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Research Article

Self-Consistent Sources and Conservation Laws for a Super Broer-Kaup-Kupershmidt Equation Hierarchy

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Based on the matrix Lie superalgebras and supertrace identity, the integrable super Broer-Kaup-Kupershmidt hierarchy with self-consistent sources is established. Furthermore, we establish the infinitely many conservation laws for the integrable super Broer-Kaup-Kupershmidt hierarchy. In the process of computation especially, Fermi variables also play an important role in super integrable systems.

1. Introduction

Soliton theory has achieved great success during the last decades; it is being applied to mathematics, physics, biology, astrophysics, and other potential fields [1–12]. The diversity and complexity of soliton theory enable investigators to do research from different views, such as Hamiltonian structure, self-consistent sources, conservation laws, and various solutions of soliton equations.

In recent years, with the development of integrable systems, super integrable systems have attracted much attention. Many scholars and experts do research on the topic and get lots of results. For example, in [13], Ma et al. gave the supertrace identity based on Lie super algebras and its application to super AKNS hierarchy and super Dirac hierarchy, and to get their super Hamiltonian structures, Hu gave an approach to generate superextensions of integrable systems [14]. Afterwards, super Boussinesq hierarchy [15] and super NLS-mKdV hierarchy [16] as well as their super Hamiltonian structures are presented. The binary nonlinearization of the super classical Boussinesq hierarchy [17], the Bargmann symmetry constraint, and binary nonlinearization of the super Dirac systems were given [18].

Soliton equation with self-consistent sources is an important part in soliton theory. They are usually used to describe interactions between different solitary waves, and they are also relevant to some problems related to hydrodynamics, solid state physics, plasma physics, and so forth. Some results have been obtained by some authors [19–21]. Very recently, self-consistent sources for super CKdV equation hierarchy [22] and super G-J hierarchy are presented [23].

The conservation laws play an important role in discussing the integrability for soliton hierarchy. An infinite number of conservation laws for KdV equation were first discovered by Miura et al. in 1968 [24], and then lots of methods have been developed to find them. This may be mainly due to the contributions of Wadati and others [25–27]. Conservation laws also play an important role in mathematics and engineering as well. Many papers dealing with symmetries and conservation laws were presented. The direct construction method of multipliers for the conservation laws was presented [28].

In this paper, starting from a Lie super algebra, isospectral problems are designed. With the help of variational identity, Yang got super Broer-Kaup-Kupershmidt hierarchy and its Hamiltonian structure [29]. Then, based on the theory of self-consistent sources, the self-consistent sources of super Broer-Kaup-Kupershmidt hierarchy are obtained by us. Furthermore, we present the conservation laws for the super Broer-Kaup-Kupershmidt hierarchy. In the calculation process, extended Fermi quantities u_1 and u_2 play an important role; namely, u_1 and u_2 satisfy $u_1^2 = u_2^2 = 0$ and $u_1u_2 = -u_2u_1$

in the whole paper. Furthermore, the operation between extended Fermi variables satisfies Grassmann algebra conditions.

2. A Super Soliton Hierarchy with Self-Consistent Sources

Based on a Lie superalgebra G,

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{1}$$

that is along with the communicative operation $[e_1, e_2] = 2e_2$, $[e_1, e_3] = -2e_3$, $[e_2, e_3] = e_1$, $[e_1, e_4] = [e_2, e_5] = e_4$, $[e_1, e_5] = [e_4, e_3] = -e_5$, $[e_4, e_5]_+ = e_1$, $[e_4, e_4]_+ = -2e_2$, and $[e_5, e_5]_+ = 2e_3$.

We consider an auxiliary linear problem

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}_x = U(u,\lambda) \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \quad U(u,\lambda) = R_1 + \sum_{i=1}^{5} u_i e_i(\lambda),$$
$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}_{t_n} = V_n(u,\lambda) \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \tag{2}$$

where $u = (u_1, \ldots, u_s)^T$, $U_n = R_1 + u_1 e_1 + \cdots + u_5 e_5$, $u_i(n,t) = u_i$ $(i = 1, 2, \ldots, 5)$, $\varphi_i = \varphi(x,t)$ are field variables defining $x \in R$, $t \in R$; $e_i = e_i(\lambda) \in \widetilde{sl}(3)$ and R_1 is a pseudoregular element.

The compatibility of (2) gives rise to the well-known zero curvature equation as follows:

$$U_{nt} - V_{nx} + [U_n, V_n] = 0, \quad n = 1, 2, \ldots. \tag{3}$$

If an equation

$$u_t = K(u) \tag{4}$$

can be worked out through (3), we call (4) a super evolution equation. If there is a super Hamiltonian operator J and a function H_n such that

$$u_t = K(u) = J\frac{\delta H_{n+1}}{\delta u}, \tag{5}$$

where

$$\frac{\delta H_n}{\delta u} = L\frac{\delta H_{n-1}}{\delta u} = \cdots = L^n \frac{\delta H_0}{\delta u},$$
$$n = 1, 2, \ldots, \quad \frac{\delta}{\delta u} = \left(\frac{\delta}{\delta u_1}, \ldots, \frac{\delta}{\delta u_5}\right)^T, \tag{6}$$

then (4) possesses a super Hamiltonian equation. If so, we can say that (4) has a super Hamiltonian structure.

According to (2), now we consider a new auxiliary linear problem. For N distinct λ_j, $j = 1, 2, \ldots, N$, the systems of (2) become as follows:

$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_x = U(u,\lambda_j) \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix} = \sum_{i=1}^{5} u_i e_i(\lambda) \begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix},$$
$$\begin{pmatrix} \varphi_{1j} \\ \varphi_{2j} \\ \varphi_{3j} \end{pmatrix}_{t_n} = V_n(u,\lambda_j) \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} \tag{7}$$
$$= \left[\sum_{m=0}^{n} V_m(u)\lambda_j^{n-m} + \Delta_n(u,\lambda_j)\right] \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}.$$

Based on the result in [30], we can show that the following equation:

$$\frac{\delta H_k}{\delta u} + \sum_{j=1}^{N} \alpha_j \frac{\delta \lambda_j}{\delta u} = 0 \tag{8}$$

holds true, where α_j are constants. Equation (8) determines a finite dimensional invariant set for the flows in (6).

From (7), we may know that

$$\frac{\delta \lambda_j}{\delta u_i} = \frac{1}{3} S\operatorname{tr}\left(\Psi_j \frac{\partial U(u,\lambda_j)}{\partial u_i}\right)$$
$$= \frac{1}{3} S\operatorname{tr}(\Psi_j e_i \lambda_j), \quad i = 1, 2, \ldots 5, \tag{9}$$

where $S\operatorname{tr}$ denotes the trace of a matrix and

$$\Psi_j = \begin{pmatrix} \psi_{1j}\psi_{2j} & -\psi_{1j}^2 & \psi_{1j}\psi_{3j} \\ \psi_{2j}^2 & -\psi_{1j}\psi_{2j} & \psi_{2j}\psi_{3j} \\ \psi_{2j}\psi_{3j} & -\psi_{1j}\psi_{3j} & 0 \end{pmatrix}. \tag{10}$$

From (8) and (9), a kind of super Hamiltonian soliton equation hierarchy with self-consistent sources is presented as follows:

$$u_{nt} = J\frac{\delta H_{n+1}}{\delta u_i} + J\sum_{j=1}^{N} \alpha_j \frac{\delta \lambda_j}{\delta u}$$
$$= JL^n \frac{\delta H_1}{\delta u_i} + J\sum_{j=1}^{N} \alpha_j \frac{\delta \lambda_j}{\delta u}, \quad n = 1, 2, \ldots. \tag{11}$$

3. The Super Broer-Kaup-Kupershmidt Hierarchy with Self-Consistent Sources

The super Broer-Kaup-Kupershmidt spectral problem associated with the Lie super algebra is given in [29]:

$$\varphi_x = U\varphi, \quad \varphi_t = V\varphi, \tag{12}$$

where

$$U=\begin{pmatrix}\lambda+r & s & u_1\\ 1 & -\lambda-r & u_2\\ u_2 & -u_1 & 0\end{pmatrix},\qquad V=\begin{pmatrix}A & B & \rho\\ C & -A & \sigma\\ \sigma & -\rho & 0\end{pmatrix},\quad(13)$$

and $A=\sum_{m\geq 0}A_m\lambda^{-m}$, $B=\sum_{m\geq 0}B_m\lambda^{-m}$, $C=\sum_{m\geq 0}C_m\lambda^{-m}$, $\rho=\sum_{m\geq 0}\rho_m\lambda^{-m}$, and $\sigma=\sum_{m\geq 0}\sigma_m\lambda^{-m}$. As u_1 and u_2 are Fermi variables, they constitute Grassmann algebra. So, we have $u_1u_2=-u_2u_1$, $u_1^2=u_2^2=0$.

Starting from the stationary zero curvature equation

$$V_x=[U,V],\quad(14)$$

we have

$$\begin{aligned}
A_{mx}&=sC_m+u_1\sigma_m-B_m+u_2\rho_m,\\
B_{mx}&=2B_{m+1}+2rB_m-2sA_m-2u_1\rho_m,\\
C_{mx}&=-2C_{m+1}-2rC_m+2A_m+2u_2\sigma_m,\\
\rho_{mx}&=\rho_{m+1}+r\rho_m+s\sigma_m-u_1A_m-u_2B_m,\\
\sigma_{mx}&=-\sigma_{m+1}-r\sigma_m+\rho_m-u_1C_m+u_2A_m,\\
&B_0=C_0=\rho_0=\sigma_0=0,\\
&A_0=1,\qquad B_1=s,\qquad C_1=r,\\
&\rho_1=u_1,\qquad \sigma_1=u_2,\qquad A_1=0,\ldots.
\end{aligned}\quad(15)$$

Then we consider the auxiliary spectral problem

$$\varphi_{tn}=V^{(n)}\varphi=(\lambda^nV)_+\varphi,\quad(16)$$

where

$$V^{(n)}=\sum_{m=0}^{n}\begin{pmatrix}A_m & B_m & \rho_m\\ C_m & -A_m & \sigma_m\\ \sigma_m & -\rho_m & 0\end{pmatrix}\lambda^{n-m},\quad(17)$$

considering

$$V^{(n)}=V_+^{(n)}+\Delta_n,\quad \Delta_n=-C_{m+1}e_1.\quad(18)$$

Substituting (18) into the zero curvature equation

$$U_{tn}-V_x^{(n)}+\left[U,V^{(n)}\right]=0,\quad(19)$$

we get the super Broer-Kaup-Kupershmidt hierarchy

$$\begin{aligned}
u_{t_n}&=\begin{pmatrix}r\\ s\\ u_1\\ u_2\end{pmatrix}_t\\
&=\begin{pmatrix}0 & \partial & 0 & 0\\ \partial & 0 & u_1 & -u_2\\ 0 & u_1 & 0 & -\frac{1}{2}\\ 0 & -u_2 & -\frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}-2A_{n+1}\\ -C_{n+1}\\ 2\sigma_{n+1}\\ -2\rho_{n+1}\end{pmatrix}\quad(20)\\
&=J\begin{pmatrix}-2A_{n+1}\\ -C_{n+1}\\ 2\sigma_{n+1}\\ -2\rho_{n+1}\end{pmatrix}=JP_{n+1},
\end{aligned}$$

where

$$P_{n+1}=LP_n,$$

$$L=\begin{pmatrix}\frac{1}{2}\partial-\partial^{-1}r\partial & -s-\partial^{-1}s\partial & \partial^{-1}u_1\partial+\frac{1}{2}u_1 & \partial^{-1}u_2\partial-\frac{1}{2}u_2\\ \frac{1}{2} & -\frac{1}{2}\partial-r & \frac{1}{2}u_2 & 0\\ -u_2 & 2u_1 & -r-\partial & -1\\ u_1-u_2\partial & 2su_2 & s+\frac{1}{2}u_1u_2 & \partial-r\end{pmatrix}.\quad(21)$$

According to super trace identity on Lie super algebras, a direct calculation reads as

$$\begin{gathered}
\frac{\delta H_n}{\delta u}=\left(-2A_{n+1},-C_{n+1},2\sigma_{n+1},-2\rho_{n+1}\right)^T,\\
H_n=\int\frac{2A_{n+2}}{n+1}dx,\quad n\geq 0.
\end{gathered}\quad(22)$$

When we take $n=2$, the hierarchy (20) can be reduced to super nonlinear integrable couplings equations

$$\begin{aligned}
r_{t_2}&=-\frac{1}{2}r_{xx}+\frac{1}{2}s_x-2rr_x+\left(u_1u_2\right)_x+\left(u_1u_{2x}\right)_x,\\
s_{t_2}&=\frac{1}{2}s_{xx}-2(rs)_x+2u_1u_{1x}+2su_2u_{2x},\\
u_{1t_2}&=u_{1xx}-\frac{3}{2}r_xu_1+\frac{1}{2}s_xu_2-2ru_{1_x}+\left(s+u_1u_2\right)u_{2x},\\
u_{2t_2}&=-u_{2xx}-\frac{1}{2}r_xu_2-2ru_{2x}-u_{1x}.
\end{aligned}\quad(23)$$

Next, we will construct the super Broer-Kaup-Kupershmidt hierarchy with self-consistent sources. Consider the linear system

$$\begin{gathered}
\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\end{pmatrix}_x=U\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\end{pmatrix},\\
\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\end{pmatrix}_t=V\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\\ \varphi_{3j}\end{pmatrix}.
\end{gathered}\quad(24)$$

From (8), for the system (12), we set

$$\frac{\delta H_n}{\delta u}=\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\quad(25)$$

and obtain the following $\delta\lambda_j/\delta u$:

$$\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u} = \sum_{j=1}^{N}\begin{pmatrix} S\,\mathrm{tr}\left(\Psi_j\frac{\delta U}{\delta q}\right) \\ S\,\mathrm{tr}\left(\Psi_j\frac{\delta U}{\delta r}\right) \\ S\,\mathrm{tr}\left(\Psi_j\frac{\delta U}{\delta \alpha}\right) \\ S\,\mathrm{tr}\left(\Psi_j\frac{\delta U}{\delta \beta}\right)\end{pmatrix} = \begin{pmatrix} 2\langle\Phi_1,\Phi_2\rangle \\ \langle\Phi_2,\Phi_2\rangle \\ -2\langle\Phi_2,\Phi_3\rangle \\ 2\langle\Phi_1,\Phi_3\rangle\end{pmatrix}, \tag{26}$$

where $\Phi_i = (\varphi_{i1},\ldots,\varphi_{iN})^T$, $i = 1, 2, 3$.

According to (11), the integrable super Broer-Kaup-Kupershmidt hierarchy with self-consistent sources is proposed as follows:

$$u_{t_n} = \begin{pmatrix} r \\ s \\ u_{1t} \\ u_{2t}\end{pmatrix}_{t_n} = J\begin{pmatrix} -2A_{n+1} \\ -C_{n+1} \\ 2\sigma_{n+1} \\ -2\rho_{n+1}\end{pmatrix} + J\begin{pmatrix} 2\langle\Phi_1,\Phi_2\rangle \\ \langle\Phi_2,\Phi_2\rangle \\ -2\langle\Phi_2,\Phi_3\rangle \\ 2\langle\Phi_1,\Phi_3\rangle\end{pmatrix}, \tag{27}$$

where $\Phi_i = (\varphi_{i1},\ldots,\varphi_{iN})^T, i = 1, 2, 3,$ satisfy

$$\begin{aligned} \varphi_{1jx} &= (\lambda + r)\varphi_{1j} + s\varphi_{2j} + u_1\varphi_{3j}, \\ \varphi_{2jx} &= \varphi_{1j} - (\lambda + r)\varphi_{2j} + u_2\varphi_{3j}, \\ \varphi_{3jx} &= u_2\varphi_{1j} - u_1\varphi_{2j}, \\ & \qquad j = 1,\ldots,N. \end{aligned} \tag{28}$$

For $n = 2$, we obtain the super Broer-Kaup-Kupershmidt equation with self-consistent sources as follows:

$$\begin{aligned} r_{t_2} &= -\frac{1}{2}r_{xx} + \frac{1}{2}s_x - 2rr_x + (u_1u_2)_x + (u_1u_{2x})_x + \partial\sum_{j=1}^{N}\varphi_{2j}^2, \\ s_{t_2} &= \frac{1}{2}s_{xx} - 2(rs)_x + 2u_1u_{1x} + 2su_2u_{2x} + 2\partial\sum_{j=1}^{N}\varphi_{1j}\varphi_{2j} \\ &\quad - 2u_1\sum_{j=1}^{N}\varphi_{2j}\varphi_{3j} - 2u_2\sum_{j=1}^{N}\varphi_{1j}\varphi_{3j}, \\ u_{1t_2} &= u_{1xx} - \frac{3}{2}r_xu_1 + \frac{1}{2}s_xu_2 - 2ru_{1_x} + (s + u_1u_2)u_{2x} \\ &\quad + u_1\sum_{j=1}^{N}\varphi_{2j}^2 - \sum_{j=1}^{N}\varphi_{1j}\varphi_{3j}, \\ u_{2t_2} &= -u_{2xx} - \frac{1}{2}r_xu_2 - 2ru_{2x} - u_{1x} - u_2\sum_{j=1}^{N}\varphi_{2j}^2 + \sum_{j=1}^{N}\varphi_{2j}\varphi_{3j}, \end{aligned} \tag{29}$$

where $\Phi_i = (\varphi_{i1},\ldots,\varphi_{iN})^T$, $i = 1, 2, 3$, satisfy

$$\begin{aligned} \varphi_{1jx} &= (\lambda + r)\varphi_{1j} + s\varphi_{2j} + u_1\varphi_{3j}, \\ \varphi_{2jx} &= \varphi_{1j} - (\lambda + r)\varphi_{2j} + u_2\varphi_{3j}, \\ \varphi_{3jx} &= u_2\varphi_{1j} - u_1\varphi_{2j}, \\ & \qquad j = 1,\ldots,N. \end{aligned} \tag{30}$$

4. Conservation Laws for the Super Broer-Kaup-Kupershmidt Hierarchy

In the following, we will construct conservation laws of the super Broer-Kaup-Kupershmidt hierarchy. We introduce the variables

$$E = \frac{\varphi_2}{\varphi_1}, \qquad K = \frac{\varphi_3}{\varphi_1}. \tag{31}$$

From (7) and (12), we have

$$\begin{aligned} E_x &= 1 - 2\lambda E - 2rE + u_2K - sE^2 - u_1EK, \\ K_x &= u_2 - \lambda K - u_1E - rK - sKE - u_1K^2. \end{aligned} \tag{32}$$

Expand E, K in the power of λ as follows:

$$E = \sum_{j=1}^{\infty}e_j\lambda^{-j}, \qquad K = \sum_{j=1}^{\infty}k_j\lambda^{-j}. \tag{33}$$

Substituting (33) into (32) and comparing the coefficients of the same power of λ, we obtain

$$\begin{gathered} e_1 = \frac{1}{2}, \qquad k_1 = u_2, \qquad e_2 = -\frac{1}{2}r, \\ k_2 = -u_{2x} - \frac{1}{2}u_1 - ru_2, \\ e_3 = \frac{1}{4}r_x - \frac{1}{2}u_2u_{2x} + \frac{1}{2}r^2 - \frac{1}{2}u_1u_2 - \frac{1}{8}s, \\ k_3 = u_{2xx} + r_xu_2 + 2ru_{2x} + \frac{1}{2}u_{1x} + ru_1 + r^2u_2 - \frac{1}{2}su_2, \ldots \end{gathered} \tag{34}$$

and a recursion formula for e_n and k_n

$$e_{n+1} = -\frac{1}{2}e_{n,x} - re_n + \frac{1}{2}u_2k_n - \frac{1}{2}s\sum_{l=1}^{n-1}e_le_{n-l} - \frac{1}{2}u_1\sum_{l=1}^{n-1}e_lk_{n-l},$$
$$k_{n+1} = -k_{n,x} - u_1e_n - rk_n - s\sum_{l=1}^{n-1}k_le_{n-l} - u_1\sum_{l=1}^{n-1}k_lk_{n-l}, \quad (35)$$

because of

$$\frac{\partial}{\partial t}\left[\lambda + r + sE + u_1K\right] = \frac{\partial}{\partial x}\left[A + BE + \rho K\right], \quad (36)$$

where

$$A = m_0\lambda^2 + m_1\lambda + \frac{1}{2}m_0s - m_0u_1u_2,$$
$$B = m_0s\lambda + \frac{1}{2}m_0s_x - m_0rs + m_1s, \quad (37)$$
$$\rho = m_0u_1\lambda + m_0u_{1x} - m_0ru_1 + m_1u_1.$$

Assume that $\delta = \lambda + r + sE + u_1K$, $\theta = A + BE + \rho K$. Then (36) can be written as $\delta_t = \theta_x$, which is the right form of conservation laws. We expand δ and θ as series in powers of λ with the coefficients, which are called conserved densities and currents, respectively,

$$\delta = \lambda + r + \sum_{j=1}^{\infty}\delta_j\lambda^{-j},$$
$$\theta = m_0\lambda^2 + m_1\lambda + m_0s + \sum_{j=1}^{\infty}\theta_j\lambda^{-j}, \quad (38)$$

where m_0, m_1 are constants of integration. The first two conserved densities and currents are read as follows:

$$\delta_1 = \frac{1}{2}s + u_1u_2,$$
$$\theta_1 = m_0\left(\frac{1}{4}s_x - sr - u_1u_{2x} + u_2u_{1x} - 2ru_1u_2\right) + m_1\left(\frac{1}{2}s + u_1u_2\right),$$
$$\delta_2 = -\frac{1}{2}sr - u_1u_{2x} - ru_1u_2, \quad (39)$$
$$\theta_2 = m_0\left(\frac{1}{4}sr_x - \frac{1}{2}su_2u_{2x} + sr^2 - su_1u_2 - \frac{1}{8}s^2 - \frac{1}{4}rs_x + u_1u_{2xx} + r_xu_1u_2 + 3ru_1u_{2x} + 2r^2u_1u_2 - u_{1x}u_{2x} - ru_{1x}u_2\right) - m_1\left(\frac{1}{2}sr + u_1u_{2x} + ru_1u_2\right).$$

The recursion relation for δ_n and θ_n are

$$\delta_n = se_n + u_1k_n,$$
$$\theta_n = m_0\left(s_{n+1} + \frac{1}{2}s_xe_n - rse_n + u_1k_{n+1} + u_{1x}k_n - ru_1k_n\right) + m_1\left(se_n + u_1k_n\right), \quad (40)$$

where e_n and k_n can be calculated from (35). The infinitely many conservation laws of (20) can be easily obtained from (32)–(40), respectively.

5. Conclusions

Starting from Lie super algebras, we may get super equation hierarchy. With the help of variational identity, the Hamiltonian structure can also be presented. Based on Lie super algebra, the self-consistent sources of super Broer-Kaup-Kupershmidt hierarchy can be obtained. It enriched the content of self-consistent sources of super soliton hierarchy. Finally, we also get the conservation laws of the super Broer-Kaup-Kupershmidt hierarchy. It is worth to note that the coupling terms of super integrable hierarchies involve fermi variables; they satisfy the Grassmann algebra which is different from the ordinary one.

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