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Research Article

Recursion Operator and Local and Nonlocal Symmetries of a New Modified KdV Equation

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The recursion operator of a new modified KdV equation and its inverse are explicitly given. Acting the recursion operator and its inverse on the trivial symmetry 0 related to the identity transformation, the infinitely many local and nonlocal symmetries are obtained. Using a closed finite dimensional symmetry algebra with both local and nonlocal symmetries of the original model, some symmetry reductions and exact solutions are found.

1. Introduction

The symmetry study plays an important role in almost all the scientific fields, especially, in mathematical physics. To find infinitely many symmetries of a given integrable model, one of the best ways is to construct a recursion operator of the studied model. Recently, Lou pointed out that the infinitely many nonlocal symmetries can be simply constructed via acting the inverse recursion operator on the identity transformation [1], infinitely many Lax pairs [2], Darboux transformations [3], Bäcklund transformations [4], conformal transformations [5], and so on.

The mKdV equation,

$$u_t - 6u^2 u_x + u_{xxx} = 0, \tag{1}$$

is one of the most important nonlinear systems and a diversity of nonlinear physical phenomena can be successively explained by this simple model [6]. To study different physical phenomena, there exist some different versions of the modified KdV models such as the Gardner equation (the combination of the KdV and the mKdV systems) and the Wadati equations. Recently, to describe the interactions between KdV solitons and cnoidal periodic waves, Lou proposed a new modified KdV (NMKdV) equation [7],

$$v_t - \frac{3}{2}\frac{v_{xx}^2}{v_x} - 2v_x^3 + v_{xxx} = 0, \tag{2}$$

which is a combination of the potential mKdV equation (without the second term of (2)) and the Schwarzian KdV equation (without the third term of (2)).

In this paper, we investigate some integrable properties and exact solutions of the NMKdV equation (2). The transformation relation between the mKdV equation (1) and the NMKdV (2) is proposed in Section 2. In Section 3, using the factorization procedure and the transformation theorem, the recursion operator and its inverse are explicitly given. Applying the recursion and inverse recursion operators on some trivial seeds such as the identity transformation and the scaling transformation, the infinitely many local and nonlocal symmetries are obtained. In Section 4, some symmetry reductions and exact solutions are obtained by using a closed finite dimensional symmetry algebra. The last section is the summary and discussions.

2. The Relation between mKdV and NMKdV

To find the relation between the mKdV equation (1) and the NMKdV (2), we study exact solution of the mKdV equation via generalized tanh function expansion method proposed by Jiao [8]. For the mKdV equation (1), the generalized truncated tanh function expansion can have the following form:

$$u = u_0 + u_1 \tanh(v), \tag{3}$$

where u_0, u_1, and v are arbitrary functions of $\{x, t\}$ and should be determined later.

The usual tanh function expansion method, with u_0 and u_1 being constants and $v = kx + \omega t$, is used to find the single kink soliton solution of the mKdV equation.

Substituting (3) into (1), we have

$$\begin{aligned} & 6u_1 v_x \left(u_1^2 - v_x^2\right) \tanh^4(v) \\ & \quad + 6\left\{v_x\left[2u_0 u_1^2 + \left(u_1 v_x\right)_x\right] - u_1^2 u_{1x}\right\} \\ & \quad \times \tanh^3(v) \\ & \quad + \left\{u_1\left[8v_x^3 - v_t - v_{xxx} - 6u_1 u_{0x}\right.\right. \\ & \qquad \left. -12u_0 u_{1x} + 6\left(u_0^2 - u_1^2\right) v_x\right] \\ & \qquad \left. -3\left(u_{1x} v_x\right)_x\right\} \tanh^2(v) \\ & \quad + \left\{u_{1t} + u_{1xxx} - 6v_x\left[2u_0 u_1^2 + \left(u_1 v_x\right)_x\right]\right. \\ & \qquad \left. - 6\left(u_0^2 u_1\right)_x\right\} \tanh(v) \\ & \quad + u_{0t} + \left(u_{0xx} - 2u_0^3\right)_x + 3\left(u_{1x} v_x\right)_x \\ & \quad + u_1\left(v_t + v_{xxx} - 2v_x^3 - 6u_0^2 v_x\right) \\ & = 0. \end{aligned} \tag{4}$$

Vanishing the coefficients of $\tanh^i(v)$ for $i = 4, 3, 2$, we have

$$u_1 = \delta v_x, \quad \delta = \pm 1, \tag{5}$$

$$u_0 = -\frac{1}{2}\delta\left(\ln\left(v_x\right)\right)_x, \tag{6}$$

$$\Delta \equiv 2v_x v_t - 4v_x^4 + 2v_x v_{xxx} - 3v_{xx}^2 = 0. \tag{7}$$

Substituting (5), (6), and (7) into (4), the remaining items become

$$\begin{aligned} & \frac{\delta}{4}\left\{2\tanh(v)\,\partial v_x^{-1} + 2 - v_x^{-1}\partial^2 v_x^{-1}\right. \\ & \qquad \left. -\left(v_x^{-1}\right)_x \partial v_x^{-1}\right\}\Delta = 0 \end{aligned} \tag{8}$$

which is clearly correct because of (7), where

$$\partial \equiv \frac{\partial}{\partial x}. \tag{9}$$

Substituting (5) and (6) into (3), we have

$$u = \frac{\delta}{2}\left(\ln\frac{\cosh^2 v}{v_x}\right)_x. \tag{10}$$

Actually, the transformation relation (10) linked with the mKdV (1) and the NMKdV as follows:

$$\begin{aligned} & u_t - 6u^2 u_x + u_{xxx} \\ & \quad = \frac{\delta}{2}\partial\left(2\tanh v - v_x^{-1}\partial\right) \\ & \qquad \times \left(v_t - \frac{3}{2}\frac{v_{xx}^2}{v_x} - 2v_x^3 + v_{xxx}\right) \\ & \quad \equiv T\left(v_t - \frac{3}{2}\frac{v_{xx}^2}{v_x} - 2v_x^3 + v_{xxx}\right). \end{aligned} \tag{11}$$

Thus, the integrability of the NMKdV is guaranteed by the integrability of the mKdV equation because the infinitely many symmetries and recursion operator (and then bi-Hamiltonian and Lax pairs) of the mKdV can be transformed to those of the NMKdV with the help of the operator T and its inverse. In the next section, we study the infinitely many local and nonlocal symmetries of the NMKdV by finding its recursion operator and its inverse recursion operator.

3. Recursion and Inverse Recursion Operators

A symmetry, σ, for a nonlinear system

$$E(u) = 0, \tag{12}$$

is defined as a solution of its linearized system:

$$E'(u)\,\sigma \equiv \frac{d}{d\epsilon}E(u + \epsilon\sigma)\Big|_{\epsilon=0} = 0. \tag{13}$$

For the mKdV equation (1) and NMKdV (2), the symmetry definition equations read

$$\sigma_t^u - 6u^2\sigma_x^u - 12uu_x\sigma^u + \sigma_{xxx}^u = 0 \tag{14}$$

and

$$\sigma_t^v - 3\frac{v_{xx}}{v_x}\sigma_{xx}^v + \frac{3}{2}\frac{v_{xx}^2}{v_x^2}\sigma_x^v - 6v_x^2\sigma_x^v + \sigma_{xxx}^v = 0, \tag{15}$$

respectively.

From the transformation relation (10), the symmetry relation between the mKdV (1) and the NMKdV (2) has the form

$$\sigma^u = T\sigma^v, \quad \sigma^v = T^{-1}\sigma^u. \tag{16}$$

It is known that for a (1 + 1)-dimensional integrable system, there exists a recursion operator, Φ, such that a set of infinitely many symmetries can be obtained by acting the recursion operator on a seed symmetry σ_0 as follows:

$$\sigma_n = \Phi^n\sigma_0, \quad n = 0, 1, 2, \ldots. \tag{17}$$

From the symmetry transformation relation (17), we can find the relation between the recursion operators Φ (the recursion operator of the mKdV equation) and $\widetilde{\Phi}$ (the recursion operator of the NMKdV):

$$\widetilde{\Phi} = T^{-1}\Phi T. \tag{18}$$

One recursion operator of the mKdV has also been given in the literature [9]:

$$\Phi = D^2 - 4u^2 - 4u_x D^{-1} u. \tag{19}$$

For convenient calculation, we firstly give out the factor forms of Φ and T:

$$\begin{aligned} \Phi &= \partial g^2 \partial g^{-4} u^{-1} \partial g^2 \partial^{-1} u, \\ T &= \frac{\delta}{2} \partial v_x^{-1} \cosh^2 v \partial \cosh^{-2} v, \end{aligned} \tag{20}$$

where $g = \exp(\partial^{-1} u)$.

Using the above results, we get the recursion operator

$$\begin{aligned} \widetilde{\Phi} &= T^{-1}\Phi T \\ &= \cosh^2 v \partial^{-1} v_x \cosh^{-2} v \partial^{-1} \\ &\quad \times \partial g^2 \partial g^{-4} u^{-1} \partial g^2 \partial^{-1} \\ &\quad \times u \partial v_x^{-1} \cosh^2 v \partial \cosh^{-2} v \\ &= v_x^2 u^{-1} \cosh^{-2} v \partial v_x^{-1} \cosh^2 v \partial^{-1} \\ &\quad \times u \partial v_x^{-1} \cosh^2 v \partial \cosh^{-2} v \end{aligned} \tag{21}$$

and the inverse recursion operator

$$\begin{aligned} \widetilde{\Phi}^{-1} &= \cosh^2 v \partial^{-1} v_x \cosh^{-2} v \partial^{-1} u^{-1} \partial v_x \\ &\quad \times \cosh^{-2} v \partial^{-1} u v_x^{-2} \cosh^2 v \end{aligned} \tag{22}$$

for the NMKdV (2).

To give out some sets of infinitely many symmetries of (2), one has to find out some seed symmetries which may be found via some types of trivial methods. For instance, using the standard classical Lie group approach [10] or the standard classical Lie symmetry approach [11], one can find that the only possible Lie point symmetries of the NMKdV (2) are the symmetries related to the space translation,

$$K_0 = v_x, \tag{23}$$

time translation,

$$K_1 = v_t, \tag{24}$$

the scaling invariance,

$$\tau_1 = x v_x + 3t v_t, \tag{25}$$

and the exponential function transformations,

$$N_0^{(\gamma)} = \exp(2\gamma v), \quad \gamma = -1, 0, 1. \tag{26}$$

Clearly, $N_0^{(-1)}$ can be obtained from $N_0^{(1)}$ by using the discrete invariant transformation $v \to -v$ of (2) and can be recombined as the following seeds:

$$\begin{aligned} N_0^{(s)} &= \sinh(2v), \\ N_0^{(c)} &= \cosh(2v). \end{aligned} \tag{27}$$

The symmetry $N_0^{(0)} \equiv 1$ is related to the field v translation invariance of (2).

Applying the recursion operator $\widetilde{\Phi}$ to the seeds K_0, τ_0, and $N_0^{(\gamma)}$, we can find five sets of infinitely many general symmetries, the K symmetries,

$$K_n = \widetilde{\Phi}^n K_0, \quad (n = 1, 2, \dots), \tag{28}$$

τ symmetries,

$$\tau_n = \widetilde{\Phi}^{n-1} \tau_1, \quad n = 0, \pm 1, \pm 2, \dots, \tag{29}$$

and $N^{(\gamma)}$ symmetries,

$$N_n^{(\gamma)} = \widetilde{\Phi}^{-n} N_0^{(\gamma)}, \quad n = 0, 1, 2, \dots, \ \gamma = 0, s, c. \tag{30}$$

For the K_n symmetries, $n \geq 0$ because K_0 is a kernel of the recursion operator $\widetilde{\Phi}$. Similarly, $n \geq 0$ for $N_n^{(\gamma)}$ symmetries because $N_0^{(\gamma)}$ are all kernels of the inverse recursion operator $\widetilde{\Phi}^{-1}$.

It is not difficult to find that the K_n symmetries for $n \geq 0$, $N_n^{(\gamma)}$, and τ_n symmetries for $n = 0$ are local symmetries while all others are nonlocal ones.

4. Group Invariant Solutions

In this section, we discuss the possible group invariant solutions via symmetry reductions by using the Lie point symmetry algebra $\mathcal{G} \equiv \{K_0, K_1, \tau_1, N_0^{(-1)}, N_0^{(0)}, N_0^{(1)}\}$ for the NMKdV equation (2).

To look for the group invariant solutions under the symmetry algebra $\mathcal{G}$, we have to solve the following symmetry constraint condition:

$$\begin{aligned} & c_1 K_0 + c_2 K_1 + c_3 \tau_1 + c_0 N_0^{(0)} + c_4 N_0^{(-1)} + c_5 N_0^{(1)} \\ &\quad = (c_3 x + c_1) v_x + (3 c_3 t + c_2) v_t + c_0 \\ &\qquad + c_4 \exp(2v) + c_5 \exp(-2v) \\ &\quad = 0, \end{aligned} \tag{31}$$

with arbitrary constants c_i, $i = 0, 1, 2, \dots, 5$.

Solving the symmetry constraint condition (31), we have two important cases.

Case I. $c_3 \neq 0$. For $c_3 \neq 0$, the general solution of the symmetry constraint condition (31) has the following form:

$$v = \frac{1}{2}\ln\left\{\frac{1}{2c_4}\left[\frac{\sqrt{c_0^2-4c_4c_5}}{3c_3} \times \tanh\left(\sqrt{c_0^2-4c_4c_5} \times \left(\ln\left(3c_3t+c_2\right)+F\left(\xi_1\right)\right)\right)-c_0\right]\right\}$$
$$\equiv \frac{1}{2}\ln\left\{\tanh\left[C\ln\left(t+t_0\right)+V\left(\xi\right)\right]-C_0\right\}+v_0, \quad (32)$$

where $\xi_1 = (c_3x+c_1)/(3c_3t+c_2)^{1/3}$,

$$\xi = \frac{x+x_0}{3(t+t_0)^{1/3}}, \quad (33)$$

x_0, t_0, v_0, C, and C_0 are arbitrary constants, and $V(\xi)$ is an function of ξ.

Substituting (32) into (2), it is straightforward to find the similarity reduction for the undetermined group invariant function $W = W(\xi) \equiv V_\xi$:

$$W_{\xi\xi} - \frac{3}{2}W_\xi^2W^{-1} + 27C - 2W^3 - 9\xi W = 0, \quad (34)$$

which is a variant form of the Painlevé II transcendent that means the solution (32) is an interaction solution between one soliton and the Painlevé II wave for the NMKdV (2).

Case II. $c_3 = 0$. In this case, the general solution of the group invariant condition (31) reads

$$v = \frac{1}{2}\ln \times\left[-\frac{1}{2c_4}\left(\sqrt{c_0^2-4c_4c_5} \times \tanh\sqrt{c_0^2-4c_4c_5}\left(t+P_1\left(\eta_1\right)\right)-c_0\right)\right]$$
$$\equiv \frac{1}{2}\ln\left[\tanh\left(Ct+P\left(\eta\right)\right)-C_0\right]+v_0, \quad (35)$$

where $\eta_1 = c_1t - c_2x$ and

$$\eta = x + c_1t, \quad (36)$$

with arbitrary constants c_1, C, C_0, and v_0.

The group invariant solution for the field $Q = Q(\eta) \equiv P_\eta$ can be obtained after substituting (35) into (2). The result reads

$$Q_\eta^2 - 4Q^4 + C_1Q^3 + 2c_1Q^2 + CQ = 0, \quad (37)$$

with C_1 being an arbitrary integrating constant.

Obviously, the solution of (37) can be expressed by means of the Jacobi elliptic functions, say,

$$Q = \frac{1+m\delta sn\left(k\eta,m\right)}{a+bsn\left(k\eta,m\right)},$$
$$\delta^2 = 1, \qquad k^2 = \frac{4\left(\delta ma-b\right)}{\left(\delta ma+b\right)\left(a^2-b^2\right)}, \quad (38)$$

where three arbitrary constants $\{c_1, C, C_1\}$ have been related to others $\{a, b, m\}$ via

$$C = \frac{8m\left(1-m^2\right)}{\left(b^2-a^2\right)\left(ma+\delta b\right)},$$
$$C_1 = \frac{8\left[\delta ab\left(1-m^2\right)+2m\left(a^2-b^2\right)\right]}{\left(b^2-a^2\right)\left(ma-\delta b\right)}, \quad (39)$$
$$c_1 = \frac{2\left[ma\left(m^2-5\right)+b\delta\left(5m^2-1\right)\right]}{\left(b^2-a^2\right)\left(ma+\delta b\right)}.$$

Thus, the second reduction solution (35) with (37) is related to the interaction solution between one soliton and elliptic periodic waves.

5. Summary and Discussions

In summary, we have studied the symmetries and symmetry reduction solutions of a new modified KdV equation proposed by Lou. The model is a combination form of the Schwarzian KdV and the potential modified KdV equation.

The recursion operator and the inverse recursion operator are explicitly given. Applying the recursion operator and its inverse to the trivial seed solution, 0, related to the identity transformation, we find one (half) set of local symmetries K_n, $n \geq 0$ and three (half) sets of nonlocal symmetries N_n^γ, $n \geq 0$, $\gamma = 0, s, c$, or equivalently $\gamma = 0, \pm 1$. Applying the recursion operators to the scaling invariance, a full set of time dependent nonlocal symmetries τ_n, $n = 0, \pm 1, \pm 2, \ldots$ can be obtained.

For the usual KdV and the modified KdV equations, to find the interaction solutions between one soliton and another kind of waves such as the elliptic periodic waves, one has to use the nonlocal symmetries related to the Darboux transformation [12] and/or the Bäcklund transformation [13]. It is interesting that for the NMKdV equation, to find the interaction solutions between soliton and other waves, we can use only the local Lie point symmetries of the model without any nonlocal symmetries.

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