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Research Article

Marichev-Saigo-Maeda Fractional Integration Operators Involving Generalized Bessel Functions

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Two integral operators involving Appell's functions, or Horn's function in the kernel are considered. Composition of such functions with generalized Bessel functions of the first kind is expressed in terms of generalized Wright function and generalized hypergeometric series. Many special cases, including cosine and sine function, are also discussed.

1. Introduction

Let α, α', β, β', $\gamma \in \mathbb{C}$ and $x > 0$; then the generalized fractional integral operators involving Appell's functions or Horn's function are defined as follows:

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3 \times \left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,dt, \tag{1}$$

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3 \times \left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,dt, \tag{2}$$

with $\mathrm{Re}(\gamma) > 0$. The generalized fractional integral operators of the types (1) and (2) have been introduced by Marichev [1] and later extended and studied by Saigo and Maeda [2]. These operators together are known as the Marichev-Saigo-Maeda operator.

The fractional integral operator has many interesting applications in various subfields in applicable mathematical analysis; for example, [3], it has applications related to a certain class of complex analytic functions. The results given in [4–6] can be referred to for some basic results on fractional calculus.

The purpose of this work is to investigate compositions of integral transforms (1) and (2) with the generalized Bessel function of the first kind $\mathscr{W}_{p,b,c}$ defined for complex $z \in \mathbb{C}$ and $b, c, p \in \mathbb{C}$ by

$$\mathscr{W}_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{\Gamma(\kappa+k)\,k!} \left(\frac{z}{2}\right)^{2k+p}, \tag{3}$$

where $\kappa := p + (b+1)/2$. More details related to the function $\mathscr{W}_{p,b,c}$ and its particular cases can be found in [7, 8] and references therein. It is worth mentioning that $\mathscr{W}_{p,1,1} = J_p$ is Bessel function of order p and $\mathscr{W}_{p,1,-1} = I_p$ is modified Bessel function of order p. Also, $\mathscr{W}_{p,2,1} = 2j_p/\sqrt{\pi}$ is spherical Bessel function of order p and $\mathscr{W}_{p,2,-1} = 2i_p/\sqrt{\pi}$ is modified spherical Bessel function of order p. Thus the study of the integral transform of $\mathscr{W}_{p,b,c}$ will give far reaching results than the result in [9, 10].

The present paper is organized as follows. In Sections 2 and 3, composition of integral transforms (1) and (2) with generalized Bessel function (3) is given in terms of generalized Wright functions and generalized hypergeometric functions, respectively. Special cases like $p=-b/2$ ($p=1-b/2$) of $\mathscr{W}_{p,b,c}$ give the composition of (1) and (2) with cosine and hyperbolic cosine (sine and hyperbolic sine) functions, which are discussed in Section 4. Some concluding remarks and comparison with earlier known work are mentioned in Section 5.

The following two results given by Saigo et al. [2, 11] are needed in sequel.

Lemma 1. *Let* $\alpha,\alpha',\beta,\beta',\gamma,\rho\in\mathbb{C}$ *be such that* $\operatorname{Re}(\gamma)>0$ *and*

$$\operatorname{Re}(\rho)>\max\left\{0,\operatorname{Re}\left(\alpha-\alpha'-\beta-\gamma\right),\operatorname{Re}\left(\alpha'-\beta'\right)\right\}. \quad (4)$$

Then there exists the relation

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right)(x) = \Gamma\left[\begin{matrix}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho+\beta'-\alpha'\\ \rho+\beta',\rho+\gamma-\alpha-\alpha',\rho+\gamma-\alpha'-\beta\end{matrix}\right]x^{\rho-\alpha-\alpha'+\gamma-1}, \quad (5)$$

where

$$\Gamma\left[\begin{matrix}a,b,c\\ d,e,f\end{matrix}\right]=\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}. \quad (6)$$

Lemma 2. *Let* $\alpha,\alpha',\beta,\beta',\gamma,\rho\in\mathbb{C}$ *be such that* $\operatorname{Re}(\gamma)>0$ *and*

$$\operatorname{Re}(\rho) < 1+\min\left\{\operatorname{Re}(-\beta),\operatorname{Re}\left(\alpha+\alpha'-\gamma\right),\operatorname{Re}\left(\alpha+\beta'-\gamma\right)\right\}. \quad (7)$$

Then there exists the relation

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right)(x) = \Gamma\left[\begin{matrix}1-\rho-\gamma+\alpha+\alpha',1-\rho+\alpha+\beta',1-\rho-\beta\\ 1-\rho,1-\rho+\alpha+\alpha'+\beta+\beta'-\gamma,1-\rho+\alpha-\beta\end{matrix}\right] \times x^{\rho-\alpha-\alpha'+\gamma-1}. \quad (8)$$

2. Representations in terms of Generalized Wright Functions

In this section composition of integral transforms (1) and (2) with generalized Bessel function (3) is given in terms of the generalized Wright hypergeometric function ${}_p\psi_q(z)$ which is defined by the series

$${}_p\psi_q(z)={}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\ \middle|\ z\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{z^k}{k!}. \quad (9)$$

Here $a_i,b_j\in\mathbb{C}$, and $\alpha_i,\beta_j\in\mathbb{R}$ ($i=1,2,\ldots,p$; $j=1,2,\ldots,q$). Asymptotic behavior of this function for large values of argument of $z\in\mathbb{C}$ was studied in [12] and under the condition

$$\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>-1 \quad (10)$$

in [13, 14]. Properties of this generalized Wright function were investigated in [9, 15, 16]. In particular, it was proved [15] that ${}_p\psi_q(z)$, $z\in\mathbb{C}$, is an entire function under the condition (10). Interesting results related to generalized Wright functions are also given in [17].

Theorem 3. *Let* $\alpha,\alpha',\beta,\beta',\gamma,\rho,p,b,c\in\mathbb{C}$ *such that* $\kappa\neq 0,-1,-2,\ldots$. *Suppose that* $\operatorname{Re}(\gamma)>0$ *and* $\operatorname{Re}(\rho+p)>\max\{0,\operatorname{Re}(\alpha+\alpha'+\beta-\gamma),\operatorname{Re}(\alpha'-\beta')\}$. *Then*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\mathscr{W}_{p,b,c}(t)\right)(x) = \frac{x^{\rho+p-\alpha-\alpha'+\gamma-1}}{2^p}\times{}_3\psi_4\left[\begin{matrix}(\rho+p,2),\left(\rho+p+\gamma-\alpha-\alpha'-\beta,2\right)\left(\rho+p+\beta'-\alpha',2\right);\\ \left(\rho+p+\beta',2\right),\left(\rho+p+\gamma-\alpha-\alpha',2\right),\\ \left(\rho+p+\gamma-\alpha'-\beta,2\right),(\kappa,1)\end{matrix}\ \middle|\ -\frac{cx^2}{4}\right]. \quad (11)$$

Proof. An application of integral transform (1) to the generalized Bessel function (3) leads to the formula

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\mathscr{W}_{p,b,c}(t)\right)(x) = \left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\sum_{k=0}^{\infty}\frac{(-c)^k(1/2)^{2k+p}}{\Gamma(\kappa+k)\,k!}t^{\rho+p+2k-1}\right)(x). \quad (12)$$

Now changing the order of integration and summation in right-hand side of (12) yields

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\mathscr{W}_{p,b,c}(t)\right)(x) = \sum_{k=0}^{\infty}\frac{(-c)^k(1/2)^{2k+p}}{\Gamma(\kappa+k)\,k!}\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho+p+2k-1}\right)(x). \quad (13)$$

Note that, for all $k = 0, 1, 2, \ldots$,

$$\operatorname{Re}(\rho + p + 2k) \geq \operatorname{Re}(\rho + p) > \max\left\{0, \operatorname{Re}\left(\alpha - \alpha' - \beta - \gamma\right), \operatorname{Re}\left(\alpha' - \beta'\right)\right\}. \tag{14}$$

Replacing ρ by $\rho + p + 2k$ in Lemma 1 and using (5), we obtain

$$\begin{aligned}&\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \mathscr{W}_{p,b,c}(t)\right)(x)\\ &= \frac{x^{\rho+p-\alpha-\alpha'+\gamma-1}}{2^p} \times \sum_{k=0}^{\infty} \Gamma\left[\begin{matrix}\rho + p + 2k, \ \rho + p + \gamma - \alpha - \alpha' - \beta + 2k, \ \rho + p + \beta' - \alpha' + 2k \\ \rho + p + \beta' + 2k, \ \rho + p + \gamma - \alpha - \alpha' + 2k, \ \rho + p + \gamma - \alpha' - \beta + 2k, \ \kappa + k\end{matrix}\right] \frac{1}{k!}\left(-\frac{cx^2}{4}\right)^k.\end{aligned} \tag{15}$$

Interpreting the right-hand side of (15), the equality (11) can be obtained from (6) and then by using the definition of generalized Wright function. □

Theorem 4. *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho, p, b, c \in \mathbb{C}$ *such that* $\kappa \neq 0, -1, -2, \ldots$. *Suppose that* $\operatorname{Re}(\gamma) > 0$ *and* $\operatorname{Re}(\rho - p) < 1 + \min\{\operatorname{Re}(-\beta), \operatorname{Re}(\alpha + \alpha' - \gamma), \operatorname{Re}(\alpha + \beta' - \gamma)\}$. *Then*

$$\begin{aligned}&\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \mathscr{W}_{p,b,c}\left(\frac{1}{t}\right)\right)(x)\\ &= \frac{x^{\rho-p-\alpha-\alpha'+\gamma-1}}{2^p} \times {}_3\Psi_4\left[\begin{matrix}\left(1-\rho+p-\gamma+\alpha+\alpha', 2\right), \left(1-\rho+p+\alpha-\beta'-\gamma, 2\right)\left(1-\rho+p-\beta, 2\right); \\ \left(1-\rho+p, 2\right), \left(1-\rho+p-\gamma+\alpha+\alpha'+\beta', 2\right), \left(1-\rho+p+\alpha-\beta, 2\right), (\kappa, k)\end{matrix} \,\middle|\, -\frac{c}{4x^2}\right].\end{aligned} \tag{16}$$

Proof. Using (2) and (3) and then changing the order of integration and summation, which is justified under the conditions with Theorem 4, yield

$$\begin{aligned}&\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \mathscr{W}_{p,b,c}\left(\frac{1}{t}\right)\right)(x)\\ &= \sum_{k=0}^{\infty} \frac{(-c)^k (1/2)^{2k+p}}{\Gamma(\kappa+k)\, k!} \left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-p-2k-1}\right)(x).\end{aligned} \tag{17}$$

Note that, for all $k = 0, 1, 2, \ldots$,

$$\begin{aligned}&\operatorname{Re}(\rho - p - 2k)\\ &\quad \leq \operatorname{Re}(\rho - p)\\ &\quad < 1 + \min\left\{\operatorname{Re}(-\beta), \operatorname{Re}\left(\alpha + \alpha' - \gamma\right), \operatorname{Re}\left(\alpha + \beta' - \gamma\right)\right\}.\end{aligned} \tag{18}$$

Hence replacing ρ by $\rho - p - 2k$ in Lemma 2 and using (8), we obtain

$$\begin{aligned}&\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \mathscr{W}_{p,b,c}\left(\frac{1}{t}\right)\right)(x)\\ &= \frac{x^{\rho-p-\alpha-\alpha'+\gamma-1}}{2^p} \times \sum_{k=0}^{\infty} \Gamma\left[\begin{matrix}1-\rho+p-\gamma+\alpha+\alpha'+2k, 1-\rho+p+\alpha+\beta'-\gamma+2k, 1-\rho+p-\beta+2k \\ 1-\rho+p+2k, 1-\rho+p-\gamma+\alpha+\alpha'+\beta'+2k, 1-\rho+p+\alpha-\beta+2k\end{matrix}\right] \frac{1}{k!}\left(-\frac{c}{4x^2}\right)^k.\end{aligned} \tag{19}$$

Now (6), (9), and (19) together imply that

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \mathscr{W}_{p,b,c}\left(\frac{1}{t}\right)\right)(x) = \frac{x^{\rho-p-\alpha-\alpha'+\gamma-1}}{2^p} \times {}_3\Psi_4\left[\begin{matrix}\left(1-\rho+p-\gamma+\alpha+\alpha',2\right),\left(1-\rho+p+\alpha-\beta'-\gamma,2\right)\left(1-\rho+p-\beta,2\right); \\ \left(1-\rho+p,2\right),\left(1-\rho+p-\gamma+\alpha+\alpha'+\beta',2\right),\left(1-\rho+p+\alpha-\beta,2\right),\left(p,k\right)\end{matrix} \Big| -\frac{c}{4x^2}\right], \tag{20}$$

and this completes the proof. □

3. Representation in terms of Generalized Hypergeometric Series

The generalized hypergeometric function ${}_pF_q(a_1,\ldots,a_p;c_1,\ldots,c_q;z)$ is given by the representation

$$ {}_pF_q\left(a_1,\ldots,a_p;c_1,\ldots,c_q;z\right) = \sum_{k=0}^{\infty} \frac{\left(a_1\right)_k\cdots\left(a_p\right)_k}{\left(c_1\right)_k\cdots\left(c_q\right)_k}\frac{z^k}{k!}, \quad (z\in\mathbb{C}), \tag{21}$$

where none of the denominator parameters is zero or a negative integer. Here p or q are allowed to be zero. The series (21) is convergent for all finite z if $p \le q$, while, for $p = q+1$, it is convergent for $|z|<1$ and divergent for $|z|>1$.

Results obtained in this section demonstrate the image formula for the generalized Bessel functions $\mathscr{W}_{p,b,c}$ under the operators (1) and (2) in terms of generalized hypergeometric functions. The well-known Legendre duplication formulas [18] given by

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \qquad (z)_{2k} = 2^{2k}\left(\frac{z}{2}\right)_k\left(\frac{z+1}{2}\right)_k, \quad \left(k\in\mathbb{N}_0\right), \tag{22}$$

are required for this purpose.

Theorem 5. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho,p,b,c\in\mathbb{C}$ such that $\kappa \neq -1,-2,\ldots$. Suppose that $\mathrm{Re}(\gamma) > 0$ and $\mathrm{Re}(\rho+p) > \max\{0,\mathrm{Re}(\alpha-\alpha'-\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\}$. Then the following formula holds:*

$$\begin{aligned}&\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\mathscr{W}_{p,b,c}(t)\right)(x) \\ &= \frac{x^{\rho+p-1}}{2^p}\frac{\Gamma(\rho+p)\Gamma(\rho+p+\gamma-\alpha-\alpha'-\beta)}{\Gamma(\rho+p+\beta')\Gamma(\rho+p+\gamma-\alpha-\alpha')\Gamma(\kappa)}\frac{\Gamma(\rho+p+\beta'-\alpha')}{\Gamma(\rho+p+\gamma-\alpha'-\beta)} \\ &\times {}_6F_7\left[\begin{matrix}\frac{\rho+p}{2},\frac{\rho+p+1}{2},\frac{\rho+p+\gamma-\alpha-\alpha'-\beta}{2},\frac{\rho+p-\alpha-\alpha'+\beta+1}{2},\frac{\rho+p+\beta'-\alpha'}{2},\frac{\rho+p+\beta'-\alpha'+1}{2}; \\ \kappa,\frac{\rho+p+\beta'}{2},\frac{\rho+p+\beta'+1}{2},\frac{\rho+p+\gamma-\alpha-\alpha'}{2},\frac{\rho+p+\gamma-\alpha-\alpha'+1}{2}, \\ \frac{\rho+p+\gamma-\alpha'-\beta}{2},\frac{\rho+p+\gamma-\alpha'-\beta+1}{2}\end{matrix}\Big| -\frac{cx^2}{4}\right].\end{aligned} \tag{23}$$

Proof. It is known that $\Gamma(z+k)=\Gamma(z)(z)_k$. Thus

$$\begin{aligned}&\Gamma\left[\begin{matrix}\rho+p+2k,\ \rho+p+\gamma-\alpha-\alpha'-\beta+2k,\ \rho+p+\beta'-\alpha'+2k \\ \rho+p+\beta'+2k,\ \rho+p+\gamma-\alpha-\alpha'+2k,\ \rho+p+\gamma-\alpha'-\beta+2k,\ \kappa+k\end{matrix}\right] \\ &= \frac{\Gamma(\rho+p)\Gamma\left(\rho+p+\gamma-\alpha-\alpha'-\beta\right)\Gamma\left(\rho+p+\beta'-\alpha'\right)}{\Gamma(\rho+p+\beta')\Gamma(\rho+p+\gamma-\alpha-\alpha')\Gamma(\rho+p+\gamma-\alpha'-\beta)\Gamma(\kappa)} \\ &\times \frac{(\rho+p)_{2k}\left(\rho+p+\gamma-\alpha-\alpha'-\beta\right)_{2k}\left(\rho+p+\beta'-\alpha'\right)_{2k}}{(\rho+p+\beta')_{2k}(\rho+p+\gamma-\alpha-\alpha')_{2k}(\rho+p+\gamma-\alpha'-\beta)_{2k}(\kappa)_k}.\end{aligned} \tag{24}$$

Now apply (3.3) on the right-hand side of the above equation and then the result follows from (15). This completes the proof. □

By adopting a similar method the next result can be obtained from (19); we omit the details.

Theorem 6. *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho, p, b, c \in \mathbb{C}$ *such that* $\kappa \neq 0, -1, -2, \ldots$. *Suppose that* $\mathrm{Re}(\gamma) > 0$ *and* $\mathrm{Re}(\rho - p) < 1 + \min\{\mathrm{Re}(-\beta), \mathrm{Re}(\alpha + \alpha' - \gamma), \mathrm{Re}(\alpha + \beta' - \gamma)\}$. *Then*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \mathscr{W}_{p,b,c}\left(\frac{1}{t}\right)\right)(x)$$
$$= \frac{x^{\rho-p-\alpha-\alpha'+\gamma-1}}{2^p} \frac{\Gamma\left(\alpha+\alpha'+p-\gamma-\rho+1\right)\Gamma\left(\alpha+\beta'-\gamma+p-\rho+1\right)\Gamma\left(-\beta+p-\rho+1\right)}{\Gamma\left(p-\rho+1\right)\Gamma\left(\kappa\right)\Gamma\left(\alpha+\alpha'+\beta'+p-\gamma-\rho+1\right)\Gamma\left(\alpha-\beta+p-\rho+1\right)}$$
$$\times {}_6F_7\left[\begin{matrix} \frac{\alpha+\alpha'+p-\gamma-\rho+1}{2}, \frac{\alpha+\alpha'+p+\gamma-\rho+2}{2}, \frac{\alpha+\beta'+p-\gamma-\rho+1}{2}, \frac{\alpha+\beta'+p-\gamma-\rho+2}{2}, \\ \frac{-\beta+p-\rho+1}{2}, \frac{-\beta+p-\rho+2}{2}; \\ \kappa, \frac{p-\rho+1}{2}, \frac{p-\rho+2}{2}, \frac{\alpha+\alpha'+\beta'+p-\gamma-\rho+1}{2}, \frac{\alpha+\alpha'+\beta'+p-\gamma-\rho+2}{2}, \\ \frac{\alpha-\beta+p-\rho+1}{2}, \frac{\alpha-\beta+p-\rho+2}{2} \end{matrix} \middle| -\frac{c}{4x^2}\right]. \tag{25}$$

4. Fractional Integration of Trigonometric Functions

4.1. Cosine and Hyperbolic Cosine Functions. For all $b \in \mathbb{C}$, if $p = -b/2$, then the generalized Bessel function $\mathscr{W}_{p,b,c}(z)$ has the form

$$\mathscr{W}_{-b/2,b,c^2}(z) = \left(\frac{2}{z}\right)^{b/2} \frac{\cos cz}{\sqrt{\pi}},$$
$$\mathscr{W}_{-b/2,b,-c^2}(z) = \left(\frac{2}{z}\right)^{b/2} \frac{\cosh cz}{\sqrt{\pi}}. \tag{26}$$

Hence the following results are a consequence of Theorems 3 and 4, respectively.

Corollary 7. *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho, c \in \mathbb{C}$ *such that* $\mathrm{Re}(\gamma) > 0$ *and*

$$\mathrm{Re}(\rho) > \max\left\{0, \mathrm{Re}\left(\alpha - \alpha' - \beta - \gamma\right), \mathrm{Re}\left(\alpha' - \beta'\right)\right\}. \tag{27}$$

Then

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \cos(ct)\right)(x)$$
$$= \pi^{1/2} x^{\rho-\alpha-\alpha'+\gamma-1} \times {}_3\Psi_4\left[\begin{matrix} (\rho,2), (\rho+\gamma-\alpha-\alpha'-\beta,2), (\rho+\beta'-\alpha',2); \\ (\rho+\beta',2), (\rho+\gamma-\alpha-\alpha',2), (\rho+\gamma-\alpha'-\beta,2), \left(\frac{1}{2},1\right) \end{matrix} \middle| -\frac{c^2x^2}{4}\right],$$
$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \cosh(ct)\right)(x)$$
$$= \pi^{1/2} x^{\rho-\alpha-\alpha'+\gamma-1} \times {}_3\Psi_4\left[\begin{matrix} (\rho,2), (\rho+\gamma-\alpha-\alpha'-\beta,2), (\rho+\beta'-\alpha',2); \\ (\rho+\beta',2), (\rho+\gamma-\alpha-\alpha',2), (\rho+\gamma-\alpha'-\beta,2), \left(\frac{1}{2},1\right) \end{matrix} \middle| \frac{c^2x^2}{4}\right]. \tag{28}$$

Proof. On setting $p = -b/2$ and replacing c by c^2 into (11) and using (37), we have

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \left(\frac{2}{t}\right)^{b/2} \frac{\cos(ct)}{\sqrt{\pi}}\right)(x)$$
$$= \frac{x^{\rho-(b/2)-\alpha-\alpha'+\gamma-1}}{2^{-b/2}} \times {}_3\Psi_4 \left[\begin{matrix} \left(\rho-\frac{b}{2},2\right),\left(\rho-\frac{b}{2}+\gamma-\alpha-\alpha'-\beta,2\right)\left(\rho-\frac{b}{2}+\beta'-\alpha',2\right); \\ \left(\rho-\frac{b}{2}+\beta',2\right),\left(\rho-\frac{b}{2}+\gamma-\alpha-\alpha',2\right),\left(\rho-\frac{b}{2}+\gamma-\alpha'-\beta,2\right),\left(\frac{1}{2},1\right) \end{matrix} \mid -\frac{cx^2}{4} \right]. \tag{29}$$

This implies that

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-(b/2)-1} \cos ct\right)(x)$$
$$= \pi^{1/2} x^{\rho-(b/2)-\alpha-\alpha'+\gamma-1} \times {}_3\Psi_4 \left[\begin{matrix} \left(\rho-\frac{b}{2},2\right),\left(\rho-\frac{b}{2}+\gamma-\alpha-\alpha'-\beta,2\right)\left(\rho-\frac{b}{2}+\beta'-\alpha',2\right); \\ \left(\rho-\frac{b}{2}+\beta',2\right),\left(\rho-\frac{b}{2}+\gamma-\alpha-\alpha',2\right),\left(\rho-\frac{b}{2}+\gamma-\alpha'-\beta,2\right),\left(\frac{1}{2},1\right) \end{matrix} \mid -\frac{cx^2}{4} \right]. \tag{30}$$

The identity (39) follows from (30) by replacing ρ by $\rho+(b/2)$.

Similarly, the identity (4.7) can be obtained from (11) by setting $p = -b/2$ and replacing c by $-c^2$. □

Corollary 8. *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ be such that* $\mathrm{Re}(\gamma) > 0$, *and*

$$\mathrm{Re}(\rho) < \min\left\{\mathrm{Re}(-\beta), \mathrm{Re}\left(\alpha+\alpha'-\gamma\right), \mathrm{Re}\left(\alpha+\beta'-\gamma\right)\right\}. \tag{31}$$

Then

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \cos\left(\frac{c}{t}\right)\right)(x)$$
$$= \pi^{1/2} x^{\rho-\alpha-\alpha'+\gamma} \times {}_3\Psi_4 \left[\begin{matrix} \left(\rho-\gamma+\alpha+\alpha',2\right),\left(-\rho+\alpha+\beta'-\gamma,2\right),\left(-\rho-\beta,2\right); \\ \left(-\rho,2\right),\left(-\rho-\gamma+\alpha+\alpha'+\beta',2\right),\left(-\rho+\alpha-\beta,2\right),\left(\frac{1}{2},1\right) \end{matrix} \mid -\frac{c^2}{4x^2} \right],$$
$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho} \cosh\left(\frac{c}{t}\right)\right)(x)$$
$$= \pi^{1/2} x^{\rho-\alpha-\alpha'+\gamma} \times {}_3\Psi_4 \left[\begin{matrix} \left(\rho-\gamma+\alpha+\alpha',2\right),\left(-\rho+\alpha+\beta'-\gamma,2\right),\left(-\rho-\beta,2\right); \\ \left(-\rho,2\right),\left(-\rho-\gamma+\alpha+\alpha'+\beta',2\right),\left(-\rho+\alpha-\beta,2\right),\left(\frac{1}{2},1\right) \end{matrix} \mid \frac{c^2}{4x^2} \right]. \tag{32}$$

The next statements show that the image formulas for cosine and hyperbolic cosine under Saigo-Maeda fractional integral operators can also be represented in terms of the generalized hypergeometric series. This result follows from Theorems 5 and 6 with taking $p = -b/2$ and replacing c by c^2 or $-c^2$, respectively.

Corollary 9. *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho, c \in \mathbb{C}$. Suppose that* $\mathrm{Re}(\gamma) > 0$ *and*

$$\mathrm{Re}(\rho) > \max\left\{0, \mathrm{Re}\left(\alpha-\alpha'-\beta-\gamma\right), \mathrm{Re}\left(\alpha'-\beta'\right)\right\}. \tag{33}$$

Then the following formula holds:

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\cos(ct)\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)} x^{\rho-\alpha-\alpha'+\gamma-1} \times {}_6F_7\left[\begin{matrix} \frac{\rho}{2}, \frac{\rho+1}{2}, \frac{\rho+\gamma-\alpha-\alpha'-\beta}{2}, \frac{\rho+\gamma-\alpha-\alpha'+\beta+1}{2}, \frac{\rho+\beta'-\alpha'}{2}, \frac{\rho+\beta'-\alpha'+1}{2}; \\ \frac{1}{2}, \frac{\rho+\beta'}{2}, \frac{\rho+\beta'+1}{2}, \frac{\rho+\gamma-\alpha-\alpha'}{2}, \frac{\rho+\gamma-\alpha-\alpha'+1}{2}, \frac{\rho+\gamma-\alpha'-\beta}{2}, \frac{\rho+\gamma-\alpha'-\beta+1}{2} \end{matrix} \mid -\frac{c^2x^2}{4}\right],$$

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\cosh(ct)\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)} x^{\rho-\alpha-\alpha'+\gamma-1} \times {}_6F_7\left[\begin{matrix} \frac{\rho}{2}, \frac{\rho+1}{2}, \frac{\rho+\gamma-\alpha-\alpha'-\beta}{2}, \frac{\rho+\gamma-\alpha-\alpha'+\beta+1}{2}, \frac{\rho+\beta'-\alpha'}{2}, \frac{\rho+\beta'-\alpha'+1}{2}; \\ \frac{1}{2}, \frac{\rho+\beta'}{2}, \frac{\rho+\beta'+1}{2}, \frac{\rho+\gamma-\alpha-\alpha'}{2}, \frac{\rho+\gamma-\alpha-\alpha'+1}{2}, \frac{\rho+\gamma-\alpha'-\beta}{2}, \frac{\rho+\gamma-\alpha'-\beta+1}{2} \end{matrix} \mid \frac{c^2x^2}{4}\right]. \quad (34)$$

Corollary 10. *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho, c \in \mathbb{C}$. *Suppose that* $\mathrm{Re}(\gamma) > 0$ *and*

$$\mathrm{Re}(\rho) < 1 + \min\left\{\mathrm{Re}(-\beta), \mathrm{Re}(\alpha+\alpha'-\gamma), \mathrm{Re}(\alpha+\beta'-\gamma)\right\}. \quad (35)$$

Then

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho}\cos\left(\frac{c}{t}\right)\right)(x) = \frac{\Gamma(\alpha+\alpha'-\gamma-\rho)\Gamma(\alpha+\beta'-\gamma-\rho)\Gamma(-\beta-\rho)}{\Gamma(-\rho)\Gamma(\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(\alpha-\beta-\rho)} x^{\rho-\alpha-\alpha'+\gamma} \times {}_6F_7\left[\begin{matrix} \frac{\alpha+\alpha'-\rho}{2}, \frac{\alpha+\alpha'-\gamma-\rho+1}{2}, \frac{\alpha+\beta'-\gamma-\rho}{2}, \frac{\alpha+\beta'-\gamma-\rho+1}{2}, \frac{-\beta-\rho}{2}, \frac{-\beta-\rho+1}{2}; \\ \frac{1}{2}, -\frac{\rho}{2}, -\frac{\rho+1}{2}, \frac{\alpha+\alpha'+\beta'-\gamma-\rho}{2}, \frac{\alpha+\alpha'+\beta'-\gamma-\rho+1}{2}, \frac{\alpha-\beta-\rho}{2}, \frac{\alpha-\beta-\rho+1}{2}, \end{matrix} \mid -\frac{c^2}{4x^2}\right], \quad (36)$$

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho}\cosh\left(\frac{c}{t}\right)\right)(x) = \frac{\Gamma(\alpha+\alpha'-\gamma-\rho)\Gamma(\alpha+\beta'-\gamma-\rho)}{\Gamma(-\rho)\Gamma(\alpha+\alpha'+\beta'-\gamma-\rho)} \frac{\Gamma(-\beta-\rho)}{\Gamma(\alpha-\beta-\rho)} x^{\rho-\alpha-\alpha'+\gamma} \times {}_6F_7\left[\begin{matrix} \frac{\alpha+\alpha'-\rho}{2}, \frac{\alpha+\alpha'-\gamma-\rho+1}{2}, \frac{\alpha+\beta'-\gamma-\rho}{2}, \frac{\alpha+\beta'-\gamma-\rho+1}{2}, \frac{-\beta-\rho}{2}, \frac{-\beta-\rho+1}{2}; \\ \frac{1}{2}, -\frac{\rho}{2}, -\frac{\rho+1}{2}, \frac{\alpha+\alpha'+\beta'-\gamma-\rho}{2}, \frac{\alpha+\alpha'+\beta'-\gamma-\rho+1}{2}, \frac{\alpha-\beta-\rho}{2}, \frac{\alpha-\beta-\rho+1}{2} \end{matrix} \mid \frac{c^2}{4x^2}\right].$$

4.2. Sine and Hyperbolic Sine Functions. For all $b \in \mathbb{C}$, if $p = 1-b/2$, then the generalized Bessel function $\mathscr{W}_{p,b,c}(z)$ has the form

$$\mathscr{W}_{1-(b/2),b,c^2}(z) = \left(\frac{2}{z}\right)^{b/2} \frac{\sin(cz)}{\sqrt{\pi}}, \qquad \mathscr{W}_{1-(b/2),b,-c^2}(z) = \left(\frac{2}{z}\right)^{b/2} \frac{\sinh(cz)}{\sqrt{\pi}}. \tag{37}$$

Thus, the composition of Saigo-Maeda fractional integral operators with sine and hyperbolic sine functions can be obtained from Theorems 3 and 4, respectively.

Corollary 11. *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho, c \in \mathbb{C}$ *such that* $\mathrm{Re}(\gamma) > 0$ *and*

$$\mathrm{Re}(\rho) > \max\left\{0, \mathrm{Re}\left(\alpha - \alpha' - \beta - \gamma\right), \mathrm{Re}\left(\alpha' - \beta'\right)\right\}. \tag{38}$$

Then

$$\begin{aligned}
&\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \sin(ct)\right)(x) \\
&= \pi^{1/2} x^{\rho-\alpha-\alpha'+\gamma-1} \times {}_3\Psi_4\left[\begin{matrix} (\rho,2),\left(\rho+\gamma-\alpha-\alpha'-\beta,2\right),\left(\rho+\beta'-\alpha',2\right); \\ \left(\rho+\beta',2\right),\left(\rho+\gamma-\alpha-\alpha',2\right),\left(\rho+\gamma-\alpha'-\beta,2\right),\left(\frac{3}{2},1\right) \end{matrix} \mid -\frac{c^2x^2}{4}\right], \\
&\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \sinh(ct)\right)(x) \\
&= \pi^{1/2} x^{\rho-\alpha-\alpha'+\gamma-1} \times {}_3\Psi_4\left[\begin{matrix} (\rho,2),\left(\rho+\gamma-\alpha-\alpha'-\beta,2\right),\left(\rho+\beta'-\alpha',2\right); \\ \left(\rho+\beta',2\right),\left(\rho+\gamma-\alpha-\alpha',2\right),\left(\rho+\gamma-\alpha'-\beta,2\right),\left(\frac{3}{2},1\right) \end{matrix} \mid \frac{c^2x^2}{4}\right].
\end{aligned} \tag{39}$$

The next result follows from Theorem 4 by setting $p = -b/2$ and replacing c by c^2 or $-c^2$, respectively.

Corollary 12. *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ *be such that* $\mathrm{Re}(\gamma) > 0$ *and*

$$\mathrm{Re}(\rho) < \min\left\{\mathrm{Re}(-\beta), \mathrm{Re}\left(\alpha+\alpha'-\gamma\right), \mathrm{Re}\left(\alpha+\beta'-\gamma\right)\right\}. \tag{40}$$

Then

$$\begin{aligned}
&\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} \sin\left(\frac{c}{t}\right)\right)(x) \\
&= \pi^{1/2} x^{\rho-\alpha-\alpha'+\gamma} \times {}_3\Psi_4\left[\begin{matrix} \left(\rho-\gamma+\alpha+\alpha',2\right),\left(-\rho+\alpha+\beta'-\gamma,2\right),\left(-\rho-\beta,2\right); \\ (-\rho,2),\left(-\rho-\gamma+\alpha+\alpha'+\beta',2\right),\left(-\rho+\alpha-\beta,2\right),\left(\frac{3}{2},1\right) \end{matrix} \mid -\frac{c^2}{4x^2}\right], \\
&\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho} \sinh\left(\frac{c}{t}\right)\right)(x) \\
&= \pi^{1/2} x^{\rho-\alpha-\alpha'+\gamma} \times {}_3\Psi_4\left[\begin{matrix} \left(\rho-\gamma+\alpha+\alpha',2\right),\left(-\rho+\alpha+\beta'-\gamma,2\right),\left(-\rho-\beta,2\right); \\ (-\rho,2),\left(-\rho-\gamma+\alpha+\alpha'+\beta',2\right),\left(-\rho+\alpha-\beta,2\right),\left(\frac{3}{2},1\right) \end{matrix} \mid \frac{c^2}{4x^2}\right].
\end{aligned} \tag{41}$$

The following result can be obtained from Theorems 5 and 6 with taking $p = -b/2$ and replacing c by c^2 or $-c^2$, respectively.

Corollary 13. *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho, c \in \mathbb{C}$. *Suppose that* $\mathrm{Re}(\gamma) > 0$ *and*

$$\mathrm{Re}(\rho) > \max\left\{0, \mathrm{Re}\left(\alpha - \alpha' - \beta - \gamma\right), \mathrm{Re}\left(\alpha' - \beta'\right)\right\}. \tag{42}$$

Then the following formula holds:

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\sin(ct)\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)} x^{\rho-\alpha-\alpha'+\gamma-1} \times {}_6F_7\left[\begin{matrix}\frac{\rho}{2},\frac{\rho+1}{2},\frac{\rho+\gamma-\alpha-\alpha'-\beta}{2},\frac{\rho+\gamma-\alpha-\alpha'+\beta+1}{2},\frac{\rho+\beta'-\alpha'}{2},\frac{\rho+\beta'-\alpha'+1}{2};\\ \frac{3}{2},\frac{\rho+\beta'}{2},\frac{\rho+\beta'+1}{2},\frac{\rho+\gamma-\alpha-\alpha'}{2},\frac{\rho+\gamma-\alpha-\alpha'+1}{2},\frac{\rho+\gamma-\alpha'-\beta}{2},\frac{\rho+\gamma-\alpha'-\beta+1}{2}\end{matrix} \mid -\frac{c^2x^2}{4}\right],$$

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\sinh(ct)\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\beta')\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)} x^{\rho-\alpha-\alpha'+\gamma-1} \times {}_6F_7\left[\begin{matrix}\frac{\rho}{2},\frac{\rho+1}{2},\frac{\rho+\gamma-\alpha-\alpha'-\beta}{2},\frac{\rho+\gamma-\alpha-\alpha'+\beta+1}{2},\frac{\rho+\beta'-\alpha'}{2},\frac{\rho+\beta'-\alpha'+1}{2};\\ \frac{3}{2},\frac{\rho+\beta'}{2},\frac{\rho+\beta'+1}{2},\frac{\rho+\gamma-\alpha-\alpha'}{2},\frac{\rho+\gamma-\alpha-\alpha'+1}{2},\frac{\rho+\gamma-\alpha'-\beta}{2},\frac{\rho+\gamma-\alpha'-\beta+1}{2}\end{matrix} \mid \frac{c^2x^2}{4}\right]. \tag{43}$$

Corollary 14. *Let* $\alpha,\alpha',\beta,\beta',\gamma,\rho,c \in \mathbb{C}$. *Suppose that* $\mathrm{Re}(\gamma) > 0$ *and* $\mathrm{Re}(\rho) < 1+\min\{\mathrm{Re}(-\beta),\mathrm{Re}(\alpha+\alpha'-\gamma),\mathrm{Re}(\alpha+\beta'-\gamma)\}$. *Then*

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho}\sin\left(\frac{c}{t}\right)\right)(x) = \frac{\Gamma(\alpha+\alpha'-\gamma-\rho)\Gamma(\alpha+\beta'-\gamma-\rho)}{\Gamma(-\rho)\Gamma(\alpha+\alpha'+\beta'-\gamma-\rho)}\frac{\Gamma(-\beta-\rho)}{\Gamma(\alpha-\beta-\rho)} x^{\rho-\alpha-\alpha'+\gamma} \times {}_6F_7\left[\begin{matrix}\frac{\alpha+\alpha'-\rho}{2},\frac{\alpha+\alpha'-\gamma-\rho+1}{2},\frac{\alpha+\beta'-\gamma-\rho}{2},\frac{\alpha+\beta'-\gamma-\rho+1}{2},\frac{-\beta-\rho}{2},\frac{-\beta-\rho+1}{2};\\ \frac{3}{2},-\frac{\rho}{2},-\frac{\rho+1}{2},\frac{\alpha+\alpha'+\beta'-\gamma-\rho}{2},\frac{\alpha+\alpha'+\beta'-\gamma-\rho+1}{2},\frac{\alpha-\beta-\rho}{2},\frac{\alpha-\beta-\rho+1}{2},\end{matrix} \mid -\frac{c^2}{4x^2}\right],$$

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho}\sinh\left(\frac{c}{t}\right)\right)(x) = \frac{\Gamma(\alpha+\alpha'-\gamma-\rho)\Gamma(\alpha+\beta'-\gamma-\rho)}{\Gamma(-\rho)\Gamma(\alpha+\alpha'+\beta'-\gamma-\rho)}\frac{\Gamma(-\beta-\rho)}{\Gamma(\alpha-\beta-\rho)} x^{\rho-\alpha-\alpha'+\gamma} \times {}_6F_7\left[\begin{matrix}\frac{\alpha+\alpha'-\rho}{2},\frac{\alpha+\alpha'-\gamma-\rho+1}{2},\frac{\alpha+\beta'-\gamma-\rho}{2},\frac{\alpha+\beta'-\gamma-\rho+1}{2},\frac{-\beta-\rho}{2},\frac{-\beta-\rho+1}{2};\\ \frac{3}{2},-\frac{\rho}{2},-\frac{\rho+1}{2},\frac{\alpha+\alpha'+\beta'-\gamma-\rho}{2},\frac{\alpha+\alpha'+\beta'-\gamma-\rho+1}{2},\frac{\alpha-\beta-\rho}{2},\frac{\alpha-\beta-\rho+1}{2}\end{matrix} \mid \frac{c^2}{4x^2}\right]. \tag{44}$$

5. Concluding Observations

In this section some consequences of the main result derived in previous sections are given in detail. Also comparison with other known results from the literature is listed.

(1) We remark that all the results given by Purohit et al. [10] are followed from the results derived in this paper by setting $b = 1 = c$.

(2) The results in Sections 2 and 3 also provide the Marichev-Saigo-Maeda fractional integration of modified Bessel function and spherical Bessel functions.

(3) Set $\alpha' = 0$ in the operators (1) and (2). Then due to identities given by Saxena and Saigo [11, P. 93], it follows that

$$\left(I_{0,+}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{0,x}^{\gamma,\alpha-\gamma,-\beta} f\right)(x)$$
$$\left(I_{0,-}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta} f\right)(x). \quad (45)$$

The generalized integral transforms that appear in the right-hand side of the above equations are due to Saigo [19] and defined as follows:

$$\left(I_{0_+}^{\alpha,\beta,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1 \times \left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)\, dt,$$
$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1 \times \left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)\, dt, \quad (46)$$

where $\Gamma(\alpha)$ is the Euler gamma function [20] and ${}_2F_1(a, b; c; x)$ is the Gauss hypergeometric function.

The above fact helps us to conclude that all the results given in [9, 21] can also be obtained from the results in this paper by setting $\alpha' = 0$.

(4) Note that Riemann-Liouville and Weyl and Erdélyi-Kober fractional calculus [22] are special case of Saigo's operator (46). Thus this paper is also useful to derive certain composition formula involving Riemann-Liouville and Weyl and Erdléyi-Kober fractional calculus and Bessel, modified Bessel, and spherical Bessel function of the first kind.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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