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Research Article

On a Periodic Solution of the 4-Body Problems

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We study the necessary and sufficient conditions on the masses for the periodic solution of planar 4-body problems, where three particles locate at the vertices of an equilateral triangle and rotate with constant angular velocity about a resting particle. We prove that the above periodic motion is a solution of Newtonian 4-body problems if and only if the resting particle is at the origin and the masses of the other three particles are equal and their angular velocity satisfies a special condition.

1. Introduction and Main Results

Celestial mechanics is related to the study of bodies (mass points) moving under the influence of their mutual gravitational attractions. The many-body problems, that is, N-body problems, are the most important problems in celestial mechanics. N-body problems are nonlinear systems of ordinary differential equations which describe the dynamical law for the motion of N-bodies. The motion equations of the Newtonian N-body problems [1–3] are

$$m_k\ddot{q}_k = \frac{\partial U}{\partial q_k} = \sum_{j\neq k} G m_k m_j \frac{q_j - q_k}{\left|q_j - q_k\right|^3}, \quad k = 1, 2 \cdots N, \tag{1}$$

where $q_k = (x_k, y_k, z_k) \in R^3$ is the position of the kth body with mass m_k, $U(q)$ is the Newtonian potential function

$$U(q) = \sum_{1\leq k<j\leq N} \frac{G m_k m_j}{\left|q_j - q_k\right|}, \tag{2}$$

and G is the gravitational constant which can be taken as one by choosing suitable units.

Definition 1 (see [1–3]). The N-bodies form a central configuration at time t if there exists a scalar $\lambda \in R$ such that

$$\lambda m_k q_k + \frac{\partial U}{\partial q_k} = \lambda m_k q_k + m_k \ddot{q}_k = 0, \quad i = 1, 2 \cdots N, \tag{3}$$

where $\lambda = U/I$ and

$$\sum_{k=1}^{N} m_k q_k = 0, \tag{4}$$

$q_i \neq q_j$ for all $i \neq j$.

A configuration $q = (q_1, \ldots, q_N)$ is a central configuration if and only if

$$\nabla\sqrt{I}U = D\sqrt{I}U = 0, \tag{5}$$

where

$$I = \sum_{k=1}^{N} m_k q_k^2, \qquad U = \sum_{1\leq k<j\leq N} \frac{m_k m_j}{\left|q_j - q_k\right|}. \tag{6}$$

It is well-known that for $N \geq 3$, the general solution of Newtonian N-body problems cannot be given until now, so great importance has been attached to searching for a particular solution from the very beginning. Central configurations have something to do with periodic solutions and collapse orbits and parabolic orbits [1–3]. So finding central configurations becomes very important. In 1772, Lagrange showed that for three masses at the vertices of an equilateral triangle,the orbit rotating about the center of masses with an appropriate angular velocity is a periodic solution of the three-body problems. In 1985, Perko and Walter [4] showed that for $N \geq 4$, N masses at the vertices of a regular polygon rotating about their common center of masses with an appropriate angular velocity describe a periodic solution of the N-body problems if and only if the masses are equal. In 1995 and in 2002, Moeckel and Simo [5] and Zhang and Zhou [6] studied the sufficient and necessary conditions for planar nested $2N$-body problems. In this paper, we study the following problems.

Let three particles locate at the vertices of a unit equilateral triangle; the orbits, describing their rotations with angular velocity ω about the 4th particle which is resting at q_4, are given by

$$q_k(t) - q_4 = (\rho_k - q_4)\exp(i\omega t)\qquad k = 1,2,3, \tag{7}$$

where ρ_k is the kth complex roots of the unit; that is.

$$\rho_k = \exp\left(\frac{i2k\pi}{3}\right),\quad i = \sqrt{-1},\ k = 1,2,3. \tag{8}$$

Theorem 2. *If $\dot{q}_4 = 0$ and $q_k(t) = (\rho_k - q_4)\exp(i\omega t) + q_4$ is a solution of Newtonian N-body problems* (1)*, then $q_4 = 0$ and the center of masses is at the origin. That is, $q = (q_1, q_2, q_3, q_4)$ form a central configuration at any time t.*

Theorem 3. *Let $\dot{q}_4 = 0$ and $q_k(t) = (\rho_k - q_4)\exp(i\omega t) + q_4$ be a solution of Newtonian N-body problems* (1) *if and only if masses (m_1, m_2, m_3) of the three particles (q_1, q_2, q_3) are equal, and the angular velocity ω is*

$$\omega = \sqrt{m_4 + \frac{m_1}{\sqrt{3}}}. \tag{9}$$

2. The Proof of Theorem 2

Differentiating twice in both sides of (7) shows that

$$\ddot{q}_k = -\omega^2(\rho_k - q_4)\exp(i\omega t),\quad k = 1,2,3. \tag{10}$$

Multiplying both sides in (10) by m_k and summing these equations over all $k = 1,2,3$, we have

$$\sum_{k=1}^{3} m_k\ddot{q}_k = -\omega^2\sum_{k=1}^{3} m_k(\rho_k - q_4)\exp(i\omega t). \tag{11}$$

If (7) is a solution of (1), then the left side of the above equation must be zero since the total force of the conservative system is zero, which is shown by (7) as

$$-\omega^2\sum_{k=1}^{3} m_k(\rho_k - q_4)\exp(i\omega t) = -\omega^2\sum_{k=1}^{3} m_k(q_k - q_4) = 0; \tag{12}$$

that is,

$$\sum_{k=1}^{3} m_k(\rho_k - q_4) = \sum_{k=1}^{3} m_k(q_k - q_4) = 0; \tag{13}$$

hence we have

$$q_4 = \frac{1}{M}\sum_{k=1}^{3} m_k q_k = \frac{1}{M}\sum_{k=1}^{3} m_k\rho_k,\qquad M = m_1 + m_2 + m_3. \tag{14}$$

Because of

$$\rho_k = \exp\left(\frac{i2k\pi}{3}\right),\quad i = \sqrt{-1},\ k = 1,2,3, \tag{15}$$

we have

$$q_4 = \frac{1}{M}\left[\left(-\frac{1}{2}m_1 - \frac{1}{2}m_2 + m_3\right) + i\left(\frac{\sqrt{3}}{2}m_1 - \frac{\sqrt{3}}{2}m_2\right)\right]. \tag{16}$$

Since the 4th particle is resting, which means that the force exerted on q_4 by the other particles is zero, then we have

$$\frac{m_1m_4(q_1(t) - q_4)}{|q_1(t) - q_4|^3} + \frac{m_2m_4(q_2(t) - q_4)}{|q_2(t) - q_4|^3} + \frac{m_3m_4(q_3(t) - q_4)}{|q_3(t) - q_4|^3} = 0. \tag{17}$$

Substituting (7) into the above equation and letting

$$d_k = \frac{1}{|q_k(t) - q_4|^3} = \frac{1}{|\rho_k - q_4|^3},\qquad k = 1,2,3, \tag{18}$$

then (17) is equivalent to the following equation:

$$m_1(\rho_1 - q_4)d_1 + m_2(\rho_2 - q_4)d_2 + m_3(\rho_3 - q_4)d_3 = 0. \tag{19}$$

Because

$$\rho_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\qquad \rho_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2},\qquad \rho_3 = 1,$$
$$q_4 = \frac{1}{M}\left[\left(-\frac{1}{2}m_1 - \frac{1}{2}m_2 + m_3\right) + i\left(\frac{\sqrt{3}}{2}m_1 - \frac{\sqrt{3}}{2}m_2\right)\right], \tag{20}$$

then (19) is equivalent to the following linear equations:

$$\begin{aligned} &m_1m_3d_1 + m_2m_3d_2 - m_3(m_1 + m_2)d_3 = 0,\\ &m_1(m_3 + 2m_2)d_1 - m_2(m_3 + 2m_1)d_2 - m_3(m_1 - m_2)d_3 = 0, \end{aligned} \tag{21}$$

where d_1, d_2, d_3 are regarded as unknowns.

It is easy to prove that the rank of the coefficient matrix of the above linear equations (21) is two, which means all the solutions of the linear equations satisfy

$$(d_1, d_2, d_3)^T = C(a_1, a_2, a_3)^T, \quad (22)$$

where C is any real number and $(a_1, a_2, a_3)^T \neq 0$ is one solution for (21).

Because $(1, 1, 1)^T$ is one solution for the linear equations (21), hence we have

$$d_1 = d_2 = d_3, \quad (23)$$

which means

$$\frac{1}{|\rho_1 - q_4|^3} = \frac{1}{|\rho_2 - q_4|^3} = \frac{1}{|\rho_3 - q_4|^3}. \quad (24)$$

Then we get $q_4 = 0$ and the common center of masses

$$\begin{aligned} &m_1 q_1 + m_2 q_2 + m_3 q_3 + m_4 q_4 \\ &\quad = (m_1 + m_2 + m_3 + m_4) q_4 = 0, \end{aligned} \quad (25)$$

and $d_1 = d_2 = d_3 = 1$, and (7) becomes

$$q_k(t) = \rho_k \exp(i\omega t) \quad k = 1, 2, 3, \quad (26)$$

so we have

$$\begin{aligned} \ddot{q}_k(t) = -\omega^2 \rho_k \exp(i\omega t) = -\omega^2 q_k(t) \\ k = 1, 2, 3. \end{aligned} \quad (27)$$

By Definition 1, q_1, q_2, q_3, q_4 form a central configuration at any time and $\omega^2 = \lambda = U/I$.

The proof of Theorem 2 is completed.

3. The Proof of Theorem 3

By Theorem 2, we have

$$q_4 = \frac{1}{M}(m_1\rho_1 + m_2\rho_2 + m_3\rho_3) = 0, \quad (28)$$

and substituting ρ_1, ρ_2, ρ_3 into the above equation, we have

$$\begin{aligned} -\frac{1}{2}m_1 - \frac{1}{2}m_2 + m_3 = 0, \\ \frac{\sqrt{3}}{2}m_1 - \frac{\sqrt{3}}{2}m_2 = 0. \end{aligned} \quad (29)$$

That is

$$m_1 = m_2 = m_3, \quad (30)$$

which means that the masses of the particles at the vertices of the equilateral triangle are equal.

By Theorem 2 and (6), we have

$$\begin{aligned} 3m_1\omega^2 &= I\omega^2 = U \\ &= \sum_{1\le k<j\le 3} \frac{m_1^2}{|q_k - q_j|} + \sum_{k=1}^{3} \frac{m_4 m_1}{|q_4 - q_k|}, \end{aligned} \quad (31)$$

and we can easily get

$$\omega = \sqrt{m_4 + \frac{m_1}{\sqrt{3}}}. \quad (32)$$

The proof of Theorem 3 is completed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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