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Research Article

On the Convergence of the Homotopy Analysis Method for Inner-Resonance of Tangent Nonlinear Cushioning Packaging System with Critical Components

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Homotopy analysis method (HAM) is applied to obtain the approximate solution of inner-resonance of tangent cushioning packaging system based on critical components. The solution is obtained in the form of infinite series with components which can be easily calculated. Using a convergence-control parameter, the HAM utilizes a simple method to adjust and control the convergence region of the infinite series solution. The obtained results show that the HAM is a very accurate technique to obtain the approximate solution.

1. Introduction

One of the most important subjects in packaging system is to investigate products damaged due to being dropped. Many researchers have investigated cushioning packaging system in this special field [1, 2]. The mechanical and electronic products are composed of large number of elements and, generally, damage at the so-called critical components. To prevent any damage, a critical component and a cushioning packaging are included in a package system [3]. The following assumptions are made basically in the last decade by the researchers [1, 4].

(1) The researchers considered that the packaging system is a spring-mass, single degree of freedom system.

(2) The use of simple linear or nonlinear springs for cushioning packaging may not be appropriate.

Wang et al. [2] considered a linear model for this system, while the oscillation in the package system is inborn nonlinearity (see [4, 5]). Our goal of this paper is to obtain the approximate solution of inner-resonance of tangent nonlinear cushioning packaging system with critical components introduced in [1, 4] using HAM, which is one of the semilinear approximate analytical methods.

In the last two decades, many researchers have employed the approximate analytical methods such as adomian decomposition method (ADM), variational iteration method (VIM), homotopy perturbation method (HPM), and HAM to solve differential equations. These methods give the solution of the differential equations in the form of infinite series. One of the advantages of approximate analytical methods is that these methods do not produce rounding-off errors. Contrary to the implicit finite difference method (FDM), the approximate analytical methods do not require the numerical solution of the systems of differential equations.

We apply the HAM to construct the series solution for the inner-resonance of tangent nonlinear cushioning packaging system with critical components. An advantage of HAM over perturbation methods is that it is not dependent on small or large parameters. As it is well known, perturbation methods cannot be applied to all nonlinear equations because these methods are based on the existence of small or large parameters. Besides, nonperturbation methods are independent of small parameters. According to [6], both of the two techniques (perturbation and nonperturbative

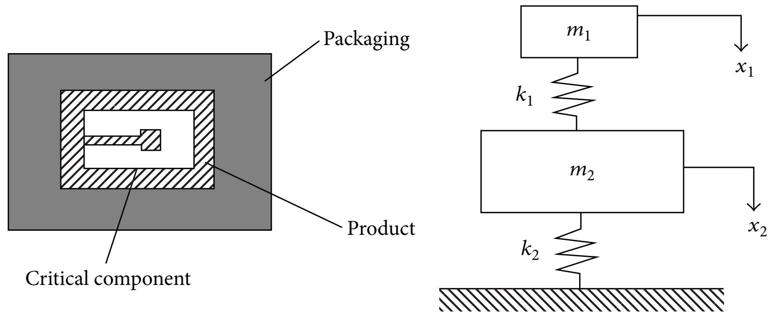

Figure 1: The model of packaging system with critical component ($m_1 \ll m_2$) and $x_1 = x(t)$, $x_2 = y(t)$.

techniques) cannot provide a simple procedure to adjust or control the convergence region and rate of given approximate series. According to [7, 8], HAM also allows for fine-tuning of convergence region and rate of convergence by allowing an auxiliary parameter $\hbar$ to vary. To the best of our knowledge, this is the first attempt at solving the inner-resonance of tangent nonlinear cushioning packaging system with components approximately using the HAM.

In recent years, the HAM has been successfully applied for solving various nonlinear systems of equations in many branches of mathematics and sciences, such as strongly coupled reaction-diffusion system [9], fractional Lorenz system [7], coupled Schrodinger-KdV equation [10], Burgers and coupled Burgers equation [11], system of second-order BVPs [8], and HIV infection $CD4^{+}$ T-cell [12]. For more studies of HAM and its applications, the readers are also referred to see [13–16].

Our paper is organized as follows.

In Section 2, we introduce mathematical modelling of the inner-resonance of tangent nonlinear cushioning packaging system with critical components. In Section 3, we present a description of the HAM on system of equations, as expanded by previous researchers in particular [9, 12], applied to the inner-resonance of tangent nonlinear cushioning packaging system with critical components. We prove the convergence of homotopy series solution for the inner-resonance of tangent nonlinear cushioning packaging system with critical components also in this section. In Section 4, we have applied the HAM to obtain the approximate solution of inner-resonance of tangent nonlinear cushioning packaging system with critical components. Finally, in Section 5, we give the conclusion of this study.

2. Modeling and Equations

Generally, imagine that everything we know and have a relationship to, including things such as art, clothes, possessions, homes, gardens, trees and fields, mountains, lakes, oceans, continents, our friends, and loved ones, are instances where resonance can and does occur. Now, consider that everything we do and think is an attempt to seek out and to return to the experience of resonance, a return to the feeling of Belonging and feeling that things feel right. Even though we may not identify the motivation for making and forming certain relationships, the real attraction and value of any relationship is whether it is fulfilling and makes us feel good. We are searching for the feeling of resonance. The profound nature of these kinds of experiences is dependent on the nature and quality of the relationship we can have to something of an external nature, but we also have the potential to experience resonance within our own body and our inner being, and this I would call a "state of Inner Resonance." When a practitioner consciously perceives resonance happening in a therapeutic relationship between themselves and patient, it is also a state of Inner Resonance. In this circumstance, the practitioners create the appropriate environment, conditions, and quality of presence so that they can receive the patients intention to be understood and received. In another way, the patient's need to experience resonance (whether consciously or unconsciously) is met. Inner Resonance, then, is a state of receptive presence or conscious empathy between two or more people.

In Figure 1, the model of packaging system with critical component is shown to be considered as a nonlinear spring with stiffness coefficient k_2. It can also idealize the joining part between the mass of critical component m_1 and the main part of the product m_2 as a linear spring with stiffness coefficient k_1. According to Figures 1 and 2, the motion of this system can be written as [1, 4]

$$m_1 \frac{d^2 x(t)}{dt^2} + k_1 (x(t) - y(t)) = 0,$$
$$m_2 \frac{d^2 y(t)}{dt^2} + \frac{2k_2 d_b}{\pi} \tan\left(\frac{\pi}{2d_b} y(t)\right) - k_1 (x(t) - y(t)) = 0, \tag{1}$$

with initial conditions

$$x(0) = 0, \qquad x'(0) = \sqrt{2gh},$$
$$y(0) = 0, \qquad y'(0) = \sqrt{2gh}. \tag{2}$$

Table 1 summarize the meanings of parameters and variables. To simplify (1), new variables are introduced as

$$X = \frac{x}{L}, \qquad Y = \frac{y}{L}, \qquad T = \frac{t}{T_0}, \tag{3}$$

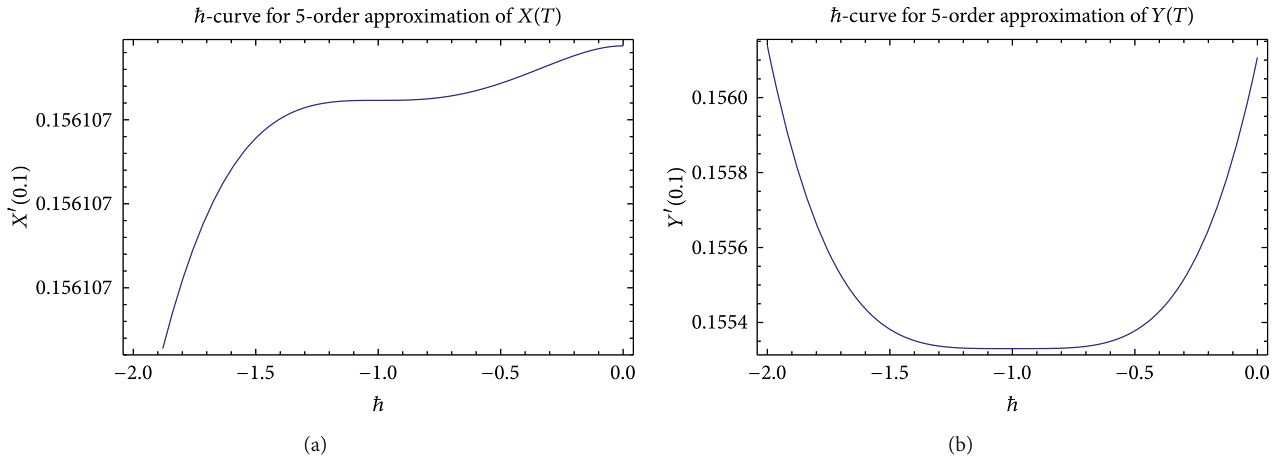

FIGURE 2: The $\hbar$-curves of $X'(0.1)$ and $Y'(0.1)$ obtained by the 5-order approximation of HAM.

TABLE 1: List of variables and parameters (modified from [1, 4]).

Parameters and variables	Illustration
x	Displacement response of critical component
y	Main body of the product
m_1	Mass of critical component
m_2	Main part of product
k_1	Stiffness coefficient
k_2	Stiffness coefficient
d_b	Compression limit of the cushioning pad
h	Dropping height
g	Gravity acceleration
$\sqrt{2gh}$	Dropping shock velocity of the product

where

$$T_0 = \sqrt{\frac{m_2}{k_2}}, \qquad L = \frac{2d_b}{\pi}. \tag{4}$$

We define the frequency parameters of the critical component and main part of product as $w_1 = \sqrt{k_1/m_1}$ and $w_2 = \sqrt{k_2/m_2}$, respectively. The notations $\lambda_1 = w_1/w_2$ and $\lambda_2 = m_1/m_2$ are considered as parameter ratio and mass ratio, respectively. By considering all parameters defined, (1) can be equivalently written in the following system of nonlinear equations [1]:

$$\begin{gathered} \frac{d^2X}{dT^2} + w_{01}^2 X - w_{01}^2 Y = 0, \\ \frac{d^2Y}{dT^2} + w_{02}^2 Y + \frac{1}{3}Y^3 + \frac{2}{15}Y^5 + \left(1 - w_{02}^2\right) X = 0, \end{gathered} \tag{5}$$

with initial conditions

$$\begin{aligned} X(0) &= 0, \qquad X'(0) = \frac{T_0}{L}\sqrt{2gh}, \\ Y(0) &= 0, \qquad Y'(0) = \frac{T_0}{L}\sqrt{2gh}, \end{aligned} \tag{6}$$

where $X = X(T)$, $Y = Y(T)$ and

$$w_{01} = \lambda_1, \qquad w_{02} = \sqrt{1 + \lambda_1^2 \lambda_2}. \tag{7}$$

3. Homotopy Analysis Method (HAM)

To apply the HAM, the nonlinear system (5) is considered. We make initial gusses on $X(T)$ and $Y(T)$ such that they satisfy the initial conditions (6) that are defined as

$$\begin{aligned} X(0) = X_0 &= \frac{T_0}{L}\sqrt{2gh}T, \\ Y(0) = Y_0 &= \frac{T_0}{L}\sqrt{2gh}T. \end{aligned} \tag{8}$$

The auxiliary linear operators $\mathscr{L}_X$ and $\mathscr{L}_Y$ are selected as

$$\mathscr{L}_X = \frac{d^2X}{dT^2}, \qquad \mathscr{L}_Y = \frac{d^2Y}{dT^2}, \tag{9}$$

satisfying the following properties:

$$\begin{aligned} \mathscr{L}_X\left(c_{1,X}T + c_{2,X}\right) &= 0, \\ \mathscr{L}_Y\left(c_{1,Y}T + c_{2,Y}\right) &= 0, \end{aligned} \tag{10}$$

where $c_{1,X}$, $c_{2,X}$, $c_{1,Y}$, and $c_{2,Y}$ are integral constants. Define the homotopy maps

$$\begin{aligned}
&\mathscr{H}_X\left(\widehat{X}(T;q),\widehat{Y}(T;q)\right)\\
&\quad=(1-q)\,\mathscr{L}_X\left[\widehat{X}(T;q)-X_0(T)\right]\\
&\qquad-q\hbar H_X(t)\,N_X\left[\widehat{X}(T;q),\widehat{Y}(T;q)\right],\\
&\mathscr{H}_Y\left(\widehat{X}(T;q),\widehat{Y}(T;q)\right)\\
&\quad=(1-q)\,\mathscr{L}_Y\left[\widehat{Y}(T;q)-Y_0(T)\right]\\
&\qquad-q\hbar H_Y(t)\,N_Y\left[\widehat{X}(T;q),\widehat{Y}(T;q)\right],
\end{aligned}\tag{11}$$

where $q\in[0,1]$ is an embedding parameter, $\hbar$ is nonzero auxiliary parameter, H_X and H_Y are auxiliary functions, and N_X and N_Y are nonlinear operators that are defined as

$$\begin{aligned}
N_X\left[\widehat{X}(T;q),\widehat{Y}(T;q)\right]&=\frac{\partial^2\widehat{X}(T;q)}{\partial T^2}+w_{01}^2\widehat{X}(T;q)\\
&\quad-w_{01}^2\widehat{Y}(T;q),\\
N_Y\left[\widehat{X}(T;q),\widehat{Y}(T;q)\right]&=\frac{\partial^2\widehat{Y}(T;q)}{\partial T^2}+w_{02}^2\widehat{Y}(T;q)\\
&\quad+\frac{\left(\widehat{Y}(T;q)\right)^3}{3}+\frac{2}{15}\left(\widehat{Y}(T;q)\right)^5\\
&\quad+\left(1-w_{02}^2\right)\widehat{X}(T;q).
\end{aligned}\tag{12}$$

Clearly, when $q=0$, we have the homotopy maps

$$\begin{aligned}
\mathscr{H}_X\left(\widehat{X}(T;0),\widehat{Y}(T;0)\right)&=\mathscr{L}_X\left[\widehat{X}(T;0)-X_0(T)\right],\\
\mathscr{H}_Y\left(\widehat{X}(T;0),\widehat{Y}(T;0)\right)&=\mathscr{L}_Y\left[\widehat{Y}(T;0)-Y_0(T)\right].
\end{aligned}\tag{13}$$

And when $q=1$, we have

$$\begin{aligned}
&\mathscr{H}_X\left(\widehat{X}(T;1),\widehat{Y}(T;1)\right)\\
&\quad=-\hbar H_X(T)\,N_X\left[\widehat{X}(T;1),\widehat{Y}(T;1)\right],\\
&\mathscr{H}_Y\left(\widehat{X}(T;1),\widehat{Y}(T;1)\right)\\
&\quad=-\hbar H_Y(T)\,N_Y\left[\widehat{X}(T;1),\widehat{Y}(T;1)\right].
\end{aligned}\tag{14}$$

Thus, by requiring

$$\mathscr{H}_X\left(\widehat{X}(T;q),\widehat{Y}(T;q)\right)=\mathscr{H}_Y\left(\widehat{X}(T;q),\widehat{Y}(T;q)\right)=0,\tag{15}$$

we can obtain

$$\begin{aligned}
&(1-q)\,\mathscr{L}_X\left[\widehat{X}(T;q)-X_0(T)\right]\\
&\quad=q\hbar H_X(T)\,N_X\left[\widehat{X}(T;q),\widehat{Y}(T;q)\right],\\
&(1-q)\,\mathscr{L}_Y\left[\widehat{Y}(T;q)-Y_0(T)\right]\\
&\quad=q\hbar H_Y(T)\,N_Y\left[\widehat{X}(T;q),\widehat{Y}(T;q)\right].
\end{aligned}\tag{16}$$

If $q=0$ and $q=1$, the homotopy equations are as follows:

$$\begin{aligned}
\widehat{X}(T;0)&=X_0, &\quad \widehat{X}(T;1)&=X(T),\\
\widehat{Y}(T;0)&=Y_0, &\quad \widehat{Y}(T;1)&=Y(T).
\end{aligned}\tag{17}$$

As q varies from 0 to 1, the solution of the nonlinear system (5) will vary from the initial guesses $X_0(T)$ and $Y_0(T)$ to the exact solutions $X(T)$ and $Y(T)$ of the nonlinear system (5). Expanding $\widehat{X}(T;q)$ and $\widehat{Y}(T;q)$ as a Taylor series with respect to q yields

$$\begin{aligned}
\widehat{X}(T;q)&=X_0+\sum_{m=1}^{\infty}X_mq^m,\\
\widehat{Y}(T;q)&=Y_0+\sum_{m=1}^{\infty}Y_mq^m,
\end{aligned}\tag{18}$$

where

$$X_m=\frac{1}{m!}\left.\frac{\partial^m\widehat{X}(T;q)}{\partial q^m}\right|_{q=0},\qquad Y_m=\frac{1}{m!}\left.\frac{\partial^m\widehat{Y}(T;q)}{\partial q^m}\right|_{q=0}.\tag{19}$$

According to [17], the convergence of the series (18) strongly depends on the auxiliary parameter $\hbar$. Note that if $q=1$, then

$$\begin{aligned}
\widehat{X}(T;1)&=X=X_0+\sum_{m=1}^{\infty}X_m,\\
\widehat{Y}(T;1)&=Y=Y_0+\sum_{m=1}^{\infty}Y_m.
\end{aligned}\tag{20}$$

According to definitions (18), the governing equations for the unknowns can be deduced from the zeroth-deformation equations (16). For further analysis, the vectors are defined as

$$\begin{aligned}
\widehat{X}_n&=\{X_0,X_1,\ldots,X_n\},\\
\widehat{Y}_n&=\{Y_0,Y_1,\ldots,Y_n\}.
\end{aligned}\tag{21}$$

Differentiating (16) m-times with respect to q, dividing by $m!$, and setting $q=0$ give the linear equations

$$\begin{aligned}
\mathscr{L}_X\left[X_m-\chi_mX_{m-1}\right]&=\hbar H_X(T)\,R_{m,X}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right),\\
\mathscr{L}_Y\left[Y_m-\chi_mY_{m-1}\right]&=\hbar H_Y(T)\,R_{m,Y}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right),
\end{aligned}\tag{22}$$

with initial conditions

$$\begin{aligned}
X_m(0)&=0, &\quad X_m'(0)&=0,\\
Y_m(0)&=0, &\quad Y_m'(0)&=0,
\end{aligned}\tag{23}$$

where

$$R_{m,X}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right)=X''_{m-1}+w_{01}^2X_{m-1}-w_{01}^2Y_{m-1}, \quad (24)$$

$$\begin{aligned}R_{m,Y}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right)&=Y''_{m-1}+w_{02}^2Y_{m-1}\\&\quad+\frac{1}{3}\sum_{i_1=0}^{m-1}Y_{i_1}\sum_{i_2=0}^{m-1-i_1}Y_{i_2}Y_{m-1-i_1-i_2}\\&\quad+\frac{2}{15}\sum_{i_1=0}^{m-1}Y_{i_1}\sum_{i_2=0}^{m-1-i_1}Y_{i_2}\sum_{i_3=0}^{m-1-i_1-i_2}Y_{i_3}\\&\quad\times\sum_{i_4=0}^{m-1-i_1-i_2-i_3}Y_{i_4}Y_{m-1-i_1-i_2-i_3-i_4}\\&\quad+\left(1-w_{02}^2\right)X_{m-1},\end{aligned} \quad (25)$$

$$\chi_m:=\begin{cases}0 & m\le 1\\ 1 & m>1.\end{cases} \quad (26)$$

Using $H_X(T)=H_Y(T)=1$, the solution of the m-order deformation equations (22) for $m\ge 1$ becomes

$$\begin{aligned}X_m&=\chi_mX_{m-1}+\hbar\iint R_{m,X}\left(\vec{X}_{m-1}(\tau),\vec{Y}_{m-1}(\tau)\right)d\tau\,d\tau\\&\quad+c_{1,X}T+c_{2,X},\\Y_m&=\chi_mY_{m-1}+\hbar\iint R_{m,Y}\left(\vec{X}_{m-1}(\tau),\vec{Y}_{m-1}(\tau)\right)d\tau\,d\tau\\&\quad+c_{1,Y}T+c_{2,Y}.\end{aligned} \quad (27)$$

The coefficients $c_{1,X}$, $c_{2,X}$, $c_{1,Y}$, and $c_{2,Y}$ are determined using initial conditions (23).

3.1. Convergence Theorem

Theorem 1. *The series $X(T)=X_0+\sum_{m=1}^{\infty}X_m$ and $Y(T)=Y_0+\sum_{m=1}^{\infty}Y_m$ converge where X_m and Y_m are governed by* (22) *under definitions* (24)–(26)*; X and Y must be the solutions of system of* (5).

Proof. If the series $\sum_{m=0}^{\infty}X_m$ and $\sum_{m=0}^{\infty}Y_m$ are convergent, we can write

$$S_X=\sum_{m=0}^{\infty}X_m,\qquad S_Y=\sum_{m=0}^{\infty}Y_m. \quad (28)$$

And it holds that

$$\lim_{m\to\infty}X_m=\lim_{m\to\infty}Y_m=0. \quad (29)$$

From (22) and using (9), we have

$$\begin{aligned}\hbar\sum_{m=1}^{\infty}R_{m,X}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right)&=\sum_{m=1}^{\infty}\mathscr{L}_X\left[X_m-\chi_mX_{m-1}\right]\\&=\lim_{m\to\infty}\sum_{m=0}^{n}\mathscr{L}_X\left[X_m-\chi_mX_{m-1}\right]\\&=\mathscr{L}_X\left[\lim_{m\to\infty}\sum_{m=0}^{n}\left(X_m-\chi_mX_{m-1}\right)\right]\\&=\mathscr{L}_X\left[\lim_{m\to\infty}X_n\right]=0,\\\hbar\sum_{m=1}^{\infty}R_{m,Y}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right)&=\sum_{m=1}^{\infty}\mathscr{L}_Y\left[Y_m-\chi_mY_{m-1}\right]\\&=\lim_{m\to\infty}\sum_{m=0}^{n}\mathscr{L}_Y\left[Y_m-\chi_mY_{m-1}\right]\\&=\mathscr{L}_Y\left[\lim_{m\to\infty}\sum_{m=0}^{n}\left(Y_m-\chi_mY_{m-1}\right)\right]\\&=\mathscr{L}_Y\left[\lim_{m\to\infty}Y_n\right]=0.\end{aligned} \quad (30)$$

Since $\hbar\neq 0$, then

$$\sum_{m=1}^{\infty}R_{m,X}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right)=0, \quad (31)$$

$$\sum_{m=1}^{\infty}R_{m,Y}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right)=0. \quad (32)$$

Substituting (24) into (31) and simplifying it, we obtain

$$\begin{aligned}&\sum_{m=1}^{\infty}R_{m,X}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right)\\&\quad=\sum_{m=1}^{\infty}\left(X''_{m-1}+w_{01}^2X_{m-1}-w_{01}^2Y_{m-1}\right)\\&\quad=\sum_{m=1}^{\infty}X''_{m-1}+w_{01}^2\sum_{m=1}^{\infty}X_{m-1}-w_{01}^2\sum_{m=1}^{\infty}Y_{m-1}\\&\quad=\frac{d^2}{dT^2}\sum_{m=0}^{\infty}X_m+w_{01}^2\sum_{m=0}^{\infty}X_m-w_{01}^2\sum_{m=0}^{\infty}Y_m\\&\quad=X''+w_{01}^2X-w_{01}^2Y=0.\end{aligned} \quad (33)$$

We repeat this process and substitute (25) into (32), and simplifying it, we obtain

$$\begin{aligned}&\sum_{m=1}^{\infty}R_{m,Y}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right)\\&\quad=\sum_{m=1}^{\infty}Y''_{m-1}+w_{02}^2\sum_{m=1}^{\infty}Y_{m-1}\\&\qquad+\sum_{m=1}^{\infty}\left(1-w_{02}^2\right)X_{m-1}\end{aligned}$$

$$+\frac{1}{3}\sum_{m=1}^{\infty}\left[\sum_{i_1=0}^{m-1}Y_{i_1}\sum_{i_2=0}^{m-1-i_1}Y_{i_2}Y_{m-1-i_1-i_2}\right]$$
$$+\frac{2}{15}\sum_{m=1}^{\infty}\left[\sum_{i_1=0}^{m-1}Y_{i_1}\sum_{i_2=0}^{m-1-i_1}Y_{i_2}\sum_{i_3=0}^{m-1-i_1-i_2}Y_{i_3}\times\sum_{i_4=0}^{m-1-i_1-i_2-i_3}Y_{i_4}Y_{m-1-i_1-i_2-i_3-i_4}\right]. \quad (34)$$

For the first three terms of (34), we can easily conclude that

$$\sum_{m=1}^{\infty}Y''_{m-1}+w_{02}^2\sum_{m=1}^{\infty}Y_{m-1}+\sum_{m=1}^{\infty}\left(1-w_{02}^2\right)X_{m-1} = Y''(T)+w_{02}^2Y(T)+\left(1-w_{02}^2\right)X(T). \quad (35)$$

For the fourth and fifth terms in (34), we have

$$\begin{aligned}
&\frac{1}{3}\sum_{m=1}^{\infty}\left[\sum_{i_1=0}^{m-1}Y_{i_1}\sum_{i_2=0}^{m-1-i_1}Y_{i_2}Y_{m-1-i_1-i_2}\right]\\
&=\frac{1}{3}\sum_{m=1}^{\infty}\sum_{i_1=0}^{m-1}Y_{i_1}\sum_{i_2=0}^{m-1-i_1}Y_{i_2}Y_{m-1-i_1-i_2}\\
&=\frac{1}{3}\sum_{i_1=0}^{\infty}\sum_{m=i_1+1}^{\infty}Y_{i_1}\sum_{i_2=0}^{m-1-i_1}Y_{i_2}Y_{m-1-i_1-i_2}\\
&=\frac{1}{3}\sum_{i_1=0}^{\infty}\sum_{j=0}^{\infty}Y_{i_1}\sum_{i_2=0}^{j}Y_{i_2}Y_{j-i_2}\\
&=\frac{1}{3}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{j=0}^{\infty}\sum_{i_2=0}^{j}Y_{i_2}Y_{j-i_2}\\
&=\frac{1}{3}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{i_2=0}^{\infty}\sum_{j=i_2}^{\infty}Y_{i_2}Y_{j-i_2}\\
&=\frac{1}{3}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{i_2=0}^{\infty}Y_{i_2}\sum_{j=i_2}^{\infty}Y_{j-i_2}\\
&=\frac{1}{3}\left(\sum_{m=0}^{\infty}Y_m\right)^3=\frac{1}{3}Y^3(T),
\end{aligned}$$

$$\begin{aligned}
&\frac{2}{15}\sum_{m=1}^{\infty}\left[\sum_{i_1=0}^{m-1}Y_{i_1}\sum_{i_2=0}^{m-1-i_1}Y_{i_2}\sum_{i_3=0}^{m-1-i_1-i_2}Y_{i_3}\times\sum_{i_4=0}^{m-1-i_1-i_2-i_3}Y_{i_4}Y_{m-1-i_1-i_2-i_3-i_4}\right]\\
&=\frac{2}{15}\sum_{m=1}^{\infty}\sum_{i_1=0}^{m-1}Y_{i_1}\sum_{i_2=0}^{m-1-i_1}Y_{i_2}\times\sum_{i_3=0}^{m-1-i_1-i_2}Y_{i_3}\sum_{i_4=0}^{m-1-i_1-i_2-i_3}Y_{i_4}Y_{m-1-i_1-i_2-i_3-i_4}\\
&=\frac{2}{15}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{m=i_1+1}^{\infty}\sum_{i_2=0}^{m-1-i_1}Y_{i_2}\times\sum_{i_3=0}^{m-1-i_1-i_2}Y_{i_3}\sum_{i_4=0}^{m-1-i_1-i_2-i_3}Y_{i_4}Y_{m-1-i_1-i_2-i_3-i_4}\\
&=\frac{2}{15}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{j=0}^{\infty}\sum_{i_2=0}^{j}Y_{i_2}\sum_{i_3=0}^{m-1-i_1-i_2}Y_{i_3}\times\sum_{i_4=0}^{m-1-i_1-i_2-i_3}Y_{i_4}Y_{m-1-i_1-i_2-i_3-i_4}\\
&=\frac{2}{15}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{i_2=0}^{\infty}Y_{i_2}\sum_{m=1+i_1+i_2}^{\infty}\sum_{i_3=0}^{m-1-i_1-i_2}Y_{i_3}\times\sum_{i_4=0}^{m-1-i_1-i_2-i_3}Y_{i_4}Y_{m-1-i_1-i_2-i_3-i_4}\\
&=\frac{2}{15}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{i_2=0}^{\infty}Y_{i_2}\sum_{l=0}^{\infty}\sum_{i_3=0}^{l}Y_{i_3}\times\sum_{i_4=0}^{m-1-i_1-i_2-i_3}Y_{i_4}Y_{m-1-i_1-i_2-i_3-i_4}\\
&=\frac{2}{15}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{i_2=0}^{\infty}Y_{i_2}\sum_{i_3=0}^{\infty}Y_{i_3}\times\sum_{m=1+i_1+i_2+i_3}^{\infty}\sum_{i_4=0}^{m-1-i_1-i_2-i_3}Y_{i_4}Y_{m-1-i_1-i_2-i_3-i_4}\\
&=\frac{2}{15}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{i_2=0}^{\infty}Y_{i_2}\sum_{i_3=0}^{\infty}Y_{i_3}\sum_{k=0}^{\infty}\sum_{i_4=0}^{k}Y_{i_4}Y_{k-i_4}\\
&=\frac{2}{15}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{i_2=0}^{\infty}Y_{i_2}\sum_{i_3=0}^{\infty}Y_{i_3}\sum_{i_4=0}^{\infty}\sum_{k=i_4}^{k}Y_{i_4}Y_{k-i_4}\\
&=\frac{2}{15}\sum_{i_1=0}^{\infty}Y_{i_1}\sum_{i_2=0}^{\infty}Y_{i_2}\sum_{i_3=0}^{\infty}Y_{i_3}\sum_{i_4=0}^{\infty}Y_{i_4}\sum_{k=i_4}^{k}Y_{k-i_4}\\
&=\frac{2}{15}Y^5(T).
\end{aligned} \quad (36)$$

Thus,

$$\sum_{m=1}^{\infty}R_{m,Y}\left(\vec{X}_{m-1},\vec{Y}_{m-1}\right)=Y''+w_{02}^2Y+\frac{1}{3}Y^3+\frac{2}{15}Y^5+\left(1-w_{02}^2\right)X=0. \quad (37)$$

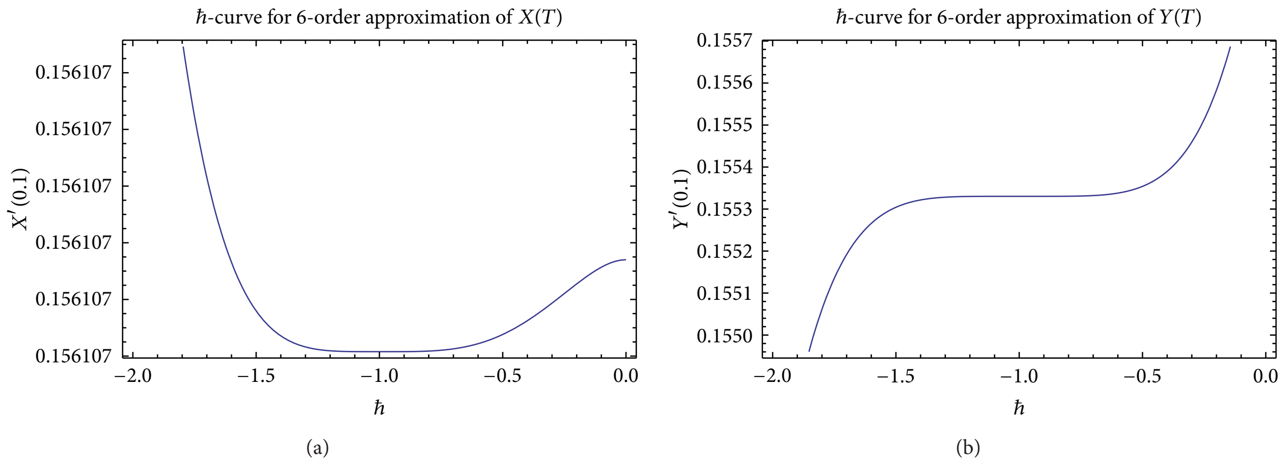

FIGURE 3: The $\hbar$-curves of $X'(0.1)$ and $Y'(0.1)$ obtained by the 6-order approximation of HAM.

From (2) and (23), it holds that

$$X_0 = \sum_{m=0}^{\infty} X_m(0) = X_0(0) + \sum_{m=1}^{\infty} X_m(0) = 0,$$
$$X_0' = \sum_{m=0}^{\infty} X_m'(0) = X_0'(0) + \sum_{m=1}^{\infty} X_m'(0) = \frac{T_0}{L}\sqrt{2gh},$$
$$Y_0 = \sum_{m=0}^{\infty} Y_m(0) = Y_0(0) + \sum_{m=1}^{\infty} Y_m(0) = 0,$$
$$Y_0' = \sum_{m=0}^{\infty} Y_m'(0) = Y_0'(0) + \sum_{m=1}^{\infty} Y_m'(0) = \frac{T_0}{L}\sqrt{2gh}. \tag{38}$$

Thus, X and Y satisfy the system (5) and it must be the exact solution for (5) with the initial conditions (6). □

4. Example

In this section, the HAM is applied to obtain the approximate solutions of the system (5) with the initial conditions (6). We have also used symbolic software Mathematica to solve the system of linear equations (22) with the initial conditions (23). Few components of the series solutions of (20) are given as follows:

$$X_0 = \frac{T_0\sqrt{gh}\pi T}{\sqrt{2}d_b},$$
$$Y_0 = \frac{T_0\sqrt{gh}\pi T}{\sqrt{2}d_b},$$
$$X_1 = 0,$$
$$Y_1 = \frac{\hbar\pi T^3\sqrt{gh}T_0}{6\sqrt{2}d_b}\left(1 + \frac{g\pi^2T^2hm_2}{20d_b^2k_2} + \frac{g^2\pi^4T^4h^2m_2^2}{210d_b^4k_2^2}\right),$$
$$X_2 = \frac{\hbar^2\pi T^5\sqrt{gh}T_0k_1m_2}{120\sqrt{2}d_bk_2m_1}\left(1 + \frac{g\pi^2T^2hm_2}{42d_b^2k_2} + \frac{g^2\pi^4T^4h^2m_2^2}{756d_b^4k_2^2}\right),$$
$$Y_2 = \frac{\hbar\pi T^3\sqrt{gh}T_0}{6\sqrt{2}d_b}\left(1 + \hbar + \frac{\hbar T^2}{2} + \frac{\hbar T^2k_1}{20k_2} + \frac{g\hbar\pi^2T^4hm_2k_1}{840d_b^2k_2^2} + \frac{g\pi^2T^2hm_2}{20d_b^2k_2} + \frac{11g\hbar\pi^2T^4hm_2}{840d_b^2k_2} + \cdots\right),$$
$$\vdots \tag{39}$$

It is clear that the HAM series solutions (20) on depend on the convergence-control parameter $\hbar$ which provides a simple way to adjust and control the convergence of the series solutions. In fact, it is very important to ensure that the series (20) are convergent. To this end, we have plotted $\hbar$-curve of $X'(0.1)$ and $Y'(0.1)$ by fifth- and sixth-order approximation of the HAM in Figures 2 and 3, respectively, for values $m_1 = 0.01$, $m_2 = 1$, $k_1 = 0.1$, $k_2 = 0.02$, $g = 0.8$, $h = 0.01$, and $d_b = 0.9$. According to these $\hbar$-curves, it is easy to discover the valid region of convergence-control parameter $\hbar$ which corresponds to the line segment nearly parallel to the horizontal axis. For clearer presentation, these valid regions have been listed in Table 2. Furthermore, these valid regions ensure us the convergence of the obtained series. Liao [18] has pointed out that when $\hbar = -1$, the solution obtained by the HAM is the same as the series solution obtained using HPM.

Now, an error analysis is introduced to obtain the optimal value of convergence-control parameter $\hbar$. Toward this end, we define $\varphi_X(T;\hbar)$ and $\varphi_Y(T;\hbar)$ to be m-order approximation HAM solution as follows:

$$\varphi_X(T;\hbar) = \sum_{j=0}^{m-1} X_j,$$
$$\varphi_Y(T;\hbar) = \sum_{j=0}^{m-1} Y_j. \tag{40}$$

We substitute (40) into nonlinear system (5) and obtain the residual error functions $\mathrm{ER}_X(X,Y,\hbar_1)$ and $\mathrm{ER}_Y(X,Y,\hbar_2)$ as follows:

$$\mathrm{ER}_X(X,Y;\hbar_1) = \frac{d^2\varphi_X(T;\hbar_1)}{dT^2} + w_{01}^2\varphi_X(T;\hbar_1) - w_{01}^2\varphi_Y(T;\hbar_1), \tag{41}$$

$$\mathrm{ER}_Y(X,Y;\hbar_2) = \frac{d^2\varphi_Y(T;\hbar_2)}{dT^2} + w_{02}^2\varphi_Y(T;\hbar_2) + \frac{1}{3}(\varphi_Y(T;\hbar_2))^3 + \frac{2}{15}(\varphi_Y(T;\hbar_2))^5 + (1-w_{02}^2)\varphi_X(T;\hbar_2). \tag{42}$$

Following [19], we define the square residual error for the m-order approximation to be

$$RX(\hbar_1) = \int_0^1 (\mathrm{ER}_X(X,Y,;\hbar_1))^2 dT,$$
$$RY(\hbar_2) = \int_0^1 (\mathrm{ER}_Y(X,Y,;\hbar_2))^2 dT. \tag{43}$$

We can obtain the values of $\hbar_1$ and $\hbar_2$ for which the $RX(\hbar_1)$ and $RY(\hbar_2)$ are minimum. The optimal values of convergence-control parameters $\hbar_1$ and $\hbar_2$ are determined by solving the system of equations as

$$\frac{dRX(\hbar_1^*)}{d\hbar_1} = 0, \qquad \frac{dRY(\hbar_2^*)}{d\hbar_2} = 0. \tag{44}$$

In [20], several methods have been introduced to find the optimal value of $\hbar$. In Table 3, the minimum values of $RX(\hbar_1)$ and $RY(\hbar_2)$ have been given with optimal values of $\hbar_1^*$ and $\hbar_2^*$ for 4-, 5-, and 6-order approximations.

In Table 4, the absolute errors ER_X and ER_Y have been calculated for various $T \in (0,1)$ when 5- and 6-order approximation HAM solutions are considered. From the table, it can be seen that the HAM provides us with the accurate approximate solution for the inner-resonance of tangent cushioning packaging system based on critical components (5).

Table 2: The admissible value of $\hbar$ derived from Figures 2 and 3.

m	5	6
$T(t)$	$-1.8 \le \hbar \le -0.7$	$-1.3 \le \hbar \le -0.8$
$I(t)$	$-1.3 \le \hbar \le -0.75$	$-1.25 \le \hbar \le -0.75$

Table 3: The minimum values of $RX(\hbar^*)$ and $RY(\hbar^*)$ for various orders of approximations.

m	$X(T)$		$Y(T)$	
	$\hbar^*$	Minimum $RX(\hbar^*)$	$\hbar^*$	Minimum $RY(\hbar^*)$
4	−0.950213	1.60718×10^{-12}	−0.950432	2.54142×10^{-8}
5	−0.973770	3.80411×10^{-15}	−0.971307	3.62018×10^{-11}
6	−0.983485	3.84892×10^{-18}	−0.993411	4.43599×10^{-14}

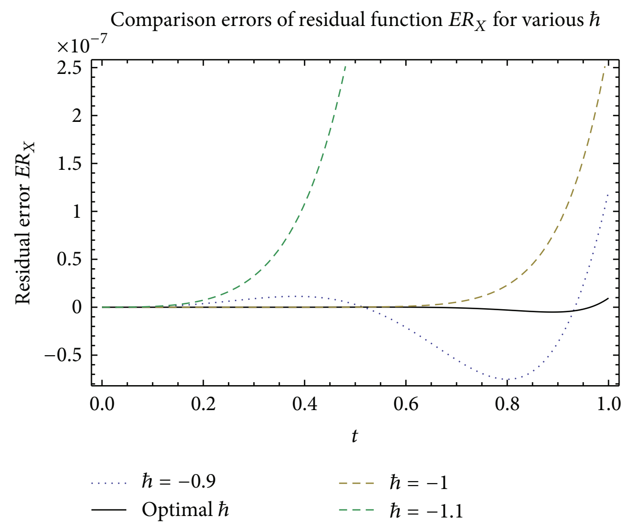

Figure 4: The errors of residual equation (41) using the sixth-order approximate solution for various $\hbar$ and $T \in (0,1)$.

The residual errors ER_X and ER_Y have been plotted in Figures 4 and 5 for $T \in (0,1)$ and various convergence-control parameters $\hbar$. By considering these Figures, it is to be noted that the solution obtained using HAM gives an analytical solution with high order of accuracy with few iterations.

5. Conclusion

The homotopy analysis method (HAM) is applied to obtain approximate solution of inner-resonance of tangent cushioning packaging system based on critical components. It is shown that the HAM solution contains the convergence-control parameter $\hbar$, which provides a simple way to adjust and control the convergence region of the resulting infinite

Table 4: The residual errors ER_X and ER_Y for various $T \in (0, 1)$.

T	5-order		6-order	
	ER_X	ER_Y	ER_X	ER_Y
0.1	7.65813×10^{-11}	1.37420×10^{-9}	3.18762×10^{-13}	4.47395×10^{-11}
0.2	3.06638×10^{-10}	2.43550×10^{-8}	1.76197×10^{-14}	4.84994×10^{-11}
0.3	1.58864×10^{-10}	8.87158×10^{-8}	3.49043×10^{-12}	1.21465×10^{-9}
0.4	2.05543×10^{-9}	5.62311×10^{-7}	9.40057×10^{-12}	7.36418×10^{-10}
0.5	2.20192×10^{-9}	1.18024×10^{-6}	7.17227×10^{-11}	1.31996×10^{-8}
0.6	1.04361×10^{-8}	6.66260×10^{-7}	5.64075×10^{-11}	4.10080×10^{-8}
0.7	5.15236×10^{-8}	3.16150×10^{-7}	6.52645×10^{-10}	1.44031×10^{-8}
0.8	1.17256×10^{-7}	1.07335×10^{-5}	2.96157×10^{-9}	2.22440×10^{-7}
0.9	1.17456×10^{-7}	1.27292×10^{-5}	4.99250×10^{-9}	5.66406×10^{-7}

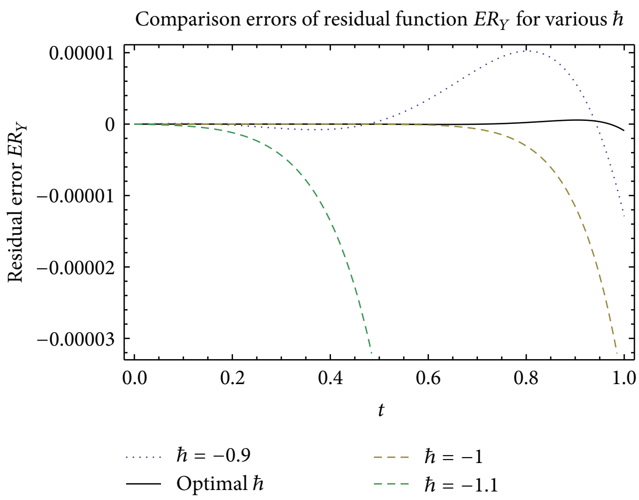

Figure 5: The errors of residual equation (42) using the sixth-order approximate solution for various $\hbar$ and $T \in (0, 1)$.

series. The convergence of HAM is also proved for inner-resonance of tangent cushioning packaging system based on critical components. The obtained results show that HAM is an accurate and effective technique for obtaining the approximate solution of inner-resonance of tangent cushioning packaging system based on critical components.

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