-1)} u\|_\infty + \|u\|_\infty$.

Define L to be the linear operator from $\operatorname{dom}(L) \bigcap X$ to E with $\operatorname{dom}(L) = \{u \in X \mid D_{0+}^{\alpha} u(t) \in E,\ u(0) = D_{0^+}^{\alpha-2} u(0) = \cdots = D_{0^+}^{\alpha-(N-1)} u(0) = 0,\ D_{0+}^{\alpha-1} u(0) = D_{0+}^{\alpha-1} u(1)\}$ and

$$Lu = D_{0+}^{\alpha} u, \quad u \in \operatorname{dom}(L). \tag{11}$$

We define $N : X \to E$ by

$$Nu(t) = f\left(t, u(t), D_{0^+}^{\alpha-1} u(t), D_{0^+}^{\alpha-2} u(t), \ldots, D_{0^+}^{\alpha-(N-1)} u(t)\right). \tag{12}$$

Then the problem (3) can be written by $Lu = Nu$.

Lemma 7. *The mapping $L : \operatorname{dom}(L) \subset E$ is a Fredholm operator of index zero.*

Proof. It is clear that

$$\ker(L) = \left\{ c_1 t^{\alpha-1} \right\} \cong \mathbb{R}^1. \tag{13}$$

Let $x \in \operatorname{Im} L$, so there exists a function $u \in \operatorname{dom} L$ which satisfies $Lu = x$. By (11) and Lemma 3, we have

$$u(t) = I_{0+}^{\alpha} x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}. \tag{14}$$

By $u(0) = D_{0^+}^{\alpha-2} u(0) = \cdots = D_{0^+}^{\alpha-(N-1)} u(0) = 0$, we can obtain $c_2 = \cdots = c_N = 0$. Hence

$$u(t) = I_{0+}^{\alpha} x(t) + c_1 t^{\alpha-1}. \tag{15}$$

Then, we have

$$\begin{aligned} D_{0+}^{\alpha-1} u(t) &= D_{0+}^{\alpha-1} \left(I_{0+}^{\alpha} x(t) + c_1 t^{\alpha-1} \right) \\ &= D_{0+}^{\alpha-1} I_{0+}^{\alpha} x(t) + c_1 \frac{\Gamma(\alpha)}{\Gamma(1)} \\ &= \int_0^t x(s)\, ds + c_1 \Gamma(\alpha). \end{aligned} \tag{16}$$

Taking into account $D_{0^+}^{\alpha-1} u(0) = D_{0^+}^{\alpha-1} u(1)$, we obtain

$$\int_0^1 x(s)\, ds = 0. \tag{17}$$

On the other hand, suppose x satisfy $\int_0^1 x(s) ds = 0$. Let $u(t) = I_{0+}^{\alpha} x(t)$, we can easily prove $u(t) \in \operatorname{dom}(L)$.

Thus, we conclude that

$$\operatorname{Im}(L) = \left\{ x : \int_0^1 x(s)\, ds = 0 \right\}. \tag{18}$$

Consider the linear operators $Q : E \to E$ defined by

$$Qx(t) = \int_0^1 x(s)\, ds. \tag{19}$$

Take $x(t) \in E$; then

$$\begin{aligned} Q(Qx(t)) &= Q\left(\int_0^1 x(s)\, ds \right) \\ &= \int_0^1 \left(\int_0^1 x(t)\, dt \right) ds \\ &= \int_0^1 x(s)\, ds = Qx(t). \end{aligned} \tag{20}$$

We can see $Q^2 = Q$.

For $x(t) \in E$ in the type $x(t) = x(t) - Qx(t) + Qx(t)$, obviously, $x(t) - Qx(t) \in \operatorname{Ker}(Q) = \operatorname{Im}(L)$ and $Qx(t) \in \operatorname{Im}(Q)$. That is to say, $E = \operatorname{Im}(L) + \operatorname{Im}(Q)$. If $u \in \operatorname{Im}(L) \bigcap \operatorname{Im}(Q)$, we have $u = c_1$; then $\int_0^1 c_1 ds = 0$. As a result $c_1 = 0$, and we get $E = \operatorname{Im}(L) \oplus \operatorname{Im}(Q)$.

Note that $\operatorname{Ind} L = \dim \ker L - \operatorname{codim} \operatorname{Im} L = 0$. Then L is a Fredholm mapping of index zero. □

We can define the operators $P : X \to X$, where

$$Pu = \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1} u(0)\, t^{\alpha-1}. \tag{21}$$

For $u \in X$,

$$\begin{aligned} P(Pu) &= P\left(\frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1} u(0)\, t^{\alpha-1} \right) \\ &= \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1} u(0)\, t^{\alpha-1} = Pu. \end{aligned} \tag{22}$$

So we have $P^2 = P$.

Note that

$$\operatorname{Ker}(P) = \left\{ u : D_{0+}^{\alpha-1} u(0) = 0 \right\}. \tag{23}$$

Since $u = u - Pu + Pu$, it is easy to say that $u - Pu \in \operatorname{Ker}(P)$ and $Pu \in \operatorname{Ker}(L)$. So we have $X = \operatorname{Ker}(P) + \operatorname{Ker}(L)$. If $u \in \operatorname{Ker}(L) \bigcap \operatorname{Ker}(P)$, then $u = c_1 t^{\alpha-1}$. We can derive $c_1 = 0$ from $D_{0+}^{\alpha-1} c_1 t^{\alpha-1}|_{t=0} = 0$. Then

$$X = \operatorname{Ker}(L) \oplus \operatorname{Ker}(P). \tag{24}$$

For $u \in X$,

$$\begin{aligned} \|Pu\|_X &= \frac{1}{\Gamma(\alpha)} \left| D_{0+}^{\alpha-1} u(0) \right| \cdot \left\| t^{\alpha-1} \right\|_X \\ &= \frac{1}{\Gamma(\alpha)} \left| D_{0+}^{\alpha-1} u(0) \right| \cdot \Big[\left\| t^{\alpha-1} \right\|_\infty + \left\| D_{0+}^{\alpha-1} t^{\alpha-1} \right\|_\infty \\ &\qquad + \cdots + \left\| D_{0+}^{\alpha-(N-1)} t^{\alpha-1} \right\|_\infty \Big] \\ &= \left(\sum_{i=1}^{N-1} \frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)} \right) \left| D_{0+}^{\alpha-1} u(0) \right| \\ &= a \left| D_{0+}^{\alpha-1} u(0) \right|, \end{aligned} \tag{25}$$

where $a = 1/\Gamma(\alpha) + \sum_{i=1}^{N-1} (1/\Gamma(i))$.

We define $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ by $K_p x = I_{0+}^{\alpha} x$.

For $x \in \operatorname{Im}(L)$, we have

$$LK_p x = L I_{0+}^{\alpha} x = D_{0+}^{\alpha} I_{0+}^{\alpha} x = x. \tag{26}$$

For $u \in \operatorname{dom}(L) \bigcap \operatorname{Ker}(P)$, we have $D_{0+}^{\alpha-1} u(0) = 0$. And for $u \in \operatorname{dom}(L)$, the coefficients $c_1, \ldots, c_N$ in the expressions

$$u = I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N} \tag{27}$$

are all equal to zero. Thus, we obtain

$$K_p L u = I_{0+}^{\alpha} D_{0+}^{\alpha} u = u. \tag{28}$$

This shows that $K_p = (L_{\operatorname{dom}(L) \bigcap \operatorname{Ker}(P)})^{-1}$. Again for each $x \in \operatorname{Im}(L)$,

$$\begin{aligned} &\left\| K_p x \right\|_X \\ &\quad = \left\| I_{0+}^{\alpha} x \right\|_X \\ &\quad = \left\| I_{0+}^{\alpha} x \right\|_\infty + \left\| D_{0+}^{\alpha-1} I_{0+}^{\alpha} x \right\|_\infty + \cdots + \left\| D_{0+}^{\alpha-(N-1)} I_{0+}^{\alpha} x \right\|_\infty; \\ &\quad \le \left(\sum_{i=1}^{N-1} \frac{1}{\Gamma(i+1)} + \frac{1}{\Gamma(\alpha+1)} \right) \|x\|_\infty \\ &\quad = b \|x\|_\infty, \end{aligned} \tag{29}$$

where $b = 1/\Gamma(\alpha+1) + \sum_{i=1}^{N-1} (1/\Gamma(i+1))$.

Lemma 8. *Assume $\Omega \subset Y$ is an open bounded subset such that* $\operatorname{dom} L \bigcap Y \neq \emptyset$*; then map N is L-compact on $\overline{\Omega}$*

Proof. By the continuity of f, we can get that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. So, in view of the Arzela-Ascoli theorem, we need only to prove that $K_P(I-Q)N(\overline{\Omega})$ is equicontinuous. From the continuity of f, there exists a constant $r > 0$, such that $|(I-Q)N(u(t))| \leq r$, for all $u \in \overline{\Omega}$, $t \in [0,1]$.

For $0 \leq t_1 \leq t_2 \leq 1$, $u \in \Omega$, we have

$$\begin{aligned}
&\left|K_{P,Q}Nu(t_2) - K_{P,Q}Nu(t_1)\right| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}(I-Q)N(u(s))\,ds - \int_0^{t_1}(t_1-s)^{\alpha-1}(I-Q)N(u(s))\,ds\right| \\
&\leq \frac{r}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]ds + \frac{r}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&= \frac{r}{\Gamma(\alpha+1)}\left(t_2^{\alpha}-t_1^{\alpha}\right).
\end{aligned} \tag{30}$$

Furthermore, we have

$$\begin{aligned}
&\left|D_{0+}^{\alpha-i}K_{P,Q}Nu(t_2) - D_{0+}^{\alpha-i}K_{P,Q}Nu(t_1)\right| \\
&= \frac{1}{\Gamma(i)}\left|\int_0^{t_2}(t_2-s)^{i-1}(I-Q)N(u(s))\,ds - \int_0^{t_1}(t_1-s)^{i-1}(I-Q)N(u(s))\,ds\right| \\
&\leq \frac{r}{\Gamma(i)}\int_0^{t_1}\left[(t_1-s)^{i-1}-(t_2-s)^{i-1}\right]ds + \frac{r}{\Gamma(i)}\int_{t_1}^{t_2}(t_2-s)^{i-1}ds \\
&= \frac{r}{\Gamma(i+1)}\left(t_2^{i}-t_1^{i}\right),
\end{aligned} \tag{31}$$

where $i = 1, 2, \ldots, N-1$. Since t^{α} and t^{i} are uniformly continuous on $[0,1]$, we can get that $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact. The proof is completed. □

To obtain our main results, we need the following conditions.

(H_1) There exist functions $\varphi, \psi_i \in L^1[0,1]$, $i = 1, N$, such that for all $u \in \mathbb{R}^2$, $t \in [0,1]$,

$$\begin{aligned}
&\left|f(t, x_1, x_2, \ldots, x_N)\right| \\
&\leq \varphi + \psi_1|x_1| + \psi_2|x_2| + \cdots + \psi_N|x_N|.
\end{aligned} \tag{32}$$

(H_2) There exists a constant $A > 0$ such that for every $y \in \mathbb{R}$, if $|x_2| > A$ for all $t \in [0,1]$, then

$$f(t, x_1, x_2, \ldots, x_N) \neq 0. \tag{33}$$

(H_3) There exists a constant $D > 0$ such that, for each c_i, $i = 1, 2$ satisfying $\min\{|c_1|, |c_2|\} > D$. We have either at least one of the following:

$$c_1 N\left(c_1 t^{\alpha-1}\right) > 0 \tag{34}$$

or

$$c_1 N\left(c_1 t^{\alpha-1}\right) < 0. \tag{35}$$

(H_4) $\sum_{i=2}^{N}\rho_i < 1$, where $\rho_{i+1} = (a+b)\|\psi_i\|_1$, $i = 1, 2, \ldots, N$.

Lemma 9. $\Omega_1 = \{u \in \operatorname{dom}(L) \setminus \operatorname{Ker}(L) \mid Lu = \lambda Nu, \lambda \in [0,1]\}$ *is bounded.*

Proof. For $u \in \Omega_1$, $\lambda \neq 0$ and $Lu = \lambda Nu$. By (12), $Lu = \lambda Nu \in \operatorname{Im}(L) = \operatorname{Ker}(Q)$; that is,

$$\begin{aligned}
&\lambda\int_0^1 f\left(t, u(t), D_{0^+}^{\alpha-1}u(t), D_{0^+}^{\alpha-2}u(t), \ldots, D_{0^+}^{\alpha-(N-1)}u(t)\right)dt \\
&= 0.
\end{aligned} \tag{36}$$

By the integral mean value theorem, there exits a constant $t_0 \in [0,1]$ such that

$$\begin{aligned}
&f\left(t_0, u(t_0), D_{0^+}^{\alpha-1}u(t_0), D_{0^+}^{\alpha-2}u(t_0), \ldots, D_{0^+}^{\alpha-(N-1)}u(t_0)\right) \\
&= 0.
\end{aligned} \tag{37}$$

Form (H_2), we can get $|D_{0+}^{\alpha-1}u(t_0)| \leq A$.

Again for $u \in \Omega_1$, $(I-P)u \in \operatorname{dom}(L) \setminus \operatorname{Ker}(L)$ and $LPu = 0$. From (29), we have

$$\|(I-P)u\|_X = \left\|K_p L(I-P)u\right\|_X = \left\|K_p Lu\right\|_X \leq b\|Nu\|_{\infty}. \tag{38}$$

Now by Lemma 4

$$\begin{aligned}
\left|D_{0+}^{\alpha-1}u(0)\right| &\leq \left|D_{0+}^{\alpha-1}u(t_0)\right| + \left|\int_0^{t_0}D_{0+}^{\alpha}u(s)\,ds\right| \\
&\leq \left|D_{0+}^{\alpha-1}u(t_0)\right| + \left|t_0\right|\max_{0\leq t\leq t_0}\left|D_{0+}^{\alpha}u(t)\right| \\
&\leq \left|D_{0+}^{\alpha-1}u(t_0)\right| + \left\|D_{0+}^{\alpha}u(t)\right\|_{\infty} \\
&\leq A + \|Lu\|_{\infty} = A + \|Nu\|_{\infty}.
\end{aligned} \tag{39}$$

That is,

$$\left|D_{0+}^{\alpha-1}u(0)\right| \leq A + \|Nu\|_{\infty}. \tag{40}$$

From (25) and (38), we have

$$\begin{aligned}\|u\|_X &= \|Pu + (I-P)u\|_X \le \|Pu\|_X + \|(I-P)u\|_X \\ &\le a\left|D_{0+}^{\alpha-1}u(0)\right| + b\|Nu\|_\infty.\end{aligned} \tag{41}$$

Furthermore, it follows from (40) and (H_1) that

$$\begin{aligned}&\|u\|_X \\ &\le \left(a\left|D_{0+}^{\alpha-1}u(0)\right| + b\|Nu\|_\infty\right) \\ &\le a\left(A + \|Nu\|_\infty\right) + b\|Nu\|_\infty = aA + (a+b)\|Nu\|_\infty \\ &\le aA + (a+b) \\ &\quad \times \left\| f\left(t, u(t), D_{0^+}^{\alpha-1}u(t), D_{0^+}^{\alpha-2}u(t), \dots, D_{0^+}^{\alpha-(N-1)}u(t)\right)\right\|_\infty \\ &\le aA + (a+b)\left(\|\varphi\|_1 + \|\psi\|_1\|u\|_\infty + \|\psi_2\|_1\left\|D_{0+}^{\alpha-1}u\right\|_\infty \right. \\ &\qquad \left. + \cdots + \|\psi_N\|_1\left\|D_{0+}^{\alpha-(N-1)}u\right\|_\infty\right) \\ &= aA + (a+b)\|\varphi\|_1 + \rho_2\|u\|_\infty + \rho_3\left\|D_{0+}^{\alpha-1}u\right\|_\infty \\ &\quad + \rho_4\left\|D_{0+}^{\alpha-2}u\right\|_\infty + \cdots + \rho_{N+1}\left\|D_{0+}^{\alpha-(N-1)}u\right\|_\infty.\end{aligned} \tag{42}$$

By the definition $\|u\|_X$ and (H_4), it is easy to see that $\|D_{0+}^{\alpha-1}u\|_\infty, \dots, \|D_{0+}^{\alpha-(N-1)}u\|_\infty$ and $\|u\|_\infty$ are bounded. So, Ω_1 is bounded. □

Lemma 10. $\Omega_2 = \{u \in \operatorname{Ker}(L) : Nu \in \operatorname{Im}(L)\}$ *is bounded.*

Proof. Let $u \in \operatorname{Ker}(L)$, so we have $u = c_1 t^{\alpha-1}$, $c_1 \in \mathbb{R}$. For $Nu \in \operatorname{Im}(L) = \operatorname{Ker}(Q)$,

$$\int_0^1 f\left(t, c_1 t^{\alpha-1}, c_1\Gamma(\alpha), \dots, \frac{\Gamma(\alpha)}{\Gamma(N-1)}c_1 t^{N-2}\right)dt = 0. \tag{43}$$

By the integral mean value theorem, there exits a constant $t_1 \in [0,1]$ such that

$$f\left(t_1, c_1 t_1^{\alpha-1}, c_1\Gamma(\alpha), \dots, \frac{\Gamma(\alpha)}{\Gamma(N-1)}c_1 t_1^{N-2}\right) = 0. \tag{44}$$

From (H_2), it follows that $|c_1| \le A/\Gamma(\alpha)$. Hence, Ω_2 is bounded. □

Lemma 11. $\Omega_3 = \{u \in \operatorname{Ker}(L) : \lambda u + (1-\lambda)QNu = 0, \lambda \in [0,1]\}$ *is bounded.*

Proof. Let $u \in \operatorname{Ker}(L)$, so we have $u = c_1 t^{\alpha-1}$, $c_1 \in \mathbb{R}$. If $\lambda = 0$, then $|c_1| \le D$. If $\lambda = 1$, we have $c_1 = 0$.

If $\lambda \ne 0$ and $\lambda \ne 1$, then

$$\lambda c_1 t^{\alpha-1} + (1-\lambda)QN(u) = 0. \tag{45}$$

It follows that

$$\begin{aligned}&\lambda c_1 t^{\beta-1} + (1-\lambda) \\ &\quad \times \int_0^1 f\left(t, c_1 t^{\alpha-1}, c_1\Gamma(\alpha), \dots, \frac{\Gamma(\alpha)}{\Gamma(N-1)}c_1 t^{N-2}\right)dt = 0.\end{aligned} \tag{46}$$

Then we get

$$\lambda c_1^2 t^{\alpha-1} + (1-\lambda)\int_0^1 c_1 f\left(t, c_1 t^{\alpha-1}, \dots, c_2\Gamma(\alpha)\right)dt = 0, \tag{47}$$

which, together with (H_3), implies $|c_1| \le D$. Here, Ω_3 is bounded. □

Remark 12. If the other parts of (H_3) hold, then the set $\Omega_3' = \{u \in \operatorname{Ker}(L) : -\lambda u + (1-\lambda)QNu = (0,0), \lambda \in [0,1]\}$ is bounded.

Theorem 13. *Suppose (H_1)–(H_4) hold; then the problem* (3) *has at least one solution in Y.*

Proof. Let Ω be a bounded open set of Y, such that $\bigcup_{i=1}^3 \overline{\Omega}_i \subset \Omega$. It follows from Lemma 8, N is L-compact on $\overline{\Omega}$. By Lemmas 9, 10, and 11, we get the following:

(1) $Lu \ne \lambda Nu$, for every $u \in [(\operatorname{dom} L \setminus \operatorname{Ker} L)\bigcap \partial\Omega] \times (0,1)$;

(2) $Nu \notin \operatorname{Im} L$ for every $u \in \operatorname{Ker} L \bigcap \partial\Omega$;

(3) let $H(u,\lambda) = \pm\lambda I u + (1-\lambda)JQNu$, where I is the identical operator. Via the homotopy property of degree, we obtain that

$$\begin{aligned}&\deg\left(JQN|_{\ker L}, \Omega \cap \ker L, 0\right) \\ &\quad = \deg\left(H(\cdot,0), \Omega \cap \ker L, 0\right) \\ &\quad = \deg\left(H(\cdot,1), \Omega \cap \ker L, 0\right) \\ &\quad = \deg\left(I, \Omega \cap \ker L, 0\right) = 1 \ne 0.\end{aligned} \tag{48}$$

Applying Theorem 6, we conclude that $Lu = Nu$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. □

Under the stronger conditions imposed on f, we can prove the uniqueness of solutions to the (3) studied above.

Theorem 14. *Suppose the conditions (H_1) in the theorem are replaced by the following conditions.*

$(H_1)'$ *There exist positive constants a_i, $i = 0, 1, \dots, N-1$, such that, for all $(x_1, x_2, \dots, x_N)$, $(y_1, y_2, \dots, y_N) \in \mathbb{R}^N$, one has*

$$\begin{aligned}&\left|f(t, x_1, x_2, \dots, x_N) - f(t, y_1, y_2, \dots, y_N)\right| \\ &\quad \le a_0|x_1 - y_1| + \cdots + a_{N-1}|x_N - y_N|.\end{aligned} \tag{49}$$

$(H_1)''$ *There exist constants l_i, $i = 1, 2, \dots, N-1$, such that for all $(x_1, x_2, \dots, x_N)$, $(y_1, y_2, \dots, y_N) \in \mathbb{R}^N$, one has*

$$\begin{aligned}&\left|f(t, x_1, x_2, \dots, x_N) - f(t, y_1, y_2, \dots, y_N)\right| \\ &\quad \ge -l_0|x_1 - y_1| + l_1|x_2 - y_2| - l_2|x_3 - y_3| \\ &\qquad - \cdots - l_{N-1}|x_N - y_N|.\end{aligned} \tag{50}$$

Then, the BVP (3) *has a unique solution, provided that*

$$\frac{al_0}{l_1} + aa_0 + a_0 c + \sum_{i=2}^{N-1} \frac{al_i}{l_1} + (a+c) \sum_{i=1}^{N-1} a_i < 1. \quad (51)$$

Proof. Let $y_i = 0$, $i = 1, 2, \ldots, N$, and $\varphi_1 = |f(t, 0, \ldots, 0)|$; then the condition (H_1) is satisfied. According to Theorem 13, BVP (3) has at least one solution. Suppose $u_i \in Y$, $i = 1, 2$ are two solutions of (3); then

$$D_{0^+}^{\alpha} u_i(t) = f\left(t, u_i(t), D_{0^+}^{\alpha-1} u_i(t), D_{0^+}^{\alpha-2} u_i(t), \ldots, D_{0^+}^{\alpha-(N-1)} u_i(t)\right), \quad i = 1, 2. \quad (52)$$

Note that $u = u_1 - u_2$, so u satisfy the equation

$$D_{0^+}^{\alpha} u = f\left(t, u_1, D_{0^+}^{\alpha-1} u_1, \ldots, D_{0^+}^{\alpha-(N-1)} u_1\right) - f\left(t, u_2, D_{0^+}^{\alpha-1} u_2, \ldots, D_{0^+}^{\alpha-(N-1)} u_2\right). \quad (53)$$

According to $\mathrm{Im}(L) = \mathrm{Ker}(Q)$, we have

$$\int_0^1 f\left(t, u_1, D_{0^+}^{\alpha-1} u_1, \ldots, D_{0^+}^{\alpha-(N-1)} u_1\right) - f\left(t, u_2, D_{0^+}^{\alpha-1} u_2, \ldots, D_{0^+}^{\alpha-(N-1)} u_2\right) dt = 0. \quad (54)$$

By the integral mean value theorem, there exists $\eta \in [0, 1]$, such that

$$f\left(\eta, u_1(\eta), D_{0^+}^{\alpha-1} u_1(\eta), \ldots, D_{0^+}^{\alpha-(N-1)} u_1(\eta)\right) - f\left(\eta, u_2(\eta), D_{0^+}^{\alpha-1} u_2(\eta), \ldots, D_{0^+}^{\alpha-(N-1)} u_2(\eta)\right) = 0. \quad (55)$$

By $(H_1)''$, we have

$$\begin{aligned} 0 &= \Big| f\left(\eta, u_1(\eta), D_{0^+}^{\alpha-1} u_1(\eta), \ldots, D_{0^+}^{\alpha-(N-1)} u_1(\eta)\right) \\ &\quad - f\left(\eta, u_2(\eta), D_{0^+}^{\alpha-1} u_2(\eta), \ldots, D_{0^+}^{\alpha-(N-1)} u_2(\eta)\right)\Big| \\ &\geq -l_0 |u(\eta)| + l_1 \left|D_{0^+}^{\alpha-1} u(\eta)\right| - l_2 \left|D_{0^+}^{\alpha-2} u(\eta)\right| \\ &\quad - \cdots - l_{N-1} \left|D_{0^+}^{\alpha-(N-1)} u(\eta)\right|. \end{aligned} \quad (56)$$

We can have

$$\begin{aligned} \left|D_{0^+}^{\alpha-1} u(\eta)\right| &\leq \frac{l_0}{l_1} |u(\eta)| + \frac{l_2}{l_1} \left|D_{0^+}^{\alpha-2} u(\eta)\right| \\ &\quad + \cdots + \frac{l_{N-1}}{l_1} \left|D_{0^+}^{\alpha-(N-1)} u(\eta)\right| \\ &\leq \frac{l_0}{l_1} \|u\|_\infty + \sum_{i=2}^{N-1} \frac{l_i}{l_1} \left\|D_{0^+}^{\alpha-i} u\right\|_\infty. \end{aligned} \quad (57)$$

Thus, we can obtain

$$\begin{aligned} \left|D_{0+}^{\alpha-1} u(0)\right| &\leq \left|D_{0+}^{\alpha-1} u(\eta)\right| + \left|\int_0^\eta D_{0+}^{\alpha} u(s)\, ds\right| \\ &\leq \left|D_{0+}^{\alpha-1} u(\eta)\right| + |\eta| \max_{0\leq t\leq \eta} \left|D_{0+}^{\alpha} u(t)\right| \\ &\leq \frac{l_0}{l_1} \|u\|_\infty + \sum_{i=2}^{N-1} \frac{l_i}{l_1} \left\|D_{0^+}^{\alpha-i} u\right\|_\infty + \left\|D_{0+}^{\alpha} u(t)\right\|_\infty \\ &= \frac{l_0}{l_1} \|u\|_\infty + \sum_{i=2}^{N-1} \frac{l_i}{l_1} \left\|D_{0^+}^{\alpha-i} u\right\|_\infty + \|Lu\|_\infty. \end{aligned} \quad (58)$$

According to (25), (38), and (58), we have

$$\begin{aligned} \|u\|_X &= \|Pu + (I-P)u\|_X \leq \|Pu\|_X + \|(I-P)u\|_X \\ &= \frac{al_0}{l_1} \|u\|_\infty + \sum_{i=2}^{N-1} \frac{al_i}{l_1} \left\|D_{0^+}^{\alpha-i} u\right\|_\infty + a\|Lu\|_\infty + c\|Lu\|_\infty; \\ &\leq \frac{al_0}{l_1} \|u\|_\infty + \sum_{i=2}^{N-1} \frac{al_i}{l_1} \left\|D_{0^+}^{\alpha-i} u\right\|_\infty \\ &\quad + (a+c)\left(a_0 \|u\|_\infty + \sum_{i=1}^{N-1} a_i \left\|D_{0^+}^{\beta-i} u\right\|_\infty\right). \end{aligned} \quad (59)$$

From the definition of $\|u\|_X$ and the assumption (51), we have $\|u\| = 0$, so that $u_1 = u_2$. □

4. Example

Let us consider the following boundary value problems:

$$\begin{gathered} D_{0^+}^{2.5} u(t) = \frac{t}{5} + \frac{1}{9} D_{0^+}^{1.5} u(t) + \sin^2\left(D_{0^+}^{0.5} u(t)\right) + \arctan u(t), \quad 0 < t < 1, \\ u(0) = D_{0^+}^{0.5} u(0) = 0, \qquad D_{0^+}^{1.5} u(0) = D_{0^+}^{1.5} u(1). \end{gathered} \quad (60)$$

Corresponding to the problem (3), we have that $\alpha = 2.5$ and

$$f(t, x, y, z) = \frac{t}{5} + \arctan x + \frac{1}{9} y + \sin^2(z). \quad (61)$$

Moreover,

$$|f(t, x, y, z)| \leq \frac{1}{5} + \frac{\pi}{2} + \frac{1}{9} |y| + 1. \quad (62)$$

We can get that the condition (H_1) holds; that is, $\varphi = (12 + 5\pi)/10$, $\psi_1 = \psi_3 = 0$, and $\psi_2 = 1/9$. Taking $A = 25$, $D = 19$, we can calculate that (H_2)–(H_4) hold.

Hence, by Theorem 13, we obtain that (60) has at least one solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors typed, read, and approved the final paper.

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